School and Workshop “Recent Advances in Fundamental Physics”

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“Evidences for a group structure of Wilson loops”

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Paper: *Group structure of Wilson loops in 2D tight-binding models with 2-band and 4-band energy spectra*, Authors: T.Supatashvili, M.Eliashvili, G.Tsitsishvili (2022)

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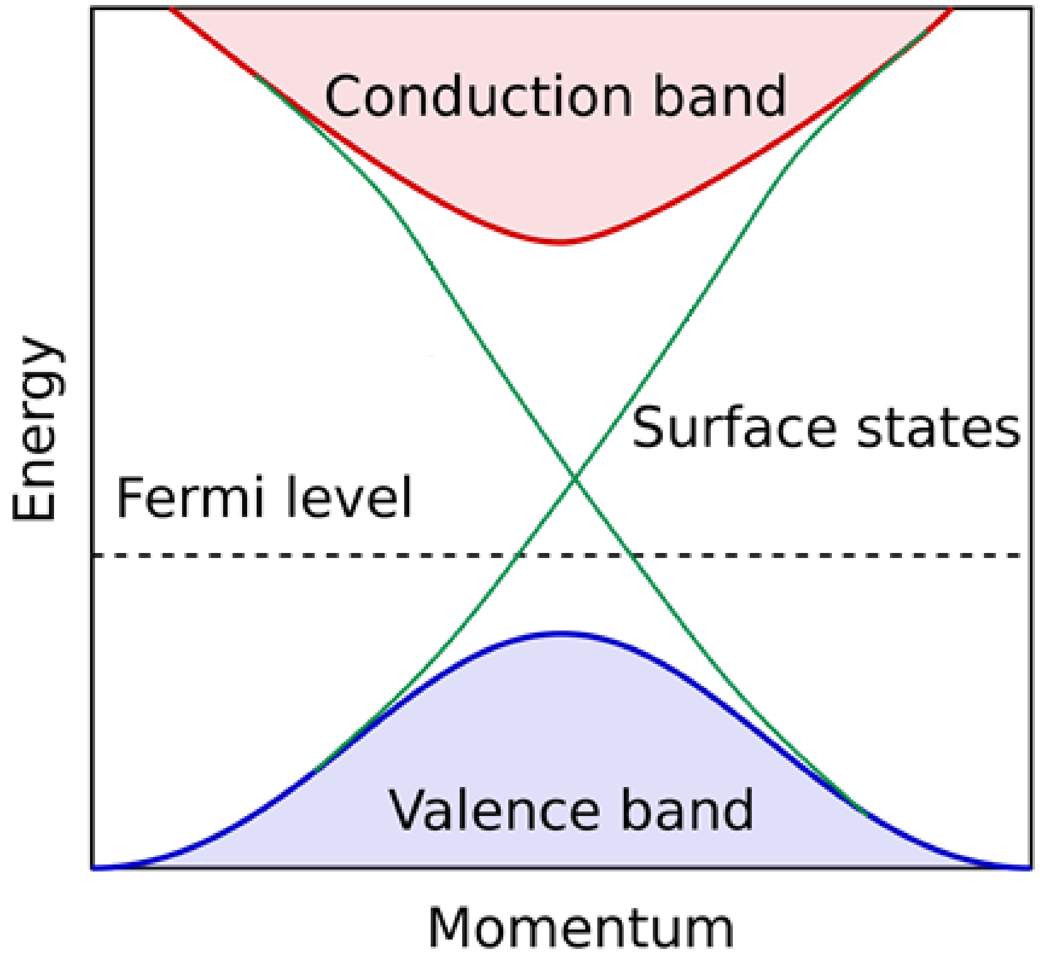
Topological Insulators (?)

- Existence of a conducting surface.
- Bulk remains insulator.
- Defining factors: topology of the eigenvectors and discrete symmetries
- Different from Landau's theory to describe phase transitions.



Getting to Wilson Loop

- $(x_1, x_2, \dots, x_n) \to (k_1, k_2, \dots, k_n)$ via Fourier transform

- $\hat{H} \to \mathcal{H}(\boldsymbol{k})_{N\times N}$ via

$$\hat{H} = \int_{\mathcal{BZ}} \psi^\dagger \mathcal{H}(\boldsymbol{k})_{N\times N} \psi \, d\boldsymbol{k} \tag{1}$$

1. Berry connection matrix

$$\left(A_\mu\right)_{mn}(\boldsymbol{k}) = i\psi_n^\dagger(\boldsymbol{k})\partial_\mu \psi_m(\boldsymbol{k}) \tag{2}$$

where $\psi_m(\boldsymbol{k})$ - eigenvectors of $\mathcal{H}(\boldsymbol{k})$, $m, n, \mu = 1, \dots, N$.

