On the definition of local spatial densities in light hadrons

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Outline

- Stating the problem;
- The charge density in the rest frame of the system;
- The charge density in moving frames;
- Summary;

Talk based on

E. Epelbaum, J. Gegelia, N. Lange, U. G. Meißner and M. V. Polyakov, Phys. Rev. Lett. **129**, no.1, 012001 (2022).

Stating the problem

It is textbook knowledge that the electric charge density of the nucleon is given by the three-dimensional Fourier transform of its electric form factor in the Breit frame.

R. Hofstadter, F. Bumiller, and M. R. Yearian, Rev. Mod. Phys. **30**, 482 (1958).
F. J. Ernst, R. G. Sachs and K. C. Wali, Phys. Rev. **119**, 1105-1114 (1960).
R. G. Sachs, Phys. Rev. **126**, 2256-2260 (1962).

Similar relations have been suggested for Fourier transforms of gravitational form factors and various local distributions in

M. V. Polyakov and A. G. Shuvaev, [arXiv:hep-ph/0207153 [hep-ph]].
M. V. Polyakov, Phys. Lett. B **555**, 57 (2003).
M. V. Polyakov and P. Schweitzer, Int. J. Mod. Phys. A **33** (2018) no.26, 1830025.

Definition of spatial density distributions via the Fourier transform of the corresponding form factors for systems whose intrinsic size is comparable with the Compton wavelength was criticized in

M. Burkardt, Phys. Rev. D **62** (2000), 071503(R), [erratum: Phys. Rev. D **66** (2002), 119903(E)].
G. A. Miller, Phys. Rev. Lett. **99**, 112001 (2007).
G. A. Miller, Phys. Rev. C **79**, 055204 (2009).
G. A. Miller, Ann. Rev. Nucl. Part. Sci. **60** (2010), 1-25.
R. L. Jaffe, Phys. Rev. D **103** (2021) no.1, 016017.
G. A. Miller, Phys. Rev. C **99**, no.3, 035202 (2019).
A. Freese and G. A. Miller, Phys. Rev. D **103**, 094023 (2021).

Miller pointed out that the derivation by Sachs implicitly assumes *delocalized* wave packet states. This would result in moments of the charge distribution governed by the size of the wave packet.

The definition of the charge density distribution for a spin-0 system was further scrutinized by Jaffe in relationship to three characteristic length scales: $\Delta^2 = 6F'(q^2)|_{q^2=0}$, the characteristic size of the wave packet R and the Compton wavelength $1/m$.

Jaffe concluded that the interpretation of the Fourier transformed form factor as the intrinsic charge density is not valid for light hadrons and argued that local density distributions cannot even be defined for systems with $\Delta \sim 1/m$.

We revisited the definition of the charge density and other spatial densities.

Using spherically symmetric wave packets in the zero averaged momentum frame (ZAMF), we showed that the charge density is defined unambiguously for sharply localized packets.

We also generalized the definition to moving frames and showed that in the infinite-momentum frame (IMF), the charge density turns into the well-known two-dimensional distribution in the transverse plane.

The charge density in the ZAMF of the system

We consider, for the sake of definiteness, a spin-0 system.

However spin plays no special role in the analysis below.

We assume that the system is an eigenstate of the charge operator $\hat{Q} = \int d^3r\, \hat{\rho}(\mathbf{r}, 0)$, $\hat{Q}|p\rangle = Q|p\rangle$, where $\hat{\rho}(\mathbf{r}, 0)$ is the electric charge density operator at $t = 0$, and we take $Q = 1$.

The momentum eigenstates $|p\rangle$ are normalized in the usual way,

$$\langle p'|p\rangle = 2E(2\pi)^3\delta^{(3)}(\mathbf{p}' - \mathbf{p})\,, \tag{1}$$

with $p = (E, \mathbf{p})$, $E = \sqrt{m^2 + \mathbf{p}^2}$.

