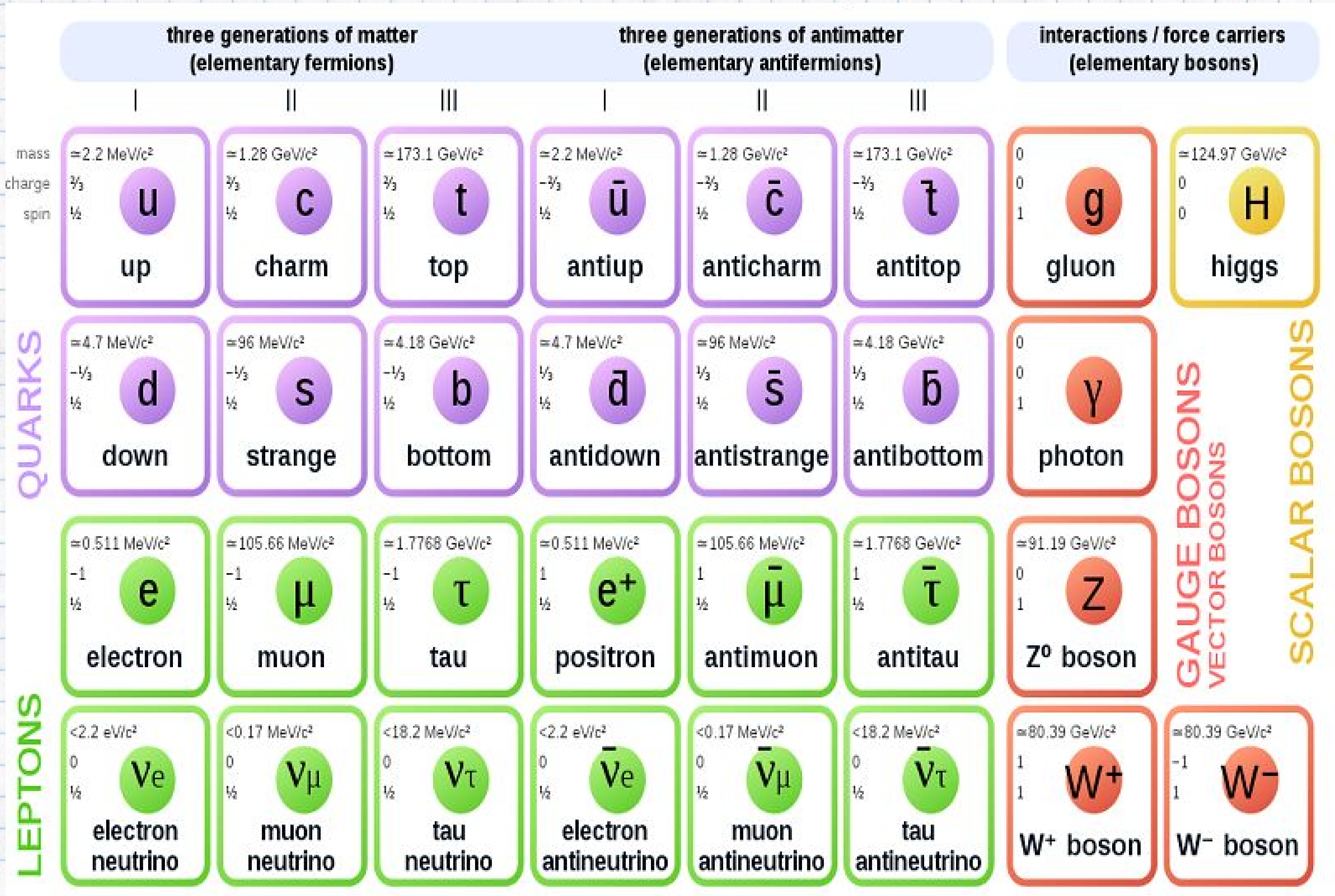
The Standard Model: theory & phenomenology

A new 'periodic table'



Symmetry: $\underbrace{ISO(1,3)}_{\text{geometric global symmetry}} \times \underbrace{SU(3)_c \times SU(2)_w \times \overbrace{\quad U(1)_Y}^{U(1)_{EM}}}_{\text{internal gauge symmetry}}$

geometric global symmetry   internal gauge symmetry

$$G_\mu^a \ (8,1,0) \qquad A_\mu^\alpha \ (1,3,0) \qquad B_\mu$$
$$g_3 \qquad g_2 \qquad g_1$$

Matter fields

leptons
$$L_i = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \sim (1, 2, -1/2)$$
$$E_i = e_L^c, \ \mu_L^c, \ \tau_L^c \sim (1, 1, 1)$$

$$Q_i = \begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L \sim (3, 2, 1/6)$$
$$U_i = u_L^c, \ c_L^c, \ t_L^c \sim (\bar{3}, 1, -2/3)$$
$$D_i = d_L^c, \ s_L^c, \ b_L^c \sim (\bar{3}, 1, 1/3)$$

$$\underbrace{H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}}_{\text{Higgs scalar doublet}} \sim (1, 2, 1/2)$$

