Euler top as a one-dimensional system and supersymmetrization

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E.Kh., S.Krivonos, A.Nersessian, *"Euler top and freedom in supersymmetrization of one-dimensional mechanics"*, Physics Letters A Volume 452, 15 November 2022, 128442

Content

- Euler top as one dimensional system
- N=2,4 supersymmetry
- N=6,8,… supersymmetry
- Conclusion

Euler Top

$$H = \sum_{i=1}^{3} \frac{x_i^2}{2I_i}, \qquad \{x_i, x_j\} = \mathrm{i}\epsilon_{ijk}, \qquad l(x_1, x_2, x_3), \qquad I_i -$$ Principal momenta of inertia

$\uparrow$

$SO(3)$

Casimir : $C = \sum_{i=1}^{3} x_i^2 ; \qquad \{C(x), x_i\} = 0.$

Its fixation leads to the Hamiltonian system with two-dimensional non-degenerated phase space, i.e., one-dimensional system.

Euler Top

It is convenient

$$j := \sqrt{\sum_{i=1}^{3} x_i^2}\,, \qquad z := \frac{x_1 + \mathrm{i}x_2}{j - x_3}$$

↑

Complete angular momentum

With the Poisson brackets

$$\{\bar{z}, z\} = -\frac{\mathrm{i}}{2j}(1 + z\bar{z})^2$$

$$\{z, j\} = \{z, z\} = 0.$$

Euler Top

It is convenient

$$j := \sqrt{\sum_{i=1}^{3} x_i^2}\,, \qquad z := \frac{x_1 + \mathrm{i}x_2}{j - x_3}$$

With the Poisson brackets

$$\{\bar{z}, z\} = -\frac{\mathrm{i}}{2j}(1 + z\bar{z})^2$$

$$\{z, j\} = \{z, z\} = 0.$$

while the momentum generators look as follows

$$x_1 \coloneqq h_1 = j\,\frac{z + \bar{z}}{1 + z\bar{z}}, \qquad x_2 \coloneqq h_2 = j\,\frac{\mathrm{i}(\bar{z} - z)}{1 + z\bar{z}}, \qquad x_3 \coloneqq h_3 = j\,\frac{z\bar{z} - 1}{1 + z\bar{z}}.$$

Euler Top

By fixing j we arrive at two dimensional phase space, equipped with one (complex) dimensional Kähler structure, which is the complex projective Plane $\mathbf{CP}^1$ with

Fubini-Studi metric

$$g(z,\bar{z})dzd\bar{z} = 2j\frac{dzd\bar{z}}{(1+z\bar{z})^2}$$

Kähler potentioal

$$K(z,\bar{z}) = 2j\log(1+z\bar{z})$$

The generators h_i become the Killing potentials of $\mathbf{CP}^1$

Hamiltonian holomorphic vector fields

$$\{h_1,\} = \mathrm{i}(1-z^2)\partial_z + c.c., \qquad \{h_2,\} = -\mathrm{i}(1+z^2)\partial_z + c.c., \qquad \{h_1,\} = 2\mathrm{i}z\partial_z + c.c.$$

Euler Top

In these terms

$$H = \sum_{i=1}^{3} \frac{x_i^2}{2I_i} = -j^2 \frac{b(z^2+\bar{z}^2)+2az\bar{z}}{2(1+z\bar{z})^2} + \frac{j^2}{2I_3}, \qquad a := \frac{2}{I_3} - \frac{1}{I_1} - \frac{1}{I_2}, b := \frac{1}{I_2} - \frac{1}{I_1}.$$

Canonical coordinates (φ, p): $z = \cot\frac{\theta}{2} \mathrm{e}^{\mathrm{i}\varphi}$

$$\{\bar{z}, z\} = -\frac{\mathrm{i}}{2j}(1+z\bar{z})^2 \quad \Rightarrow \quad \{\varphi, j\cos\theta\} = 1 \qquad p := j\cos\theta$$

$$h_1 + \mathrm{i}h_2 = j\sin\theta\mathrm{e}^{\mathrm{i}\varphi} = \sqrt{j^2 - p^2}\mathrm{e}^{\mathrm{i}\varphi}, \qquad h_3 = j\cos\theta = p,$$

$$H = \frac{1}{4}(a + b\cos 2\varphi)p^2 + \frac{j^2}{4}\left(\frac{2}{I_3} - (a + b\cos 2\varphi)\right)$$

Euler Top

Canonical transformation $(\varphi, p) \longrightarrow (Q, P)$

$$P = \sqrt{\frac{a + b\cos 2\varphi}{2}}\, p,$$

Type equation here.

$$Q = \sqrt{\frac{2}{a+b}} \int_0^{\varphi} \frac{d\theta}{\sqrt{1 - \frac{2b}{a+b}\sin^2\theta}} = \sqrt{\frac{2}{a+b}} F\left(\varphi, \sqrt{2b/(a+b)}\right): \quad \{Q, P\} = 1.$$

Where $F(\varphi, k)$ is an elliptic integral of the first kind, with $k = \sqrt{2b/(a+b)}$.