2. Curvature tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\left[A_\mu, A_\nu\right] \tag{3}$$

Wilson loop

- Wilson loop:

$$W_\gamma = \mathcal{P} \exp\left\{-i \oint_\gamma A_\mu \, \mathrm{d}k^\mu\right\} \tag{5}$$

- "$\mathcal{P}$“ – path ordering, γ – a loop on torus.
- Benefits: determinant, trace, eigenvalues – Gauge invariants.

- Non-Abelian Stokes Theorem (R.L. Karp, F. Mansouri, J.S. Rno (1999)).

$$W(\boldsymbol{k}_0) = \mathcal{P} \exp\left\{-i \oint_{\partial S} A_\mu \, \mathrm{d}k^\mu \right\} = \mathcal{P}_{k_2} \exp\left\{-\frac{i}{2} \int_S T^{-1}(\boldsymbol{k}) F_{\mu\nu} T(\boldsymbol{k}) \, \mathrm{d}k_\mu \wedge \mathrm{d}k_\nu\right\}. \tag{6}$$

Connection and Curvature

- Expressing $F_{\mu\nu}$ with eigenvectors using the expression of $\left(A_\mu\right)_{mn}(\boldsymbol{k})$

$$\left(A_\mu\right)_{mn}(\boldsymbol{k}) = i\psi_n^\dagger(\boldsymbol{k})\partial_\mu\psi_m(\boldsymbol{k}),$$

$$\boxed{\left(F_{\mu\nu}\right)_{mn} = i\left(\psi_n^\dagger\right)_k(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)(\psi_m)_k} \tag{4}$$

- *Result:* $F_{\mu\nu}$ is equal to zero everywhere on T^2 except the points where ψ's are singular.

A Model

- 2D infinite lattice with fermions on its sites.
- Choosing the first-quantized Hamiltonian to be: $\mathcal{H}(\boldsymbol{k}) = \boldsymbol{h}(\boldsymbol{k}) \cdot \boldsymbol{\sigma}$ (10)

  $\boldsymbol{h} = (\mathrm{h}_1, \mathrm{h}_2, \mathrm{h}_3), \boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ – Pauli matrices.

- $E_{1,2} = \pm|\boldsymbol{h}| \equiv \pm h.$ Eigenvectors:

$$\psi_1 = \frac{1}{\sqrt{2h(h-h_3)}}\begin{pmatrix} h_1 - ih_2 \\ h - h_3 \end{pmatrix}, \qquad \psi_2 = \frac{1}{\sqrt{2h(h-h_3)}}\begin{pmatrix} -h + h_3 \\ h_1 - ih_2 \end{pmatrix}.$$

- Source of singularities:
- $h = 0$
- $h = h_3$.