Higgs scalar doublet

Q. What are the quantum # for right-h. particles?

Constructing the invariant Lagrangian

Covariant derivatives:

- Left-handed fermions $\psi_L = Q_{iL}, L_{iL}$

$$D_\mu \psi_L = \left(\partial_\mu - ig_3 \frac{\lambda^a}{2} G^a_\mu - ig_2 \frac{\sigma^\alpha}{2} A^\alpha_\mu - ig_1 Y B_\mu\right)\psi_L$$

Note: $-ig_3 \frac{\lambda^a}{2} G^a_\mu \cdot L_{iL} \equiv 0$

$$-ig_1 Y B_\mu \cdot (Q_{iL}, L_{iL}) =$$
$$= -ig\left(1/6, -1/2\right) B_\mu (Q_{iL}, L_{iL})$$

- Left-handed anti-fermions $\psi^c_L = U_{iL}, D_{iL}, E_{iL}$

$$D_\mu \psi^c_L = \left(\partial_\mu + ig_3 \frac{\lambda^a}{3} G^a_\mu + ig_1 Y' B_\mu\right)\psi^c_L$$

$Y(U_i, D_i, E_i) = (-2/3, -1/3, 1)$

Gauge transformations:

$$D'_\mu = U D_\mu U^\dagger \Rightarrow F'_{\mu\nu} = U F_{\mu\nu} U^\dagger$$

Hence, the invariant kinetic term is $\propto Tr(F_{\mu\nu}F^{\mu\nu})$

$$(Tr\, F_{\mu\nu}F^{\mu})' = Tr(U F_{\mu\nu} \underbrace{U^\dagger U}_{1} F^{\mu\nu} U^\dagger) = Tr(\underbrace{U^\dagger U}_{1} F_{\mu\nu}F^{\mu\nu})$$

$$= Tr\, F_{\mu\nu}F^{\mu\nu}$$

Fermions: $\psi' = U\psi$, $(D_\mu\psi)' = U D_\mu \underbrace{U^\dagger U}_{1}\psi = U D_\mu \psi$

$\bar{\psi}' = \bar{\psi} U^\dagger$

Hence, kinetic term $-i\bar{\psi}\gamma^\mu D_\mu \psi$

Higgs: $(D_\mu H)^\dagger (D^\mu H)$

Covariant derivative for the Higgs doublet:

$$D_\mu H = \left(\partial_\mu + i g_2 \frac{\sigma^\alpha}{2} A_\mu^\alpha + i g_1 \frac{1}{2} B_\mu\right) H$$

Matrix-valued gauge fields & field strengths

$U(1)_Y$: B_μ, $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$

$SU(2)_W$: $A_\mu \equiv A_\mu^\alpha \frac{\tau^\alpha}{2}$, $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i g_2 [A_\mu, A_\nu]$

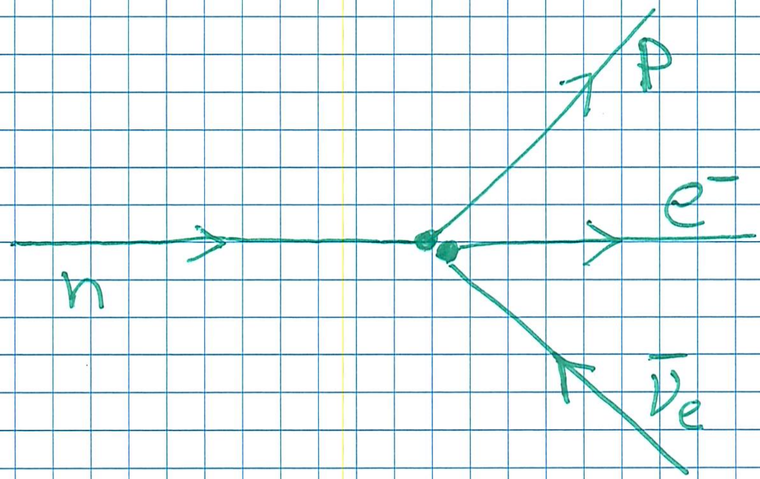
$SU(3)_c$: $G_\mu = G_\mu^a \frac{\lambda^a}{2}$, $G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - i g_3 [G_\mu, G_\nu]$

