School and Workshop "Recent Advances in Fundamental Physics"

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Split octonionic field Lagrangian

PhD student: Alexandre Gurchumelia

PhD Supervisor: Assoc. Prof. Merab Gogberashvili

Ivane Javakhishvili Tbilisi State University

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Outline

- Split octonions $\mathbb{O}'$
- Triality in $(4+4)$-space
- Analysis of $\mathbb{O}' \to \mathbb{O}'$ funcitons
- Construct Lagrangian with $\mathbb{O}'$
- Field equations in $\mathbb{O}'$, Dirac and dyonic Maxwell equaitons

Octonions in physics

Octonions

- Color symmetry[Gunaydin, Gursey 1973; Morita 1981], Quantum mechanics[Gunaydin, Piron, Ruegg 1978], GUT[Sudbery 1984, Dixon 1990; Castro 2007], wave equations[Kurdgelaidze 1985, Gogberashvili 2006], associator quantization[Lohmus, Paal, Sorgsepp 1998], M theory[Lukierski, Toppan 2002], quantum Hall effect[Bernevig, Hu, *et al* 2003]

Split octonions

- Particle generations[Gunaydin, Gursey 1974, Silagadze 1995], electrodynamics[Nash 1989], gravity[Nash 2010], geometry[Gogberashvili 2009, 2015].

Cayley-Dickson constructions

Given an algebra $\mathbb{A}$ we double it $\mathbb{A} \times \mathbb{A}$ and equip it with a product

$$\begin{aligned} x\,y &= (x_1,\, x_2)\,(y_1,\, y_2) \\ &= (x_1 y_1 - s\,\overline{y}_2 x_2,\; y_2 x_1 + x_2 \overline{y}_1) \qquad \text{where } s = \pm 1 \end{aligned} \tag{1}$$

We get $\quad \mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O} \to \mathbb{S} \to \cdots \quad$ or $\quad \mathbb{R} \to \mathbb{C}' \to \mathbb{H}' \to \cdots$

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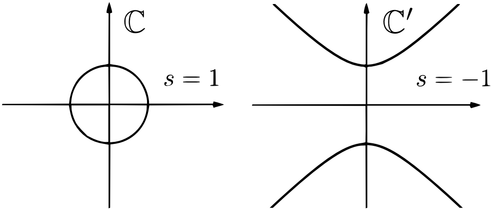
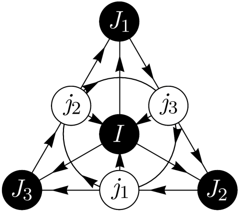
$$\begin{aligned} &\text{for } s = 1 \text{ we have} && i^2 = -1 && \to && e^{i\varphi} = \cos\varphi + i\sin\varphi \\ &\text{for } s = -1 \text{ we have} && j^2 = 1 && \to && e^{j\varphi} = \cosh\varphi + j\sinh\varphi \end{aligned} \tag{2}$$



Split octonion algebra $\mathbb{O}'$

Algebraic relations	Fano plane
$\begin{aligned} j_m j_n &= -\delta_{mn} + \sum_k \epsilon_{mnk} j_k \ , \\ I^2 &= 1 \ , \\ J_m J_n &= \delta_{mn} + \sum_k \epsilon_{mnk} j_k \ , \\ j_n I &= J_n \ , \\ J_m j_n &= \delta_{mn} I - \sum_k \epsilon_{mnk} J_k \ . \end{aligned}$	

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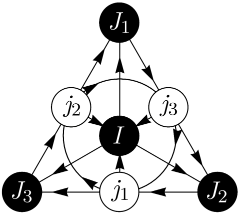
$j_m j_n = -\delta_{mn} + \sum_k \epsilon_{mnk} j_k \ ,$ $I^2 = 1 \ ,$ $J_m J_n = \delta_{mn} + \sum_k \epsilon_{mnk} j_k \ ,$ $j_n I = J_n \ ,$ $J_m j_n = \delta_{mn} I - \sum_k \epsilon_{mnk} J_k \ .$	