Euler Top

$k = \sqrt{2b/(a+b)}$ --- modulus of elliptic function,
$\varphi = F^{-1}(F,k) = \mathbf{amp}(F,k)$ --- Jacobi amplitude.

$$\sin\varphi = \sin(\mathbf{amp}(F,k)) = \mathbf{sn}(F,k)$$

In canonical coordinates: $H = \frac{1}{2}p^2 + \frac{j^2 b}{2}\mathbf{sn}^2\left(\sqrt{\frac{a+b}{2}}Q, \sqrt{\frac{2b}{a+b}}\right) + \frac{j^2}{2I_1}.$

So, the Euler top is the one-dimensional Hamiltonian system with $\mathbf{CP}^1$ *phase space and with the Hamiltonian given by the quadratic function of its Killing potentials. In the canonical coordinates it results into one-dimensional nonlinear oscillator.*

Supersymmetry

We will consider the systems with generic two-(real) dimensional phase space

➡ One (complex) dimensional Kähler structure

➡ Poisson brackets will be given by $\{z, \bar{z}\} = \frac{\mathrm{i}}{g(z, \bar{z})}$

For construction of $\mathcal{N}$-supersymmetric extensions (for even $\mathcal{N}$) of a system with $H(z, \bar{z}) \succ 0$ we extend phase space by canonical complex Grassmann variables $\psi_a, \quad a = 1, \dots, \mathcal{N}/2$

$$\{\psi_a, \bar{\psi}^b\} = \mathrm{i}\delta_a^b.$$

$$\{Q_a, \bar{Q}^b\} = \mathcal{H}\delta_a^b, \qquad \mathcal{H} = H + fermionic\ terms.$$

$\mathcal{N} = 2$ supersymmetry

An ansatz for supercharges

$$Q = \sqrt{H} e^{i\Phi} \psi, \qquad \bar{Q} = \sqrt{H} e^{-i\Phi} \bar{\psi}, \qquad \Longrightarrow \qquad \mathcal{H} = H + \{\Phi, H\} \psi \bar{\psi}.$$

Specifying the Poisson brackets and Hamiltonian we will get the respective supersymmetric extension of the Euler top.

Direct construction of $\mathcal{N} \geq 4$ supersymmetries ⟹ Trivial result

Namely, $Q_a = \sqrt{H} e^{i\Phi} \psi_a$ to fulfill $\{Q_a, \bar{Q}^b\} = \mathcal{H} \delta_a^b$

⇓

$$\{H, \Phi\} = 0, \qquad \mathcal{H} = H.$$

$\mathcal{N} = 4$ supersymmetry

$$Q_a = f_1(z,\bar{z})\psi_a + f_2(z,\bar{z})\psi_a \sum_{b=1}^{\mathcal{N}/2} \psi_b\,\bar{\psi}^b$$

$$\bar{Q}^a = \bar{f}_1(z,\bar{z})\bar{\psi}^a + \bar{f}_2(z,\bar{z})\bar{\psi}^a \sum_{b=1}^{\mathcal{N}/2} \psi_b\,\bar{\psi}^b$$

with

$$f_1(z,\bar{z}) = \sqrt{H}\mathrm{e}^{-\mathrm{i}\Phi_1}$$
$$f_2(z,\bar{z}) = R\mathrm{e}^{-\mathrm{i}(\Phi_1-\Phi_2)}$$
$$R(z,\bar{z}) = \bar{R}(z,\bar{z})$$
$$\Phi_a(z,\bar{z}) = \bar{\Phi}_a\,(z,\bar{z})$$

Obey superalgebra $\{Q_a, \bar{Q}^b\} = \mathcal{H}\,\delta_a^b$

⬇

$$\mathrm{i}\{f_1, \bar{f}_1\} = f_1\bar{f}_2 + \bar{f}_1 f_2 \quad \Leftrightarrow \quad \{\sqrt{H}, \Phi_1\} = R\cos\Phi_2$$