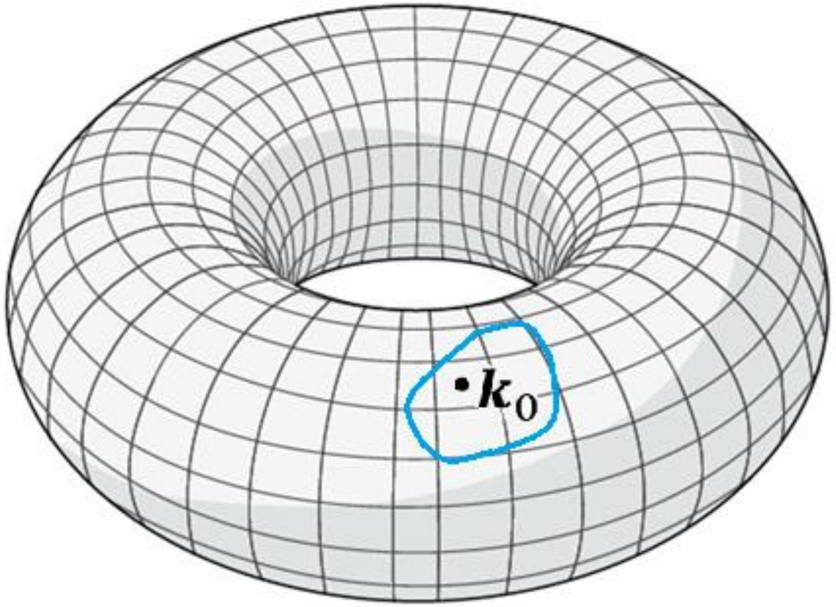
Calculating some of the Wilson Loops

- When γ's are contractible the result is trivial (identity matrix), since

$$\forall \boldsymbol{k} \in T^2, \; F_{\mu\nu} = \mathbb{O}$$

- If there is $\boldsymbol{k}_0$ inside the loop

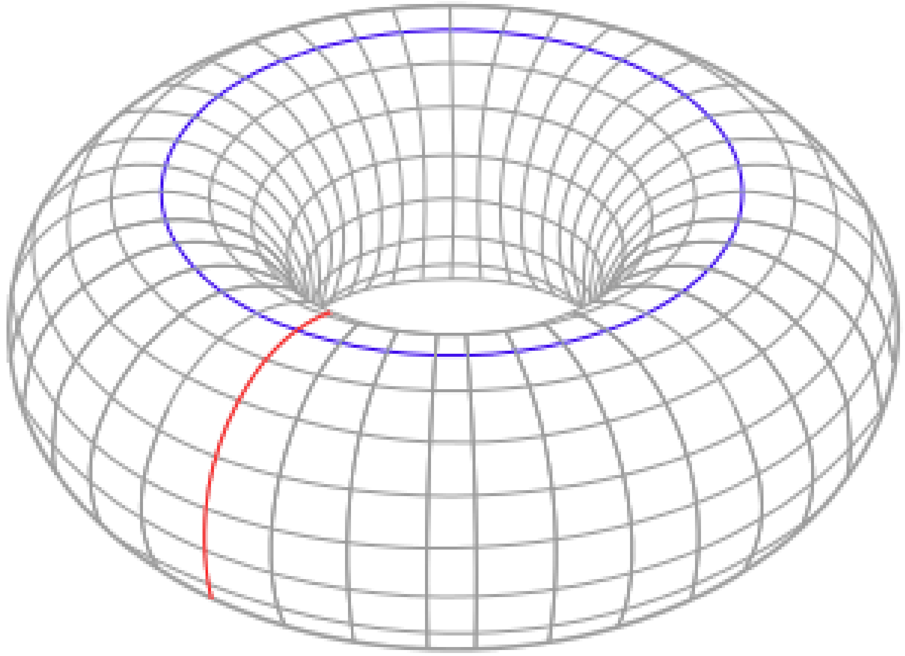
$$\boxed{\begin{aligned} W(\boldsymbol{k}_0) &= T^{-1} e^{-2\pi i \Phi(\boldsymbol{k}_0)} T = \\ &= T^{-1} e^{-2\pi i n (\in \mathbb{Z}) \sigma_3} T = \mathbb{I}_{2\times 2} \end{aligned}} \tag{11}$$



- The same is true when the number of such points inside a loop is more than one.

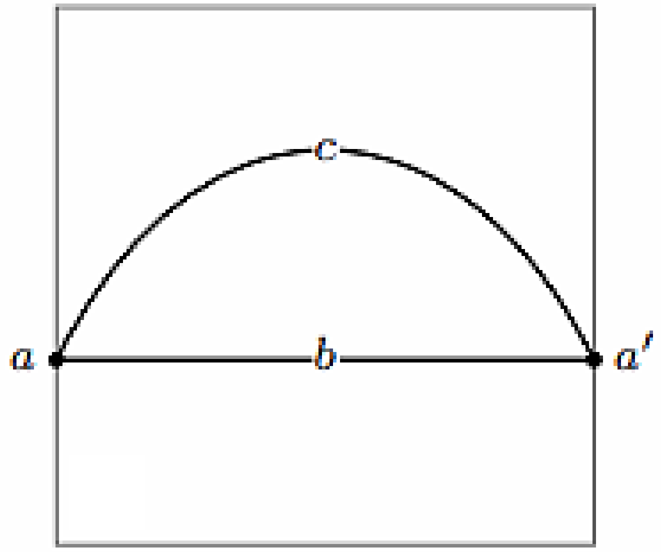
Fundamental group of Torus

- $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$.
- Each loop can be characterized by two integers (m, n), where m counts a winding number around a big principal circle of torus and n – around a small principal circle.
- For example: blue loop – (1,0),
  red loop – (0,1)



Group Structure of Wilson Loops

- Let $\mathcal{W}_{\boldsymbol{k}_{01}}$ be the set of Wilson loops with $\boldsymbol{k}_{01}$ as a starting (and ending) point. <u>It can be showed that for it group axioms are satisfied.</u>
- For each element $W_{\boldsymbol{k}_{01}}$ of this group we have an inverse: $W_{\boldsymbol{k}_{01}}^{-1} = W_{\boldsymbol{k}_{01}}^{\dagger}$
- We can characterize each element of this group by the loop labels (m,n).



