Theories of multiple spin-2 fields as ghost-free multimetric gravity

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Workshop on Recent Advances in Fundamental Physics
Sept 27 – Oct 01, 2022, Tbilisi, Georgia

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Disclaimer: Many people have contributed to this field, but I willl only focus of a few works)

Outline of the talk

Motivation: Why spin 2 fields?

Ghost-free Bimetric theory

Uniqueness and the local structure of spacetime

Ghost-free multi spin-2 theories

Discussion

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Motivation: Why spin 2 fields?

General relativity:

The gravitational metric $g_{\mu\nu}(x)$ is a field of $spin = 2$ and $mass = 0$.

Bimetric & multimetric theories:

Gravity ($g_{\mu\nu}$) coupled to other spin-2 fields ($f_{\mu\nu}, \cdots$).

Spectrum:

A massless spin-2 state + massive spin-2 states

Why are these theories interesting?
What are the challenges?
What is the progress?

Recall: Significance of spin

Fields/particles are classified by their *spin* s, *mass* m, $\cdots$

Spin s determines the basic form of field equations:

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- $s = 0$: $(\Box + m^2)\phi = 0$ — *Klein-Gordon*
- $s = \frac{1}{2}$: $(i\gamma^\mu \partial_\mu - m)\psi = 0$ — *Dirac*
- $s = 1$: $D_\mu F^{\mu\nu} = 0$ — *Maxwell* (+ *Yang-Mills*)
- $s = 2$: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$ — *Einstein*

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How come they still work? *spin* $\Rightarrow$ *unique form!*

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Standard Model: multiplets of $s = 0, \frac{1}{2}, 1$ + *intricate structures*

General Relativity: The *simplest possible* theory of $s = 2$

Physics beyond GR and SM: What are the possibilities?

Recall: Spin based classification of theories

- $s < 2$: Well known field theories *(e.g. in Standard Model)*
- $s > 2$: Local theories with finite field content may not exist *(cf. Higher spins, String theory)*

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  (*The spin-2 equivalent of* $\Box\phi = 0$ *&* $\partial_\mu F^{\mu\nu} = 0$)

  By contrast, in SM: $\phi \rightarrow$ *Higgs multiplet,*
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Do theories of *multiple* spin-2 fields exist? Or, is GR unique?

(Unexplored corner of the theory space)

Digression: the ghost problem

Ghost: A field with negative kinetic energy

Example:

$$\mathcal{L} = T - V = (\partial_t \phi)^2 \cdots \qquad (\textit{healthy})$$

But

$$\mathcal{L} = T - V = -(\partial_t \phi)^2 \cdots \qquad (\textit{ghostly})$$

Consequences:

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Consequences:

- Instability: unlimited energy transfer from ghost to other fields
- Negative probabilities, violation of unitarity in quantum theory

Why are higher spins $s \geq 2$ difficult?

Number of propagating helicities (n_h) for spin s:

$$mass = 0 : \; n_h = 2 \text{ or } 1 \;, \qquad\qquad mass \neq 0 : \;\; n_h = 2s + 1$$

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But, Lorentz invariance (general covariance) requires a field with $s \geq 1$ to have more than $2s + 1$ components. Examples:

$s = 1 : n_h < 4$ components of A_μ

$s = 2 : n_h < 10$ components of $g_{\mu\nu}$

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The extra components contain ghost instabilities. Need to be eliminated by symmetries+constraints.

(Are there enough of these?)

For massive spin-2, $n_h = 5 + 1$: The Boulware-Deser ghost (1972)

Recap: why are multiple spin-2 theories interesting?

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- Uncharted corner of local field theories, difficult to probe.
- New features. Relevant to gravity, dark matter, dark energy, inflation, etc.
- Not guided by experiments, but
  motivated by experience!

  (Precedents: Einstein-Hilbert, KG, Dirac, Proca, YM, Higgs)

Outline of the talk

GR + a generic spin-2 field

A dynamical theory of the metric $g_{\mu\nu}$ & spin-2 field $f_{\mu\nu}$

$$\mathcal{L} = m_p^2 \sqrt{|g|} R -$$

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No dynamics for $f_{\mu\nu} = \eta_{\mu\nu}$: Massive Gravity

describes a massive spin-2 (5 helicities) + a ghost (1 helicity)

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A very special $V(g^{-1}\eta) \Rightarrow$ ghost-free massive gravity:

[Creminelli, Nicolis, Papucci, Trincherini, (2005)]

[de Rham, Gabadadze (2010); de Rham, Gabadadze, Tolley (2010)]

[SFH, Rosen (2011); SFH, Rosen, Schmidt-May (2011)]