Q. - check that field strength is

$$F_{\mu\nu} = \frac{1}{ig}[D_\mu, D_\nu],$$ D_μ - covariant derivative

From Fermi theory to the Standard Model – A lesson from the past

Fermi theory of β-decay: $\quad n \to p\, e^- \bar{\nu}_e$ (Fermi, 1934)



$$\mathcal{L}_F \propto - G_F\, J_\mu^+ J^{-\mu}$$

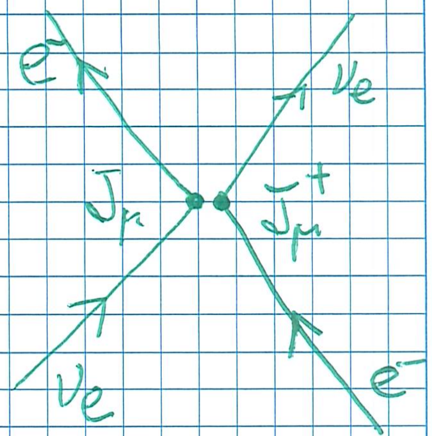
$$J_\mu^- = \bar{p}\,\gamma_\mu\, n + \bar{\nu}_e\, \gamma_\mu\, e^- \quad [n \to p,\ e^- \to \nu_e$$

$$J_\mu^+ = \bar{n}\,\gamma_\mu\, p + \bar{e}\,\gamma_\mu\, \nu_e \quad [p \to n,\ \nu_e \to e^-$$

$$\times\ \bar{\nu}_e e^-$$

$$G_F \approx 1.17 \cdot 10^{-5} \frac{1}{\mathrm{GeV}^2} \quad \text{(Fermi constant)}$$

- Othe processes, e.g. $\nu_e e^- \to \nu_e e^-$

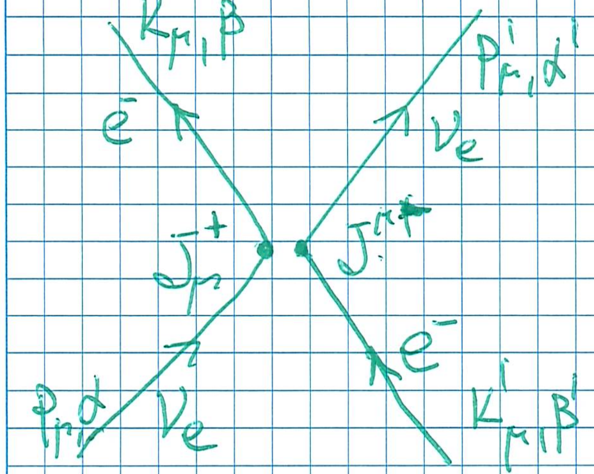


- Extention: include mor leptons (μ, τ) and quarks...

Fermi theory accurately describes (at low energies!):

- $n \to p e^- \bar{\nu}_e$, $e^- p \to \nu_e n$
- $\mu^- \to e^- \bar{\nu}_e \nu_\mu$, $\tau \to \mu^- \bar{\nu}_\mu \nu_\tau$, $\nu_\tau \pi^-$, ...
- decays of mesons π, K ...
- ν scattering ($\nu_\mu e^- \to \mu^- \nu_e$, $\nu_\mu n \to \mu^- p$, ...)

Excellent theory, but...



$$A \propto G_F \, \bar{e}(k,\beta)\gamma_\mu \nu_e(p,\alpha)\, \bar{\nu}_e(p',\alpha')\gamma^\mu e(k',\beta')$$

$$\frac{1}{2}\times\frac{1}{2}\sum_{\alpha,\beta,\alpha',\beta'} |A|^2 = \frac{1}{4}\sum_{\alpha,\beta,\alpha',\beta'} \bar{e}(k,\beta)\gamma_\mu \nu_e(p,\alpha)\,\bar{\nu}_e(p',\alpha')\gamma^\mu e(k',\beta')\,\bar{e}(k',\beta')\gamma^\nu \nu_e(p',\alpha')$$

$$\times\, \bar{\nu}_e(p,\alpha)\gamma_\nu e(k,\beta)$$

$$\boxed{\text{Note: } \sum_\alpha f(\alpha)\bar{f}(\alpha) = p_\mu\gamma^\mu - m_f}$$

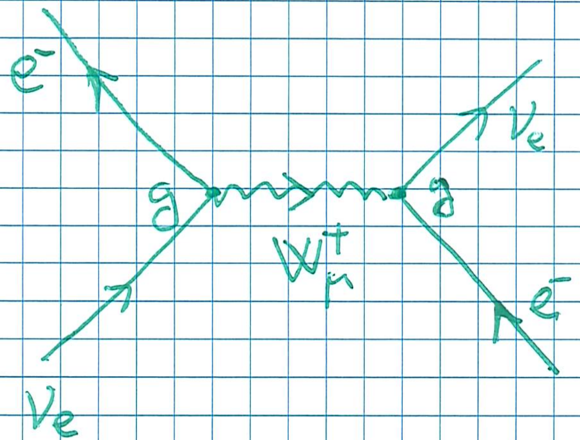
Hence, $|A|^2 \propto p^4$

More accurately $\sigma(\nu_e e^- \to e^- \nu_e) = \dfrac{G_F^2 E_{CM}^2}{\pi}$

Perturbative unitarity at high energies ($E \sim 500\,\text{GeV}$) is violated!!!