The matrix elements of $\hat{\rho}(\mathbf{r}, 0)$ between momentum eigenstates of a spin-0 system can be written as

$$\langle p'|\hat{\rho}(\mathbf{r}, 0)|p\rangle = e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{r}}(E + E')F(q^2), \tag{2}$$

where $F(q^2)$ is the electric form factor and $q = p' - p$.

We define a normalizable Heisenberg-picture state of the system in terms of the wave packet

$$|\Phi, \mathbf{X}\rangle = \int \frac{d^3p}{\sqrt{2E(2\pi)^3}}\, \phi(\mathbf{p})\, e^{-i\mathbf{p}\cdot\mathbf{X}}|p\rangle, \tag{3}$$

where the profile function $\phi(\mathbf{p})$ is required to satisfy

$$\int d^3p\, |\phi(\mathbf{p})|^2 = 1 \tag{4}$$

in order to ensure the proper normalization of the wave packet.

For later use, we define a dimensionless profile function $\tilde{\phi}$ via

$$\phi(\mathbf{p}) = R^{3/2}\, \tilde{\phi}(R\mathbf{p})\,, \tag{5}$$

where R denotes the characteristic size of the wave packet with $R \to 0$ corresponding to a sharp localization.

The charge density distribution in this state has the form

$$\langle\Phi,\mathbf{X}|\hat{\rho}(\mathbf{r},0)|\Phi,\mathbf{X}\rangle = \int \frac{d^3p\,d^3p'(E+E')}{(2\pi)^3\sqrt{4EE'}} F\left(q^2\right)\phi^\star(\mathbf{p}')\phi(\mathbf{p})e^{i\mathbf{q}\cdot(\mathbf{X}+\mathbf{r})},$$

where $\mathbf{q}=\mathbf{p}'-\mathbf{p}$ and $q^2=(E'-E)^2-\mathbf{q}^2$.
Without loss of generality we choose $\mathbf{X}=0$.

Finally, introducing the total and relative momentum variables via $\mathbf{p}=\mathbf{P}-\mathbf{q}/2$ and $\mathbf{p}'=\mathbf{P}+\mathbf{q}/2$, the charge density is written as

$$\begin{aligned}\rho_\phi(\mathbf{r}) &\equiv \langle\Phi,\mathbf{0}|\hat{\rho}(\mathbf{r},0)|\Phi,\mathbf{0}\rangle \\ &= \int \frac{d^3P\,d^3q}{(2\pi)^3\sqrt{4EE'}}(E+E')\,F\left[(E-E')^2-\mathbf{q}^2\right] \\ &\times\ \phi\left(\mathbf{P}-\frac{\mathbf{q}}{2}\right)\phi^\star\left(\mathbf{P}+\frac{\mathbf{q}}{2}\right)e^{i\mathbf{q}\cdot\mathbf{r}},\end{aligned} \tag{6}$$

where $E=\sqrt{m^2+\mathbf{P}^2-\mathbf{P}\cdot\mathbf{q}+\mathbf{q}^2/4}$ and
$E'=\sqrt{m^2+\mathbf{P}^2+\mathbf{P}\cdot\mathbf{q}+\mathbf{q}^2/4}$.

The traditional ("naive") interpretation of the charge density in terms of $F(-\mathbf{q}^2)$, emerges by first taking the static limit by substituting $E = E' = m$ in the integrand,

$$\rho_{\phi,\,\mathrm{naive}}(\mathbf{r}) = \int \frac{d^3P\, d^3q}{(2\pi)^3}\, \phi\left(\mathbf{P} - \frac{\mathbf{q}}{2}\right) \phi^\star\left(\mathbf{P} + \frac{\mathbf{q}}{2}\right) F\left(-\mathbf{q}^2\right)\, e^{i\mathbf{q}\cdot\mathbf{r}}, \quad (7)$$

and subsequently taking the limit $R \to 0$.