$\mathcal{N} = 4$ supersymmetry

$$\mathcal{H} = f_1 \bar{f}_1 + \mathrm{i}\{f_1, \bar{f}_1\} \sum_{a=1}^{\mathcal{N}/2} \psi_a \bar{\psi}^a + \frac{\mathrm{i}}{2}\left(\{f_1, \bar{f}_2\} + \{f_2, \bar{f}_1\}\right)\left(\sum_{a=1}^{\mathcal{N}/2} \psi_a \bar{\psi}^a\right)^2$$

$$H = f_1 \bar{f}_1$$

single functional degree of freedom

$\mathcal{N} = 6$: $\{f_1, \bar{f}_2\} + \{f_2, \bar{f}_1\} = 2\mathrm{i} f_2 \bar{f}_2$

$\mathcal{N} = 8$: $\{f_1, \bar{f}_2\} + \{f_2, \bar{f}_1\} = 2\mathrm{i} f_2 \bar{f}_2,\quad \{f_2, \bar{f}_2\} = 0$

no functional degree of freedom

To construct $\mathcal{N} = 6$ and $\mathcal{N} = 8$ with a wide functional freedom is to extend the supercharges ansatz by 5- and7-fermionic terms.

$\mathcal{N} = 6,8, \ldots$ supersymmetry

$\mathcal{N} = 6$

$$Q_a = f_1(z,\bar{z})\psi_a + f_2(z,\bar{z})\psi_a\left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\,\bar{\psi}^b\right) + f_3(z,\bar{z})\psi_a\left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\,\bar{\psi}^b\right)^2$$

$$\bar{Q}^a = \bar{f}_1(z,\bar{z})\bar{\psi}^a + \bar{f}_2(z,\bar{z})\bar{\psi}^a\left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\,\bar{\psi}^b\right) + \bar{f}_3(z,\bar{z})\bar{\psi}^a\left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\,\bar{\psi}^b\right)^2$$

Where

$$f_1 = \sqrt{H}\mathrm{e}^{\mathrm{i}\Phi_1}, \qquad f_2 = R_2\mathrm{e}^{\mathrm{i}(\Phi_1-\Phi_2)}, \qquad f_3 = R_3\mathrm{e}^{\mathrm{i}(\Phi_1-\Phi_2-\Phi_3)}.$$

$\mathcal{N} = 6,8, \ldots$ supersymmetry

$\mathcal{N} = 6$

$$f_1\bar{f}_2 + \bar{f}_1 f_2 - \mathrm{i}\{f_1, \bar{f}_1\} = 0$$

$$2f_3\bar{f}_1 + 2f_2\bar{f}_2 + 2f_2\bar{f}_3 - \mathrm{i}\{f_1, \bar{f}_2\} + \mathrm{i}\{f_2, \bar{f}_1\} = 0$$

$$\mathcal{H} = \frac{1}{2} f_1\bar{f}_1 + \frac{1}{2}(f_1\bar{f}_2 + f_2\bar{f}_1)\sum_{a=1}^{\mathcal{N}/2}\psi_a\bar{\psi}^a +$$

$$\frac{1}{2}(f_3\bar{f}_1 + f_1\bar{f}_3 + f_2\bar{f}_2)\left(\sum_{a=1}^{\mathcal{N}/2}\psi_a\bar{\psi}^a\right)^2 + \frac{1}{2}(f_2\bar{f}_3 + f_3\bar{f}_2)\left(\sum_{a=1}^{\mathcal{N}/2}\psi_a\bar{\psi}^a\right)^3$$

(Φ_1, Φ_2, Φ_3)

$\mathcal{N} = 6,8,\ldots$ supersymmetry

$\mathcal{N} = 8$

$$Q_a = f_1(z,\bar{z})\psi_a + f_2(z,\bar{z})\psi_a\left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\,\bar{\psi}^b\right) + f_3(z,\bar{z})\psi_a\left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\,\bar{\psi}^b\right)^2$$

$$+f_4(z,\bar{z})\psi_a\left(\sum_{b=1}^{\mathcal{N}/2}\psi_b\,\bar{\psi}^b\right)^4$$

$$f_4 = R_4 \mathrm{e}^{\mathrm{i}(\Phi_1-\Phi_2-\Phi_3-\Phi_4)}.$$

$$f_1\bar{f}_2 + \bar{f}_1 f_2 - \mathrm{i}\{f_1,\bar{f}_1\} = 0$$

$$2f_3\bar{f}_1 + 2f_2\bar{f}_2 + 2f_2\bar{f}_3 - \mathrm{i}\{f_1,\bar{f}_2\} - \mathrm{i}\{f_2,\bar{f}_1\} = 0$$

$$3f_4\bar{f}_1 + 3f_3\bar{f}_2 + 3f_2\bar{f}_3 + 3f_1\bar{f}_4 - \mathrm{i}\{f_1,\bar{f}_3\} - \mathrm{i}\{f_3,\bar{f}_1\} - \mathrm{i}\{f_2,\bar{f}_2\} = 0$$