Introduce intermediate massive vector bosons (Glashow & others)

$$\mathcal{L} = -g J^{+}_{\mu} W^{-\mu} - g W^{+}_{\mu} J^{-\mu} + M_W^2 W^{+}_{\mu} W^{-\mu}$$



additional part in the previous amplitude

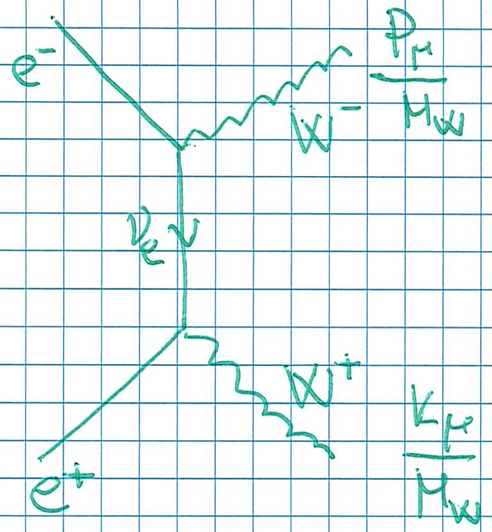
$$A \propto \frac{g^2}{p^2 - M_W^2}$$

- For $|p| \gg M_W^2$ cancells p^4
- For $|p| \ll M_W$, reproduces Fermi theory $\boxed{\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 M_W^2}}$

This particle has indeed been discovered, $M_W \simeq 80$ GeV!

(1983)

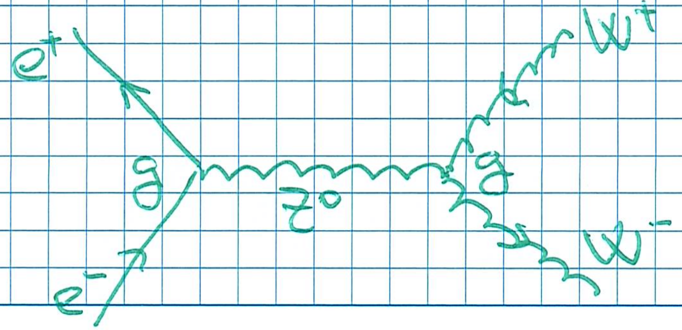
However,



Violation of perturbative unitarity, e.g. in $e^+e^- \to W^+W^-$ at $E \sim 500\,\text{GeV}$!!!

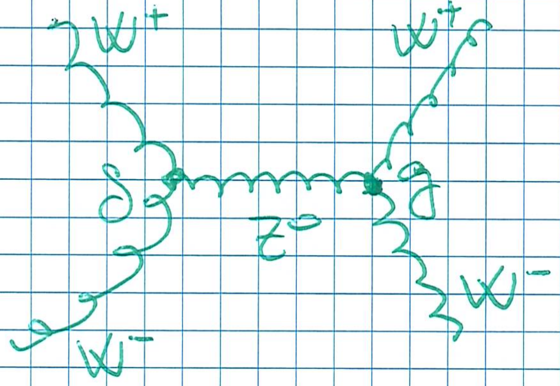
Introduce another massive vector boson Z^0

Fix $Z^0W^+W^-$ and $e^+e^-Z^0$ couplings to cancell out bad behaviour of amplitudes



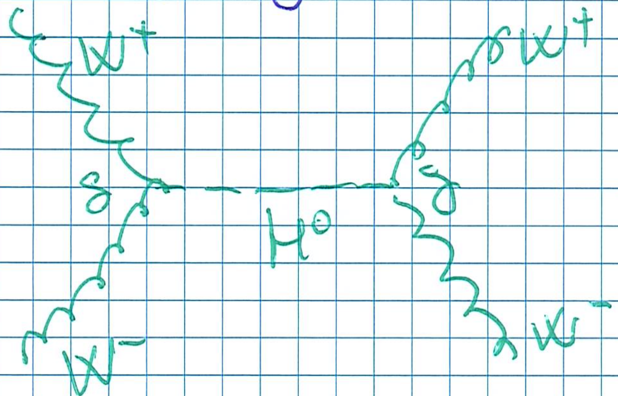
This is neutral Z-particle, has been also discovered (1983), $M_Z \simeq 91\,\text{GeV}$

However,



Violates unitarity at $E \sim$ few TeV

Introduce new massive scalar particle H^0, adjust couplings to 'restore' unitarity



This particle has been discovered in 2012, $M_H \simeq 125$ GeV!!!