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- what is $V(g^{-1} f)$?
- what is $\mathcal{L}(f, \nabla f)$?
- proof of absence of the Boulware-Deser ghost

Ghost-free "bi-metric" theory

[SFH, Rosen (1109.3515,1111.2070)]

Ghost-free combination of *kinetic* and *potential* terms:

$$\mathcal{L} = m_g^2 \sqrt{|g|} R_g - \sqrt{|g|} \sum_{n=0}^{4} \beta_n \, e_n \left(\sqrt{g^{-1} f} \right) + m_f^2 \sqrt{|f|} \, R_f$$

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- Bimetric structure
- $7 = 2 + 5$ nonlinear propagating modes, no BD ghost!
- No ghost $\Rightarrow$ minimal matter couplings:

$$\mathcal{L}_{min}(g, \psi) + \mathcal{L}_{min}(f, \psi')$$

Mass spectrum & Limits

[SFH, Schmidt-May, von Strauss (arXiv:1208.1515)]

$$\bar{f} = c^2 \bar{g}\,, \quad g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,, \quad f_{\mu\nu} = \bar{f}_{\mu\nu} + \delta f_{\mu\nu}$$

Linear modes:

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Linear modes:

Massless spin-2: $$\delta G_{\mu\nu} = \left(\delta g_{\mu\nu} + \frac{m_f^2}{m_g^2}\,\delta f_{\mu\nu}\right) \qquad (2)$$

Massive spin-2 : $$\delta M_{\mu\nu} = \left(\delta f_{\mu\nu} - c^2 \delta g_{\mu\nu}\right) \qquad (5)$$

$g_{\mu\nu}$, $f_{\mu\nu}$ are mixtures of *massless* and *massive* modes

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General Relativity limit: $m_g = M_P\,, \quad m_f/m_g \to 0$

Massive gravity limit: $m_g = M_P\,, \quad m_f/m_g \to \infty$

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Uniqueness and the local structure of spacetime

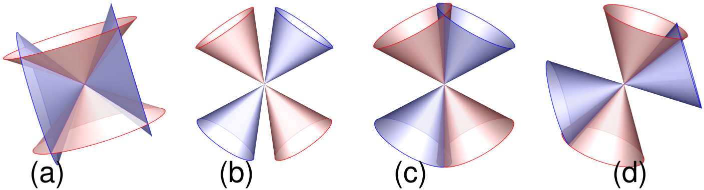
Ghost-free multi spin-2 theories

Discussion

Potential consistency problems and their solutions

Potential problem 1: Incompatible spacetimes

$g_{\mu\nu}$ & $f_{\mu\nu}$ may not admit compatible notions of *space* and *time*
(3+1 splits)



Then:

No consistent time evolution, no Hamiltonian formulation

Potential consistency problems and their solutions

Potential problem 2: Uniqueness, Reality, Covariance

Matrix square root: $S^{\mu}_{\ \nu} = \left(\sqrt{g^{-1}f}\right)^{\mu}_{\ \nu}$.

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Potential problem 2: Uniqueness, Reality, Covariance

Matrix square root: $S^{\mu}_{\ \nu} = \left(\sqrt{g^{-1}f}\right)^{\mu}_{\ \nu}$.

- Not unique: Multiple roots (*primary, non-primary*)
- Possibly non-real
- No general covariance (not a $(1,1)$ tensor)

Both problems happen to have a common solution

Uniqueness and the local structure of spacetime

[SFH, M. Kocic (arXiv:1706.07806)]

General Covariance:

$S =$ *principal root* $\Rightarrow$ Unique.

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General Covariance:

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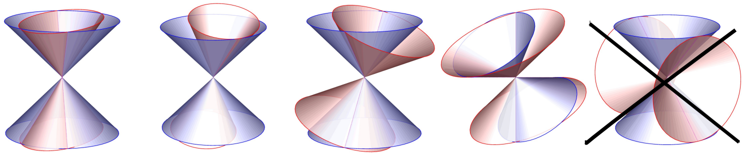
Reality (theorem):

S is real *iff* the null cones of $g_{\mu\nu}$ and $f_{\mu\nu}$ intersect as:



Type I Type IIa Type IIb Type III Type IV

Types I-III: Allowed, proper 3+1 decompositions possible.