Since $W_{aba'ca} = \mathbb{1}$, $\quad W_{aba'} = W_{aca'}$

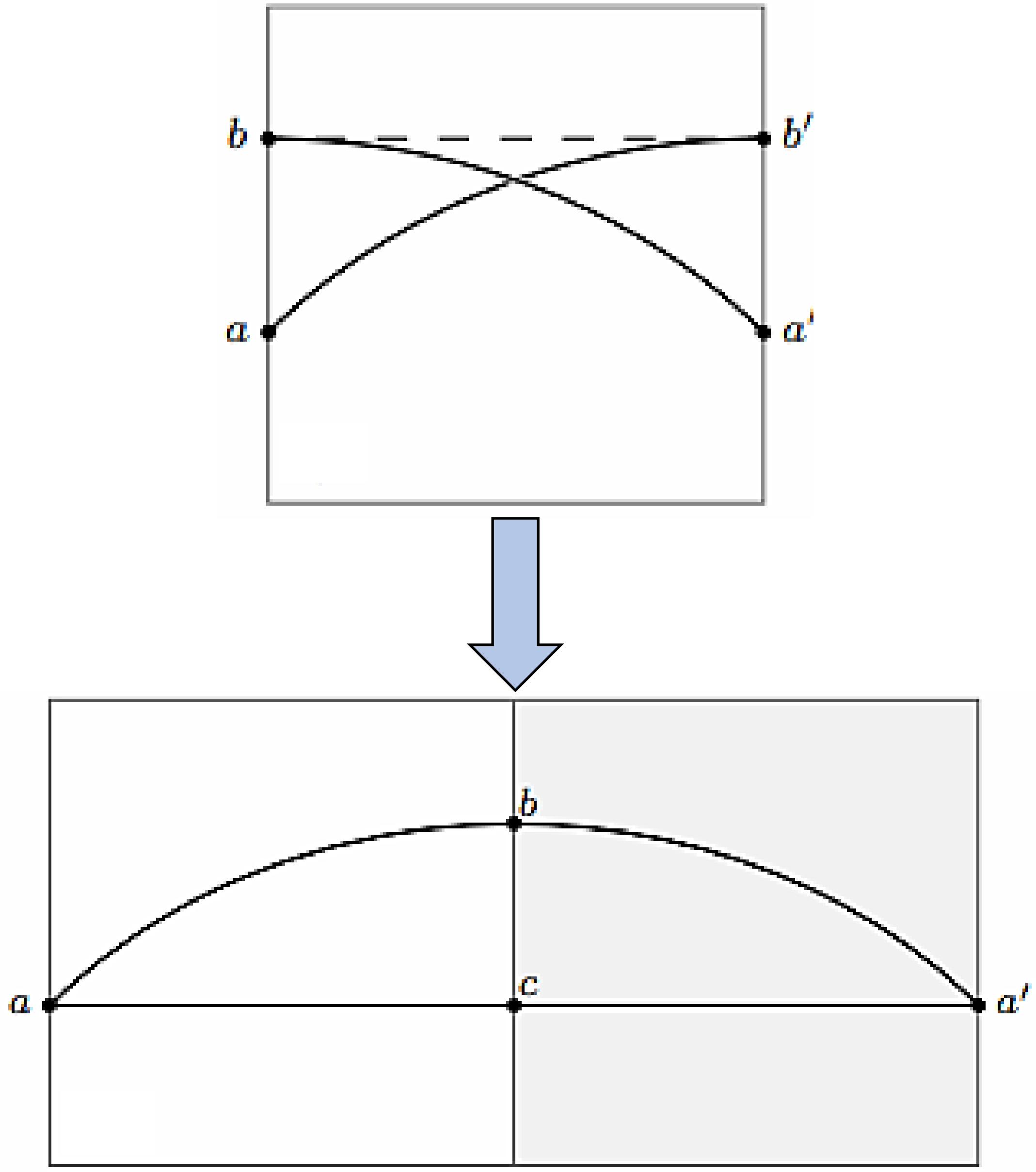
Label of the loop (and the corresponding element in $\mathcal{W}_{\boldsymbol{k}_{01}}$)