This way you can 'theoretically' discover the Standard Model!!!

$$\mathcal{L}_{SM} = i\,\overline{\Psi}_{iL}\,\gamma^\mu D_\mu \psi_{iL} + i\,\overline{\Psi}^c_{iL}\,\gamma^\mu D_\mu \psi^c_{iL}$$

$$+ (D_\mu H)^+ (D^\mu H)$$

$$- \frac{1}{2}\,\mathrm{Tr}\,G_{\mu\nu}G^{\mu\nu} - \frac{1}{2}\,\mathrm{Tr}\,A_{\mu\nu}A^{\mu\nu} - \frac{1}{4}\,B_{\mu\nu}B^{\mu\nu}$$

Yukawa interactions

$$+ y^{(up)}_{ij}\,\overline{Q}_i\,U_j\,H + y^{(down)}_{ij}\,\overline{Q}_i\,D_j\,\overline{H}$$

$$+ y^{(lept)}_{ij}\,\overline{L}_i\,E_j\,\overline{H} + \text{h.c.}$$

$$\overline{H}^\alpha = \epsilon^{\alpha\beta} H^*_\beta$$

Higgs potential

$$- \frac{\lambda}{2}\left(H^+H - \frac{v^2}{2}\right)^2$$

Q. – check that this is invariant under rel. symetries

Few observations

- Only terms with $d \leq 4$
- No explicit masses for fermions & gauge fields

$$m\bar{\Psi}\Psi = m(\bar{\Psi}_L \Psi_R + h.c) = m(\bar{\Psi}_L \Psi_L^c + h.c.)$$

$\bar{\Psi}_L \Psi_L^c$ cannot form invariant $\Psi_L \sim 2$, $\Psi_L^c \sim 1$

- v - is the only mass parameter!

$v \to 0$ - scale-invariant limit

- Higgs mass term $\propto -\frac{\lambda}{2} v^2 H^+ H$

Since, $\lambda > 0$

$-\frac{\lambda}{2} v^2 < 0$ - tachyon! (instability)

It turns out that we can rewrite the above Lagrangian in new variables such that mass terms are explicit and no Higgs instability occurs $\Rightarrow$ the Higgs mechanism

First, one must carefully define the field configurations that correspond to the vacuum state of the theory:

$$\langle 0|\psi|0\rangle = \langle 0|A_\mu|0\rangle \text{ (up to a gauge transf.)} = 0$$

$$\boxed{\langle 0|H|0\rangle = \frac{v}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}}$$

$SU(2)_W \times U(1)_Y \rightarrow U(1)_{EM}$

spontaneous symm. breaking

If this is a global symmetry, we would expect

$$\dim\left(SU(2)_W \times U(1)_Y\right) - \dim\left(U(1)_{EM}\right) = 4 - 1 = 3$$

massless Goldstone bosons

However, $SU(2)_W \times U(1)_Y$ is a gauge symmetry, hence the would-be Goldstone bosons are gauge-variant and can be fixed-away by a gauge choice

What remains are 3 massive gauge fields. that precisely carry additional 3 degrees of freedom relative to massless gauge fields.

More explicitly, we inspect 'seagull' terms in the Higgs kinetic part of the Lagrangian:

$$\left[\left(i g_2 \frac{\sigma^\alpha}{2} A^\alpha_\mu + i g_1 \frac{1}{2} B_\mu\right) H\right]^* \times$$

$$\times \left[\left(i g_2 \frac{\sigma^\alpha}{2} A^\alpha_\mu + i g_1 \frac{1}{2} B_\mu\right) H\right]$$

Using Pauli matrices $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$g_2 \frac{\sigma^\alpha}{2} A^\alpha_\mu + g_1 \frac{1}{2} B_\mu = \begin{bmatrix} \frac{g_2}{2} A^3_\mu + \frac{g_1}{2} B_\mu , & \frac{g_2}{\sqrt{2}} W^+_\mu \\ \frac{g_2}{\sqrt{2}} W^-_\mu , & -\frac{g_2}{2} A^3_\mu + \frac{g_1}{2} B_\mu \end{bmatrix}$$

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(A^1_\mu \mp i A^2_\mu\right)$$

Introduce:

$$A^3_\mu = \cos\theta_W Z_\mu + \sin\theta_W A_\mu$$

$$B_\mu = -\sin\theta_W Z_\mu + \cos\theta_W A_\mu$$

$$g_1 \cos\theta_W = g_2 \sin\theta_W, \quad \tan\theta_W = \frac{g_1}{g_2}$$

$$g_2 \frac{\sigma^\alpha}{2} A^\alpha_\mu + \frac{1}{2} g_1 B_\mu = \begin{bmatrix} \frac{g_2}{2\cos\theta_W}\left[\sin^2\theta_W - \cos^2\theta_W\right] Z_\mu - A_\mu, & \frac{g_2}{\sqrt{2}} W^+_\mu \\ \frac{g_2}{\sqrt{2}} W^-_\mu, & \frac{g_2}{2\cos\theta_W} Z_\mu \end{bmatrix}$$