This can be done without specifying the functions $F\left(q^2\right)$ and $\phi(\mathbf{p})$ using the method of dimensional counting

J. Gegelia, G. S. Japaridze and K. S. Turashvili, Theor. Math. Phys. **101**, 1313-1319 (1994).

For $F\left(q^2\right)$ decreasing at large q^2 faster than $1/q^2$, the only non-vanishing contribution to $\rho_{\phi,\,\text{naive}}(\mathbf{r})$ in the $R \to 0$ limit is obtained by substituting $\mathbf{P} = \tilde{\mathbf{P}}/R$, expanding the integrand in R around $R = 0$ and keeping the zeroth order term.

The resulting naive charge density has the familiar form

$$\begin{aligned}
\rho_{\text{naive}}(r) &= \int \frac{d^3\tilde{P}\, d^3q}{(2\pi)^3} F\left(-\mathbf{q}^2\right) |\tilde{\phi}(\tilde{\mathbf{P}})|^2 e^{i\mathbf{q}\cdot\mathbf{r}} \\
&= \int \frac{d^3q}{(2\pi)^3} F\left(-\mathbf{q}^2\right) e^{i\mathbf{q}\cdot\mathbf{r}},
\end{aligned}$$

This expression corresponds to $R \gg \frac{1}{m}$.

On the other hand, the method of dimensional counting allows one to take the $R \to 0$ limit without employing the static approximation.

Following the same steps as before but for arbitrary m, we obtain

$$\rho_\phi(\mathbf{r}) = \int \frac{d^3\tilde{P}\, d^3q}{(2\pi)^3}\, F\left[\frac{(\tilde{\mathbf{P}}\cdot\mathbf{q})^2}{\tilde{\mathbf{P}}^2} - \mathbf{q}^2\right] |\tilde{\phi}(\tilde{\mathbf{P}})|^2\, e^{i\mathbf{q}\cdot\mathbf{r}}. \tag{8}$$

The resulting density depends on the shape of the wave packet unless it is spherically symmetric.

Since there is no preferred direction in ZAMF of the system, we *define* the charge density distribution by employing spherically symmetric wave packets with $\tilde{\phi}(\tilde{\mathbf{P}}) = \tilde{\phi}(|\tilde{\mathbf{P}}|)$.

Using spherical coordinates to perform the integration over $\tilde{\mathbf{P}}$, we arrive at the final form of the charge density distribution

$$\rho(r) = \int \frac{d^3q}{(2\pi)^3}\, e^{i\mathbf{q}\cdot\mathbf{r}} \int_{-1}^{+1} d\alpha\, \frac{1}{2}\, F\left[(\alpha^2 - 1)\,\mathbf{q}^2\right]. \tag{9}$$

As argued by Jaffe the traditional result $\rho_{\text{naive}}(r)$ is valid for the hierarchy of scales $\Delta \gg 1/m$, because we have to take $\Delta \gg R \gg 1/m$.

The validity of the new definition, does not depend on the relation between the intrinsic size of the system Δ and its Compton wavelength $1/m$.

Discussion:

A striking feature of the obtained result for $\rho(r)$ is its independence of the particle's mass.

This implies that the traditional expression for the charge density, $\rho_{\rm naive}(r)$, does *not* emerge from $\rho(r)$ by taking the static limit: $\rho_{\rm naive}(r) \neq \lim_{m\to\infty} \rho(r)$.

The reason for this mismatch is the non-commutativity of the $R \to 0$ and $m \to \infty$ limits of $\rho_\phi(\mathbf{r})$.

While the static limit and, more generally, the non-relativistic approximation is perfectly valid when calculating the form factor provided $-q^2 \ll m^2$, *it is violated in certain momentum regions when performing the integration over momenta*.

To demonstrate the non-commutativity of the $m \to \infty$ and $R \to 0$ limits consider the wave packet in one spatial dimension with

$$\phi(p) = \sqrt{\frac{2R}{\pi}} \frac{1}{1 + R^2 p^2}, \tag{10}$$

and the form factor

$$F(q_0^2 - q^2) = \frac{2}{2 - \Delta^2(q_0^2 - q^2)}, \tag{11}$$

so that $F(0) = 1$ and $F'(0) = \Delta^2/2$.