(1,0)

Group Structure of Wilson Loops

- $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z} \to \mathcal{W}_{\boldsymbol{k}_{01}}$.
- $\Rightarrow \mathcal{W}_{\boldsymbol{k}_{01}}$ is an abelian group with two generators that correspond to the loops (1,0) and (0,1).
- Any element of the group can be written as

$$W_{(m,n)} = W_{(1,0)}^{m} \cdot W_{(0,1)}^{n}.$$



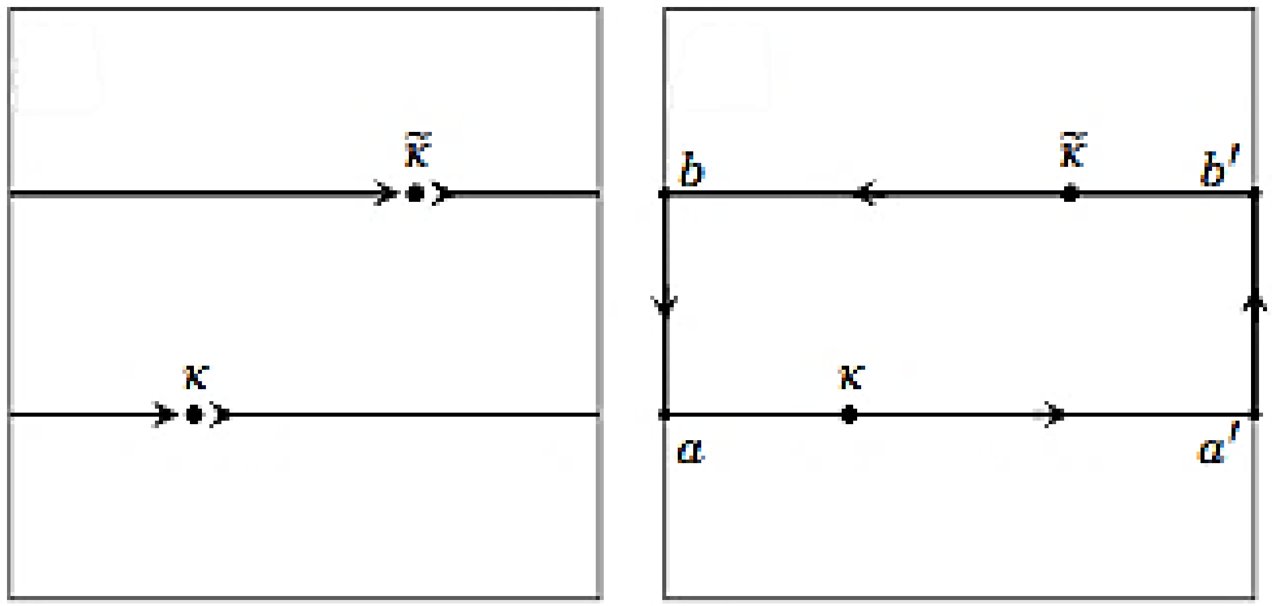
Group Structure of Wilson Loops

- Relation between $\mathcal{W}_{\boldsymbol{k}}$ and $\mathcal{W}_{\tilde{\boldsymbol{k}}}$, both isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

- Since $W_{ka'b'\tilde{k}bak} = \mathbb{1}$,

$$W_{ka'a} = U^{\dagger} W_{aca'} U,$$

- Where $U = W_{\tilde{k}bak}$.



Holonomy Group

Principal bundle	$(E = T^2 \times SU(2), \pi, T^2)$
Sections	$\Psi = 1/\sqrt{2}(\psi_1 \;\; \psi_2), \Phi = 1/\sqrt{2}\left(\psi'_1 \;\; \psi'_2\right)$
Connection 1-form	$A_{mn} = A_\mu(\in \mathfrak{su}(2))dk^\mu$
Curvature (2-form)	$F = dA + A \wedge A = 1/2F_{\mu\nu}dk^\mu \wedge dk^\nu$

- $\Phi = \Psi \mathrm{g}$, where $g \in SU(2)$.
- $\boldsymbol{h} = (0,0,h_3)$ – problem for Ψ;
- $\boldsymbol{h} = (0,0,-h_3)$ – problem for Φ
- Assume $h \neq 0$.