Introduce, $\quad H(x) = \frac{v}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ h(x) \end{pmatrix} \quad \Big| \quad \langle 0 | h(x) | 0 \rangle = 0$

$$\mathcal{L}_{mass} = \mathcal{L}_{seagull}\Big|_{H=\langle H\rangle} =$$

$$= \frac{v}{\sqrt{2}}(0,1)\begin{bmatrix} \frac{g_2}{2\cos\theta_W}(\sin^2\theta_W - \cos^2\theta_W)Z_\mu - A_\mu, & \frac{g_2}{\sqrt{2}}W_\mu^+ \\ \frac{g_2}{\sqrt{2}}W_\mu^- & , \frac{g_2}{2\cos\theta_W}Z_\mu \end{bmatrix} \times$$

$$\times \begin{bmatrix} \frac{g_2}{2\cos\theta_W}(\sin^2\theta_W - \cos^2\theta_W)Z^\mu - A^\mu, & \frac{g_2}{\sqrt{2}}W^{\mu+} \\ \frac{g_2}{\sqrt{2}}W^{\mu-} & , \frac{g_2}{2\cos\theta_W}Z^\mu \end{bmatrix} \frac{v}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= M_W^2\, W_\mu^- W^{\mu+} + \frac{1}{2}M_Z^2\, Z_\mu Z^\mu$$

$$M_W^2 = \frac{1}{4} g_2^2 v^2$$

$$M_Z^2 = \frac{1}{4} \frac{g_2^2}{\cos^2\theta_W} v^2$$

$$\boxed{\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = 1}$$ (+ quantum corrections)

$$\rho^{exp} \simeq 1,0107 \pm 0.0006$$

Define the Higgs boson mass:

$$m_h^2 = \frac{\partial V}{\partial H \partial H^+}\bigg|_{H=\langle H\rangle} = \lambda \frac{v^2}{2}(0,1)\begin{pmatrix}0\\1\end{pmatrix}$$

$$= +\lambda \frac{v^2}{2} > 0$$

Let's inspect now charged current interactions:

$$\mathcal{L}_{cc} = \frac{g_2}{2\sqrt{2}}\left[W_\mu^+ J_-^\mu + W_\mu^- J_+^\mu\right]$$

$$J_\mu^\alpha = \bar{L}_i \gamma_\mu \frac{\sigma^\alpha}{2} L_i + \bar{Q}_i \gamma_\mu \frac{\sigma^\alpha}{2} Q_i$$

$$J_\pm^\mu = 2(J^{1\mu} \mp i J^{2\mu})$$

$$J_+^\mu = 2\bar{L}_1 \gamma_\mu \frac{\sigma^1 - i\sigma^2}{2} L_1 + 2\bar{L}_2 \gamma_\mu \frac{\sigma^1 - i\sigma^2}{2} L_2 + \ldots$$

$$= 2(\bar{\nu}_e, \bar{e})_L \gamma_\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L +$$

$$+ 2(\bar{\nu}_\mu, \bar{\mu})_L \gamma_\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L + \ldots$$

$$= 2\bar{e}_L \gamma_\mu \nu_{eL} + 2\bar{\mu}_L \gamma_\mu \nu_{\mu L} + \ldots$$

$$= \bar{e}\gamma_\mu (1-\gamma_5)\nu_e + \bar{\mu}\gamma_\mu(1-\gamma_5)\nu_\mu + \ldots$$

[Q. show this]

$$J_-^\mu = (J_+^\mu)^\dagger$$

Now consider processes mediated by $W_\mu^\pm$ bosons at low-energies $q^2 \ll M_W^2$

W-boson propagator:

$$\Delta_{\mu\nu}(q) = \frac{1}{q^2 - M_W^2}\left[\eta_{\mu\nu} - \frac{q_\mu q_\nu}{M_W^2}\right]$$

$$\xrightarrow{q^2 \ll M_W} -\frac{\eta_{\mu\nu}}{M_W^2}$$

So the amplitude of the process mediated by $W^\pm$ is:

$$S_{cc}^{eff} = \frac{i}{2}\int d^4x\, d^4y\, \frac{g_2^2}{8} J_+^\mu(x) \underbrace{\langle 0|T\, W_\mu^-(x) W_\nu^+(y)|0\rangle}_{\Delta_{\mu\nu}(x-y)} \times J_-^\nu(y)$$

$$= \frac{i}{2} \int d^4x \, \frac{g_2^2}{8 M_W^2} J_+^{\mu} J_{-\mu}$$

$$\Downarrow$$

$$\mathcal{L}_{cc}^{eff} = \frac{g_2^2}{8 M_W^2} J_+^{\mu} J_{-\mu}$$

This decribes Fermi's interactions

$$\boxed{G_F = \frac{\sqrt{2}}{8} \frac{g_2^2}{M_W^2}}$$

Recall, $M_W^2 = \frac{1}{4} g_2^2 v^2$

$$\underline{v = (\sqrt{2} G_F)^{1/2} \simeq 246 \text{ GeV}}$$

Now, we inspect neutral current interactions:

$$\mathcal{L}_{NC} = \left\{ (g_2 \cos\theta_W + g_1 \sin\theta_W) J_3^\mu - g_1 \sin\theta_W J_{em}^\mu \right\} Z_\mu$$

$$+ \left\{ g_1 \cos\theta_W J_{em}^\mu + \overbrace{(g_1 \cos\theta_W - g_2 \sin\theta_W)}^{\to 0} J_3^\mu \right\} A_\mu$$