We calculate the second order moment of the charge distribution

$$\langle x^2 \rangle_\phi = \int_{-\infty}^{+\infty} dx\, x^2 \int_{-\infty}^{+\infty} \frac{dP\, dq}{2\pi \sqrt{4EE'}} (E + E') \tag{12}$$
$$\times F\left[(E - E')^2 - k\, q^2\right] \phi\left(P - \frac{q}{2}\right) \phi^\star\left(P + \frac{q}{2}\right) e^{iqx}.$$

For demonstration purposes, we have introduced a control parameter k to be set to $k = 1$ in the final result.

The resulting expression has the form

$$\langle x^2 \rangle_\phi = k\Delta^2 - \frac{\Delta^2}{(1+mR)^2} + \frac{R^2}{2} - \frac{R}{4m(1+mR)^3} \,. \tag{13}$$

Taking the limit $R \to 0$ in Eq. (13) leads to

$$\langle x^2 \rangle = (k-1)\Delta^2 = 0 \,, \tag{14}$$

which does not depend on the mass m.

On the other hand, taking first the static limit $m \to \infty$ and subsequently the $R \to 0$ limit we obtain a different result

$$\langle x^2 \rangle_{\mathrm{naive}} = k\Delta^2 = \Delta^2 \,. \tag{15}$$

The dependence of $\rho(r)$ on form factor $\frac{1}{2}\int_{-1}^{+1} d\alpha\, F\left[(\alpha^2-1)\mathbf{q}^2\right]$ rather than on $F(-\mathbf{q}^2)$ affects the radial profile of the charge density.

We compare $\rho(r)$ and $\rho_{\rm naive}(r)$ for a charged and a neutral particles.

We employ form factors

$F_{\rm p}(q^2) = G_{\rm D}(q^2) = (1 - q^2/\Lambda^2)^{-2}$

with $\Lambda^2 = 0.71$ GeV^2,

and

$F_{\rm n}(q^2) = A\tau/(1+B\tau)\, G_{\rm D}(q^2)$,

where $\tau = -q^2/(4m_p^2)$, $A = 1.70$, $B = 3.30$.

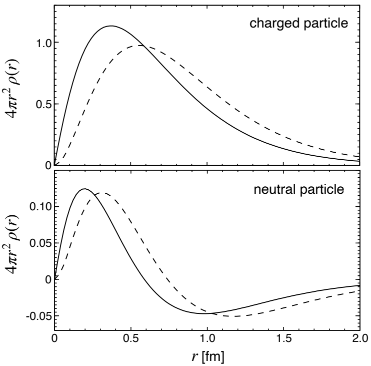

Figure: Radial charge density distributions $4\pi r^2 \rho(r)$ (solid lines) and $4\pi r^2 \rho_{\rm naive}(r)$ (dashed lines) for a charged and a neutral particles.

To gain further insights into the relationship between the charge density and the form factor it is instructive to use coordinate independent form:

$$\rho(r) = \frac{1}{4\pi} \int d^2\hat{n}\, \rho_{\hat{\mathbf{n}}}(\mathbf{r})\,, \tag{16}$$

where $\hat{\mathbf{n}} \equiv \mathbf{n}/|\mathbf{n}|$ is a unit vector and

$$\rho_{\hat{\mathbf{n}}}(\mathbf{r}) = \int \frac{d^3q}{(2\pi)^3} F(-\mathbf{q}_\perp^2)\, e^{i\mathbf{q}\cdot\mathbf{r}} = \rho_{\hat{\mathbf{n}}}(r_\parallel)\, \rho_{\hat{\mathbf{n}}}(r_\perp)\,. \tag{17}$$