$A_\mu = (\Psi)_{nk}\partial_\mu \Psi_{km} \, dk^\mu$
$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

Holonomy Group

$$Hol_{p\in T^2} = \{g_\gamma | \gamma_h^\uparrow(end\ point) = hg_\gamma\},$$

- where $\gamma_h^\uparrow$ means horizontal lift of γ (loop on T^2), $h = \gamma_h^\uparrow(starting\ point)$.
- $Hol_\gamma^0 -$ when γ's are contractible
- Useful features:

1) If connected, then $Hol_q(A) = g^{-1}Hol_p g$. ✓
2) If simply connected, then $\mathrm{H}ol(A) = Hol^0(A)$.
3) A is flat if and only if $Hol^0(A)$ is trivial. ✓
4) Natural surjective group homomorphism:
   $\pi_1(base\ sp.) \to Hol(A)/\ Hol^0(A)$. ✓

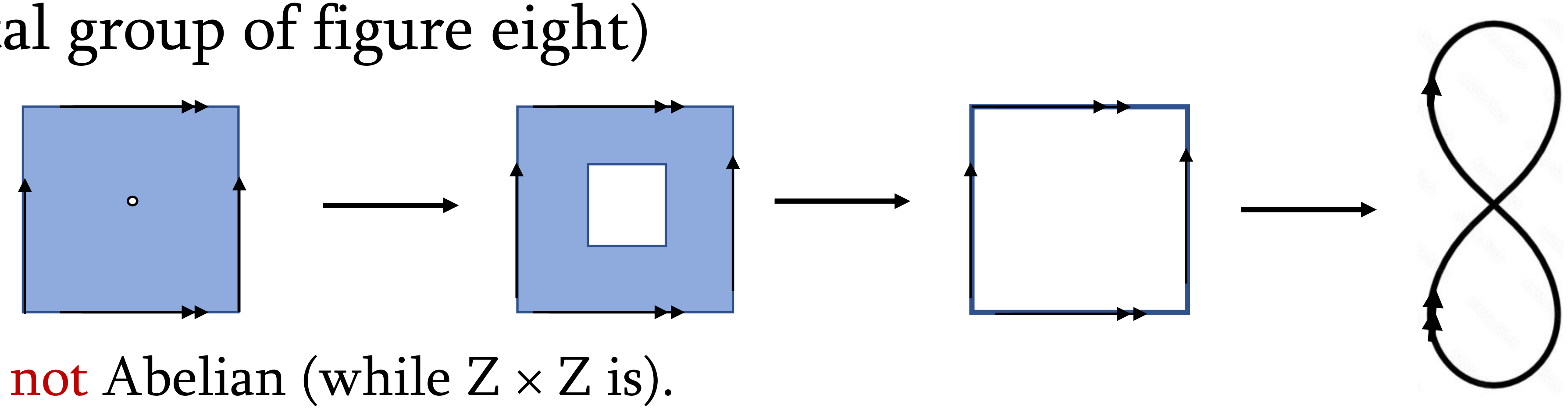
Simulating the Remaining Singular Points

- Consider a manifold $T^2/\{k_{01}, k_{02}, \dots, k_{0n}\}$ for the base space.
- Since

$$\pi_1(T^2/\{k_{01}, k_{02}, \dots, k_{0n}\}) = \prod_{i=1}^{n+1} Z \, *$$

$*$ denotes a free product.

- E.g. Z $*$ Z is a free group with two generators (fundamental group of figure eight)



Note: Z $*$ Z is not Abelian (while Z $\times$ Z is).

Summary:

- $F_{\mu\nu}$ is equal to zero everywhere on T^2 except the points where ψ's are singular.
- Using the Non-Abelian Stokes theorem and the behaviour of $F_{\mu\nu}$, calculations are simplified.
- $W = \mathbb{1}$ for all contractible loops that do not contain any of the singular points or contain singular points in which the energy gap is open.
- The set of $\{W_{\boldsymbol{k}}\}$ has a group structure and is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.
- The same results are achieved if we look at the set $\{W_{\boldsymbol{k}}\}$ as the holonomy group of A.
- Including the gap closing points we arrive to non-Abelian groups.

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Thank You!