Recall, $g_1 \cos\theta_W = g_2 \sin\theta_W$

Identify, $e = g_1 \cos\theta_W$

$$\mathcal{L}_{NC} = \frac{e}{2 \cos\theta_W \sin\theta_W} J_\mu^{NC} Z^\mu$$

$$+ \underbrace{e J_{em}^\mu A_\mu}_{\text{electromagnetic interactions}}$$

$$J_\mu^{NC} = 2\left(J_\mu^3 - \sin^2\theta_W \, J_{em\,\mu}\right)$$

Recall, $J_\mu^3 = \overline{L}_i \gamma_\mu \frac{\sigma^3}{2} L_i + \overline{Q}_i \gamma_\mu \frac{\sigma^3}{2} Q_i$

At low energies, the processes mediated by the massive Z-boson is described by the effective ~~Fermi~~ 4 current-current Lagrangian:

$$\mathcal{L}_{NC}^{eff} = \frac{G_F}{\sqrt{2}} \rho \, J_\mu^{NC} J^{NC\,\mu}$$

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} \simeq 1$$

Fermion masses

$$\mathcal{L}_{Yukawa} = y^{(up)}_{ij}\,\overline{Q}_i\, U_j\, H + y^{(down)}_{ij}\,\overline{Q}_i\, D_j\,\overline{H}$$

$$+ y^{(lept)}_{ij}\,\overline{L}_i\, E_j\,\overline{H} \quad + h.c.$$

$$\mathcal{L}^f_{mass} = \mathcal{L}_{Yukawa}\Big|_{H=\langle H\rangle} = M^{(up)}_{ij}\,\overline{U}_{iL}\, U^c_{jL} + h.c.$$

$$+ M^{(down)}_{ij}\,\overline{d}_{iL}\, d^c_{jL} + h.c$$

$$+ M^{(lept)}_{ij}\,\overline{e}_{iL}\, e^c_{iL} + h.c.$$

$$M^{(up)}_{ij} = \frac{v}{\sqrt{2}}\, y^{(up)}_{ij} \;;\; M^{(down)}_{ij} = \frac{v}{\sqrt{2}}\, y^{(down)}_{ij} \;,\; M^{(lept)}_{ij} = \frac{v}{\sqrt{2}}\, y^{(lept)}_{ij}$$

$$\rightarrow \sum_{ij} \overline{d}_{iL} \gamma^\mu V_{ij}^{CKM} u_{jL} W_\mu^- + h.c.$$

$$\hat{V}_{CKM} = S_L^{d\dagger} \cdot S_L^u$$

This is the unitary matrix – the Cabibbo-Kobayashi-Maskawa (CKM) matrix.

- Charged weak bosons $W_\mu^\pm$ mediate flavour-changing interactions at leading order (tree-level).

$$L_i \to (S_L^e)_{ij} L_j \;, \quad E_i \to (S_R^e)_{ij} E_j$$

$$U_{iL} \to (S_L^u)_{ij} U_{jL} \;, \quad d_{iL} \Rightarrow (S_L^d)_{ij} d_{jL}$$

$$U_i \to (S_R^u)_{ij} U_j \;, \quad D_i \to (S_R^d)_{ij} D_j$$

Let's inspect charged current interactions:

$$\sum_{i=1}^{3} \bar{Q}_{iL} \gamma^\mu \frac{\sigma_1 \mp i\sigma_2}{2} Q_{iL} W_\mu^\mp =$$

$$= \sum_{i=1}^{3} \left[\bar{d}_{iL} \gamma^\mu U_{iL} W_\mu^- + \bar{U}_{iL} \gamma^\mu d_{iL} W_\mu^+ \right] \longrightarrow$$

In this basis, quark & lepton states are not eigenstates of quadratic Casimir operator $C_2 = P_\mu P^\mu$. We must change the basis such that the mass matrices are diagonal:

$$S_L^{u+} \hat{M}^{up} S_R^u = \begin{pmatrix} m_u & & 0 \\ & m_c & \\ 0 & & m_t \end{pmatrix}$$

$$S_L^{d+} \hat{M}^{down} S_R^d = \begin{pmatrix} m_d & & 0 \\ & m_s & \\ 0 & & m_b \end{pmatrix}$$

$$S_L^{e+} \hat{M}^{lept} S_R^e = \begin{pmatrix} m_e & & 0 \\ & m_\mu & \\ 0 & & m_\tau \end{pmatrix}$$

- Since, Z_μ (and A_μ) are coupled to the diagonal $SU(2)_w \times U(1)_Y$ generators the neutral current gauge interactions remain flavour-diagonal.
  - No flavour-changing neutral current processes (at leading order).