Here, $\mathbf{q}_\perp = \hat{\mathbf{n}} \times (\mathbf{q} \times \hat{\mathbf{n}})$, $\mathbf{r}_\perp = \hat{\mathbf{n}} \times (\mathbf{r} \times \hat{\mathbf{n}})$, $r_\parallel = \mathbf{r} \cdot \hat{\mathbf{n}}$, $r_\perp \equiv |\mathbf{r}_\perp|$, and $\hat{\mathbf{n}}$ and $\mathbf{r}_\perp$ directions are given by

$$\begin{aligned} \rho_{\hat{\mathbf{n}}}(r_\parallel) &= \int \frac{dq_\parallel}{2\pi}\, e^{iq_\parallel r_\parallel} = \delta(r_\parallel)\,, \\ \rho_{\hat{\mathbf{n}}}(r_\perp) &= \int \frac{d^2q_\perp}{(2\pi)^2} F(-\mathbf{q}_\perp^2)\, e^{i\mathbf{q}_\perp\cdot\mathbf{r}_\perp}\,, \end{aligned} \tag{18}$$

with $q_\parallel = \mathbf{q} \cdot \hat{\mathbf{n}}$. These expressions establish a geometric interpretation of $\rho(r)$ to be discussed below.

The charge density in moving frames

While the static approximation $\rho_{\text{naive}}(r)$ does not depend on the frame, the expressions for $\rho(r)$ are valid in ZAMF only.

It is straightforward to generalize these results to a boosted frame.

We start with the general expression for $\rho_\phi(\mathbf{r})$ and replace $\phi(\mathbf{p})$ with $\phi_{\mathbf{v}}(\mathbf{p})$, where $\mathbf{v}$ denotes the boost velocity.

$\phi_{\mathbf{v}}(\mathbf{p})$ is expressed in terms of the rest frame quantity $\phi(\mathbf{p})$

$$\phi_{\mathbf{v}}(\mathbf{p}) = \sqrt{\gamma\left(1 - \frac{\mathbf{v}\cdot\mathbf{p}}{E}\right)}\,\phi\left[\mathbf{p}_\perp + \gamma(\mathbf{p}_\parallel - \mathbf{v}E)\right], \tag{19}$$

where $\gamma = (1 - v^2)^{-1/2}$, $\mathbf{p}_\parallel = (\mathbf{p}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}$, $\mathbf{p}_\perp = \mathbf{p} - \mathbf{p}_\parallel$ and $E = \sqrt{m^2 + \mathbf{p}^2}$.

S. E. Hoffmann, [arXiv:1804.00548 [quant-ph]].

Following the same steps as in the ZAMF and using the method of dimensional counting to evaluate the $R \to 0$ limit we arrive at

$$\rho_{\phi,\mathbf{v}}(\mathbf{r}) = \int \frac{d^3\tilde{P}'\, d^3q}{(2\pi)^3} \left|\tilde{\phi}(\tilde{\mathbf{P}}')\right|^2 e^{i\mathbf{q}\cdot\mathbf{r}} \times F\left\{ \frac{[\tilde{\mathbf{P}}'_\perp \cdot \mathbf{q}_\perp + \gamma(\tilde{\mathbf{P}}'_\parallel + \mathbf{v}\tilde{P}') \cdot \mathbf{q}_\parallel]^2}{\gamma^2(\tilde{P}' + v\tilde{P}'_\parallel)^2} - \mathbf{q}^2 \right\}. \tag{20}$$