General element $x \in \mathbb{O}' \simeq \mathbb{R}^{4,4}$ and its conjugate

$$
\begin{aligned}
x &= x_0 + I x_4 + \sum_{1,2,3} (j_n x_n + J_n x_{4+n}) \\
\overline{x} &= x_0 - I x_4 - \sum_{1,2,3} (j_n x_n + J_n x_{4+n})
\end{aligned}
\tag{3}
$$

Linear forms on $\mathbb{O}' \times \mathbb{O}' \times \cdots$

Symmetric nondegenerate bilinear form $\langle \cdot, \cdot \rangle : \mathbb{O}' \times \mathbb{O}' \to \mathbb{R}$ is just an *"inner product"* in $(4+4)$-space

$$\begin{aligned} \langle x, y \rangle &= \frac{1}{2} \left(\overline{x}\, y + \overline{y}\, x \right) \\ &= \sum_{1,2,3} \left(x_n\, y_n - x_{4+n}\, y_{4+n} \right) \end{aligned} \tag{4}$$

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Trilinear form $\mathcal{F} : \mathbb{O}' \times \mathbb{O}' \times \mathbb{O}' \to \mathbb{R}$

$$\mathcal{F}\,(\Phi, X, \Psi) = \langle \overline{\Phi}, X\, \Psi \rangle \tag{6}$$

Triality and pseudo-orthogonal groups in $\mathbb{O}'$

$SO\,(4,4)$ vector $X \in \mathbb{O}'$, $\quad$ $Spin\,(4,4)$ spinor chiral parts $\Phi, \Psi \in \mathbb{O}'$

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$$\begin{cases} \varPhi \mapsto (\varPhi u)\,\overline{T}_{uv}\,, \\ X \mapsto T_{uv}\,(uXv)\,T_{uv}\,, \\ \varPsi \mapsto \overline{T}_{uv}\varPsi\,, \end{cases} \qquad T_{uv} = \begin{cases} u\cos\frac{\vartheta}{2} + v\sin\frac{\vartheta}{2}, & u^2v^2 = -1 \\ u\cosh\frac{\vartheta}{2} + v\sinh\frac{\vartheta}{2}, & u^2v^2 = 1 \end{cases} \tag{7}$$

for $u, v \in$ basis $(\mathbb{O}')$

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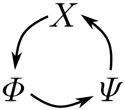
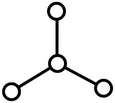
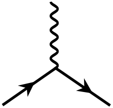
The invariants: $\mathcal{Q}\left(X\right)$, $\mathcal{Q}\left(\varPhi\right)$, $\mathcal{Q}\left(\varPsi\right)$ and $\mathcal{F}\left(\varPhi, X, \varPsi\right)$. Transformations for $\varPhi, X, \varPsi \in \mathbb{O}'$ are interchangable. This follows from triality symmetry of D_4







Triality in $\mathbb{O}'$: Gurchumelia, Gogberashvili (2021) & Mikosz (2013)
Triality in $\mathbb{O}$: Gamba (1967), Peculiarities of the Eight-Dimensional Space
Supersym D_4: Frank, Smith (1993) & Baez (2001) Spinors and Trialities

Analysis of $\mathbb{O}' \to \mathbb{O}'$ functions

Derivatives are

$$\begin{aligned} \partial &= \frac{1}{2}\left(\partial_0 + I\partial_4\right) + \frac{1}{2}\sum_{1,2,3}\left(j_n\partial_n + J_n\partial_{4+n}\right) \\ \overline{\partial} &= \frac{1}{2}\left(\partial_0 - I\partial_4\right) - \frac{1}{2}\sum_{1,2,3}\left(j_n\partial_n + J_n\partial_{4+n}\right) \end{aligned} \tag{8}$$

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But usual properties hold only for linear functions

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Extention of Cauchy-Riemann-Fueter equations to $f : \mathbb{O}' \to \mathbb{O}'$

$$\begin{aligned} \partial f &= 0 \quad \text{analyticity} \\ \overline{\partial} f &= 0 \quad \text{anti-analyticity} \end{aligned} \tag{10}$$