- Note, S^u_R, S^d_R & S^e_R do not show up in the Lagrangian $\rightarrow$ these rotations are unobservable.

- Neutrinos are massless

- Observable parameters:

  9 quark & lepton masses

  some elements of $\hat{V}^{CKM}$

- Unitary matrix $N \times N$ has N^2 parameters

$$N^2 = \underbrace{\frac{N(N-1)}{2}}_{\text{rotation angles}} + \underbrace{\frac{N(N+1)}{2}}_{\text{complex phases}}$$

- $(2N-1)$ phases are not observable as they can be removed by the change in phase of quark states

- Hence, $\frac{N(N-1)}{2}$ – angles

$\frac{(N-1)(N-2)}{2}$ – complex phases

For $N=3$ generation of quarks

3 – angles

1 – complex phase (CP-violation!)

Kobayashi-Maskawa

Quantum Chromodynamics

$SU(3)_c$ gauge theory that describes interactions of quarks mediated by the corresponding massless gauge bosons – gluons

$$\frac{d g_3}{d \ln \mu} = \beta_3 < 0$$

g_3 ($\mu \to \infty$, small scales) $\to 0$ (asymptotic freedom)

g_3 ($\mu \to 0$, large scales) $\to \infty$ (confinement)

Perturbation theory breaks $\mu \sim 1$ GeV

Covariant derivative

$$(D_\mu)_{ij} = \delta_{ij}\partial^\mu - i\,g_3\left(\frac{\lambda_a}{2}\right)_{ij} G^a_\mu$$

Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Let's focus on low-energy QCD with the two lightest quarks only, u & d.

In the limit $m_u, m_d \to 0$, we can organise these quarks intro strong iso-doublet:

$$Q_i = \begin{pmatrix} u \\ d \end{pmatrix}_i$$

— denotes colours $i = 1, 2, 3$

$$\mathcal{L}_{QCD}^{2\,\text{flavour}}\Big|_{m_u = m_d = 0} = \bar{Q}_i \, i\gamma^\mu (D_\mu)_{ij} Q_j - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}$$

- 2-flavour QCD Lagrangian in the $m_u, m_d \to 0$ limit exhibits a global $U(2)_V \times U(2)_A$ symmetry (Note: $U(2) = SU(2) \times U(1)$)

$$Q' = \exp\left[i \alpha^V_\alpha T_\alpha \right] Q$$

$$Q' = \exp\left[i \alpha^A_\alpha T_\alpha \right] Q \qquad \text{(Q. verify invariance)}$$

$$\alpha = 1, 2, 3, 4 \qquad 2\, T_\alpha = \{ \sigma_\alpha, \mathbb{1} \}$$

- Note, even if $m_u, m_d \neq 0$, the Lagrangian is invariant under phase transformations of u & d-quark states. This corresponds to a conserved charge - baryon number $\quad U(2)_V = SU(2)_V \times \underset{\text{exact!}}{\underline{U(1)_V}}$

• In the case $m_u = m_d \neq 0$, $SU(2)_V$ is also an exact symmetry. This is nothing but the strong isospin symmetry of nuclear physic, e.g. neutron & proton form a doublet of $SU(2)_V$, with (almost) the same mass.

• What about $U(2)_A = SU(2)_A \times U(1)_A$. If (approximately) exact, we should have seen another doublet of nucleons with an opposite parity and (almost) degenerate with (n, p). No such states are seen $\rightarrow$ $U(2)_A$ must be spontaneously broken

- This turns out to be the case!

$$\langle 0 | \bar{u} u | 0 \rangle = \langle 0 | \bar{d} d | 0 \rangle \neq 0$$

$$U(2)_V \times U(2)_A \longrightarrow U(2)_V$$

(Q. verify this)

Let's count Goldstone bosons

$$\dim\left(U(2)_V \times U(2)_A\right) - \dim\left(U(2)_V\right)$$

$$= (4+4) - 4 = 4$$

- (pseudo) Goldstones $\pi^0, \pi^\pm$, $m_\pi \ll \Lambda_{QCD} \sim 1\ \text{GeV}$
- where's the fourth Goldstone? $U(1)_A$ puzzle.

- $U(1)_A$ is broken by (non-pertorbative) quantum effects (instantons) strongly enough to generate large mass for the fourth (psedo) Goldstone

$$m_\eta \sim \Lambda_{QCD}$$

- Note, instatons also generate CP-violating term

$$\mathcal{L}_\theta \propto \theta\, G^a_{\mu\nu} \tilde{G}^{a\mu\nu}$$

$$\tilde{G}^{a\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} G^a_{\rho\sigma}$$

the strong CP problem