Using spherical symmetry of $\tilde{\phi}(\tilde{\mathbf{P}}')$, the integration over $\tilde{P}'$ becomes trivial.
The remaining angular integration over $\hat{\tilde{\mathbf{P}}}'$ can be done in spherical coordinates. Our final result then reads:

$$\rho_{\mathbf{v}}(\mathbf{r}) = \int \frac{d^3q}{(2\pi)^3} \, \bar{F}\left(q_{\parallel}, q_{\perp}\right) \, e^{i\mathbf{q}\cdot\mathbf{r}} \,, \tag{21}$$

with $q_{\parallel} \equiv \hat{\mathbf{v}} \cdot \mathbf{q}$, $q_{\perp} \equiv |\mathbf{q}_{\perp}|$ and

$$\begin{aligned} \bar{F}(q_{\parallel}, q_{\perp}) =& \frac{1}{4\pi} \int_{-1}^{+1} d\eta \int_{0}^{2\pi} d\phi \\ & \times F\left\{ \frac{\left[\sqrt{1-\eta^2}\cos\phi \, q_{\perp} + \gamma(\eta + v) q_{\parallel}\right]^2}{\gamma^2(1+v\eta)^2} - \mathbf{q}^2 \right\} . \end{aligned} \tag{22}$$

In the IMF with $v \to 1$ and $\gamma \to \infty$, the charge density turns into the usual two-dimensional distribution in the transverse plane, $\rho_{\rm IMF}(\mathbf{r}) = \delta(r_{||})\, \rho_{\rm IMF}(r_\perp)$ with

$$\rho_{\rm IMF}(r_\perp) = \int \frac{d^2 q_\perp}{(2\pi)^2}\, F\left(-\mathbf{q}_\perp^2\right)\, e^{i\mathbf{q}_\perp \cdot \mathbf{r}_\perp}\,. \tag{23}$$

It is instructive to use a coordinate independent form similar to the FAMF expressions.

The charge density $\rho_{\mathbf{v}}(\mathbf{r})$ can be written as

$$\rho_{\mathbf{v}}(\mathbf{r}) = \frac{1}{4\pi} \int d^2\hat{m}\, \rho_{\hat{\mathbf{n}}(\mathbf{v},\hat{\mathbf{m}})}(\mathbf{r})\,, \tag{24}$$

where $\rho_{\hat{\mathbf{n}}(\mathbf{v},\hat{\mathbf{m}})}(\mathbf{r}) \equiv \rho_{\hat{\mathbf{n}}}(\mathbf{r})$ is defined above and

$$\mathbf{n}(\mathbf{v},\hat{\mathbf{m}}) = \hat{\mathbf{v}} \times (\hat{\mathbf{m}} \times \hat{\mathbf{v}}) + \gamma(\hat{\mathbf{m}} \cdot \hat{\mathbf{v}} + v)\hat{\mathbf{v}}\,. \tag{25}$$

In this form, both extreme limits for the boosting velocity become particularly transparent by using the relations $\hat{\mathbf{n}}(\mathbf{v},\hat{\mathbf{m}}) \xrightarrow{v\to 0} \hat{\mathbf{m}}$ and $\hat{\mathbf{n}}(\mathbf{v},\hat{\mathbf{m}}) \xrightarrow{v\to 1} \hat{\mathbf{v}}$, leading evidently to the ZAMF and IMF expressions, respectively.

The interpretation of the obtained results follows directly from the corresponding coordinate independent expressions.

The ZAMF expression $\rho(r)$ is given by a continuous (isotropic) superposition of the two-dimensional "images" of the system, $\rho_{\mathrm{IMF}}(\mathbf{r})$, made in all possible IMFs.

It is intuitively clear that the full image of a three-dimensional object can be reconstructed by putting together all possible two-dimensional projections.

Summary

- We introduced an unambiguous definition of a spatial distribution of the expectation values of local operators independent of the specific form of the wave packet.
- New definition also applies to systems whose intrinsic size is comparable or even smaller than the Compton wavelength.
- Our relationships between the electric form factor and the charge density in moving frames agrees with the well-known result in the IMF.
- Our results suggest an unconventional $\langle r^2 \rangle = 4F'(0)$ in contrast to the usual relationship $\langle r^2 \rangle_{\rm naive} = 6F'(0)$ motivated by the Breit frame distribution.
- The approximation $\rho_{\rm naive}(r)$ does not emerge in the static limit of the exact expression for $\rho(r)$, accuracy of which is independent of the particle's mass.