For $\mathbb{H}$: DeLeo (2003) Quaternionic analysis. Appl. Math. Letters
For $\mathbb{O}$: Kauhanen, Orelma (2018) Cauchy-Riemann Operators in Octonionic Analysis. Adv. Appl. Cliff. Algebras

Lagrangian and EoM: Dirac

In $\mathcal{F}(\Phi, X, \Psi) = \left\langle \overline{\Phi}, X\Psi \right\rangle$ we replace $X \to D = I\partial I$ and set Φ and Ψ to be functions of $x \in \mathbb{O}'$.

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$$\mathcal{L} = \left\langle \overline{\Phi}, D\Psi \right\rangle + m \left\langle \overline{\Phi}, J_3\Psi \right\rangle \tag{11}$$

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gives two equations

$$\begin{cases} \left(\overline{D} - J_3 m\right) \overline{\Phi} = 0 \\ (D - J_3 m)\, \Psi = 0 \end{cases} \tag{12}$$

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Second eq gives Dirac equation when we remove dimensions $0, 5, 6, 7$ i.e.

$$D \to \nabla = I\partial_t - \sum_{1,2,3} j_n \partial_n \tag{13}$$

gives

$$\left(\nabla - J_3 m\right) \varPsi = 0 \tag{14}$$

Lagrangian and EoM: Maxwell

We add $\mathcal{Q}(\mathit{\Phi}) = \langle \mathit{\Phi}, \mathit{\Phi} \rangle = \overline{\mathit{\Phi}}\mathit{\Phi}$ term

$$\mathcal{L} = \left\langle \overline{\mathit{\Phi}},\ D\,\mathit{\Psi} \right\rangle + \frac{1}{2}\lambda \left\langle \mathit{\Phi},\ \mathit{\Phi} \right\rangle \tag{15}$$

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and get EoM

$$\begin{cases} \overline{D}\,\overline{\varPhi} = 0 \\ D\,\varPsi = -\lambda\overline{\varPhi} \end{cases} \longrightarrow \qquad \langle D, D \rangle\,\varPsi = 0 \tag{16}$$

which is free dyonic Maxwell equation in the same limit as in the Dirac case.

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which is free dyonic Maxwell equation in the same limit as in the Dirac case. Regular Maxwell is obtained when

$$\Psi \to A = I A_0 + \sum_{1,2,3} j_n A_n \tag{17}$$

$$\langle \nabla, \nabla \rangle\, A = 0 \tag{18}$$

Summary and conclusion

1. $\mathbb{O}'$ have interesting properties:
   - might be an important algebra for describing geomety, dynamics and particle properties,
   - $(4+4)$-space triality might determines fermion-fermion-boson interaction structure,
   - Lagrangians exhibit SUSY-like behavior.
2. We constructed generalizaions of Dirac and Maxwell 1-st order differential equation systems as $\mathbb{O}' \to \mathbb{O}'$ analyticity condition

Publications

- Gurchumelia, A., & Gogberashvili, M. (2021, April). Split Octonions and Triality in (4+4)-Space. In RDP online workshop" Recent Advances in Mathematical Physics (p. 8).
- Gogberashvili, M., & Gurchumelia, A. (2019). Geometry of the non-compact G(2). Journal of Geometry and Physics, 144, 308-313.
- Gogberashvili, Merab, and Otari Sakhelashvili. "Geometrical applications of split octonions." Advances in Mathematical Physics 2015 (2015).
- Gogberashvili, M., & Sakhelashvili, O. (2015). Geometrical applications of split octonions. Advances in Mathematical Physics, 2015.
- Gogberashvili, M. (2009). Rotations in the space of split octonions. Advances in Mathematical Physics, 2009.
- Gogberashvili, M. (2006). Octonionic electrodynamics. Journal of Physics A: Mathematical and General, 39(22), 7099.
- Gogberashvili, M. (2002). Observable algebra. arXiv preprint hep-th/0212251.