$\mathcal{N} = 6,8, \ldots$ supersymmetry

$\mathcal{N} = 8$

$$\mathcal{H} = \frac{1}{2} f_1 \bar{f}_1 + \frac{1}{2}\left(f_1 \bar{f}_2 + f_2 \bar{f}_1\right) \sum_{a=1}^{\mathcal{N}/2} \psi_a \bar{\psi}^a + \frac{1}{2}\left(f_3 \bar{f}_1 + f_1 \bar{f}_3 + f_2 \bar{f}_2\right) \left(\sum_{a=1}^{\mathcal{N}/2} \psi_a \bar{\psi}^a\right)^2$$

$$+ \frac{1}{2}\left(f_4 \bar{f}_1 + f_1 \bar{f}_4 + f_2 \bar{f}_3 + f_3 \bar{f}_2\right) \left(\sum_{a=1}^{\mathcal{N}/2} \psi_a \bar{\psi}^a\right)^3 \qquad (\Phi_1, \Phi_2, \Phi_3, \Phi_4)$$

$$+ \frac{\mathrm{i}}{8}\left(\{f_1, \bar{f}_4\} + \{f_2, \bar{f}_3\} + \{f_3, \bar{f}_2\} + \{f_4, \bar{f}_1\}\right) \left(\sum_{a=1}^{\mathcal{N}/2} \psi_a \bar{\psi}^a\right)^4$$

$\mathcal{N} = 6,8,\ldots$ supersymmetry

Specifying the formulae to the particular case of Euler top we will get its integrable N = 2, 4, 6, 8 supersymmetric extensions.

$\mathcal{N} = 2k$

$$Q_a = f_1(z,\bar{z})\psi_a + \psi_a \sum_{l=2}^{\mathcal{N}/2} f_l(z,\bar{z}) \left(\sum_{b=1}^{\mathcal{N}/2} \psi_b \bar{\psi}^b \right)^{l-1}, \qquad \bar{Q}^a = c.c.$$

Depending k complex functions $f_a(z,\bar{z})$

Obeying superalgebra $\{Q_a, \bar{Q}^b\} = \mathcal{H}\delta_a^b$, ➡ $k-1$ constraints

$\mathcal{N} = 6,8, \ldots$ supersymmetry

In a particular cases of $\mathcal{N} = 2,4,6,8$

# of supersymmetries	Constraints
$\mathcal{N} = 2$	-
$\mathcal{N} = 4$	$f_1\bar{f}_2 + \bar{f}_1 f_2 - \mathrm{i}\{f_1, \bar{f}_1\} = 0$
$\mathcal{N} = 6$	$f_1\bar{f}_2 + \bar{f}_1 f_2 - \mathrm{i}\{f_1, \bar{f}_1\} = 0$ $2f_3\bar{f}_1 + 2f_2\bar{f}_2 + 2f_2\bar{f}_3 - \mathrm{i}\{f_1, \bar{f}_2\} - \mathrm{i}\{f_2, \bar{f}_1\} = 0$
$\mathcal{N} = 8$	$f_1\bar{f}_2 + \bar{f}_1 f_2 - \mathrm{i}\{f_1, \bar{f}_1\} = 0$ $2f_3\bar{f}_1 + 2f_2\bar{f}_2 + 2f_2\bar{f}_3 - \mathrm{i}\{f_1, \bar{f}_2\} - \mathrm{i}\{f_2, \bar{f}_1\} = 0$ $3f_4\bar{f}_1 + 3f_3\bar{f}_2 + 3f_2\bar{f}_3 + 3f_1\bar{f}_4 - \mathrm{i}\{f_1, \bar{f}_3\} - \mathrm{i}\{f_3, \bar{f}_1\} - \mathrm{i}\{f_2, \bar{f}_2\} = 0$

Having in mind that $f_1\bar{f}_1 = H(z, \bar{z})$, we conclude that in this way we will arrive the family of N = 2k supersymmetric extensions of H, which are parameterized by k arbitrary real functions.

Thank you!