Type IV: Non-primary, excluded by general covariance

(Implication for accausality arguments in the literature)

Bimetric theory in the vielbein formulation

[B. Zumino (1970)]

[K. Hinterbichler, R. A. Rosen (arXiv:1203.5783)]

$$g_{\mu\nu} = \eta_{AB} e^A_{\ \mu} e^B_{\ \nu} , \qquad f_{\mu\nu} = \eta_{AB} E^A_{\ \mu} E^B_{\ \nu}$$

$$\begin{aligned}\mathcal{L} = & m_g^2 \det(e) R_e + m_f^2 \det(E) R_E \\ & - m^4 \Big(\beta_0 e \wedge e \wedge e \wedge e + \beta_1 e \wedge e \wedge e \wedge E + \beta_2 e \wedge e \wedge E \wedge E \\ & \qquad + \beta_3 e \wedge E \wedge E \wedge E + \beta_4 E \wedge E \wedge E \wedge E \Big)\end{aligned}$$

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Eqns of motion imply:

$$e^A_{\ \mu} \eta_{AB} E^B_{\ \nu} - E^A_{\ \mu} \eta_{AB} e^B_{\ \nu} = 0\,, \quad \Longleftrightarrow \quad \text{evaluation of } \sqrt{g^{-1} f}$$

No real simplification!

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2) Proposed multi-vielbein interactions

[K. Hinterbichler, R. A. Rosen (arXiv:1203.5783)]

$$V = M^4 \sum_{I,J,K,L=1}^{\mathcal{N}} \beta^{IJKL}\, \epsilon_{ABCD}\, (e_I)^A \wedge (e_J)^B \wedge (e_K)^C \wedge (e_L)^D\,,$$

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3) But, generally, not ghost free

[C. de Rham, A. J. Tolley (arXiv:1505.01450)]

Do genuine ghost-free multi spin-2 interaction exist? Yes.

Ghost-free multi spin-2 theories

[SFH, Angnis Schmidt-May (arXiv:1804.09723)]

[SFH, Joakim Flinckman (to appear)]

Certain genuine multi spin-2 interactions for $(e_I)^A_{\ \mu}$ can be constructed. E.g.,

$$\mathcal{L} = \sum_{I=1}^{N} m_I^2 \sqrt{|g^I|} R(g^I) - M^4 \det \left(\sum_{I=1}^{N} \beta^I e_I \right)$$

- Has the correct number of constraints to eliminate the ghosts.

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- Has the correct number of constraints to eliminate the ghosts.
- A subset of the general veilbein interactions: $\beta^{IJKL} = \beta^I \beta^J \beta^K \beta^L$

Ghost-free multi spin-2 theories

$$\mathcal{L} = \sum_{I=1}^{N} m_I^2 \sqrt{|g^I|} R(g^I) - M^4 \det \left(\sum_{I=1}^{N} \beta^I e_I \right)$$

- Mass eigenstates and eigenvalues
- Is there a formulation in terms of the metrics?
- Certain "basic" extensions can be constructed and argued to be ghost free
- Compatible space and time decompositions?

[SFH, Joakim Flinckman (to appear)]

What is the most general form? Systematics not known

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The beginning of understanding theories of spin-2 fields beyond General Relativity.

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The beginning of understanding theories of spin-2 fields beyond General Relativity.

- Causal structure?
- Superluminality? (yes, not necessarily harmful, *inflation?*)
- Unavoidable mixings of mass eigenstates (unlike neutrino mixings)
- Systematics of multispin-2 interactions?
- A more fundamental formulation, say, (*a la* Higgs)
- Implications for cosmology, blackholes, GW, etc.
- Extra symmetries $\Rightarrow$ Modified kinetic terms?
  MacDowell-Mansouri type theories, More interesting but less understood.

Thank you!

EXTRA MATERIAL

Can Bimetric be a fundamental theory?

- Similar to Proca theory in curved background,

$$\sqrt{|\det g|}(F_{\mu\nu}F^{\mu\nu} - m^2\, g^{\mu\nu}A_\mu A_\nu + R_g)$$

- May need the equivalent of Higgs mechanism with the extra fields for better quantum or even classical behaviour

Extra: Elementary symmetric polynomials $e_n(S)$

For a 4×4 matrix S with eigenvalues $\lambda_1, \cdots, \lambda_4$,

$$
\begin{aligned}
e_1(S) &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \,, \\
e_2(S) &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 , \\
e_3(S) &= \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 \,, \\
e_4(S) &= \lambda_1\lambda_2\lambda_3\lambda_4 \,, \qquad e_{n>4}(S) = 0 \,.
\end{aligned}
$$

$$
\begin{aligned}
e_0(S) &= 1 \,, \\
e_1(S) &= \mathrm{Tr}(S) \equiv [S] \,, \\
e_2(S) &= \tfrac{1}{2}([S]^2 - [S^2]), \\
e_3(S) &= \tfrac{1}{6}([S]^3 - 3[S][S^2] + 2[S^3]) \,, \\
e_4(S) &= \det(S) \,, \qquad\qquad e_{n>4}(S) = 0 \,.
\end{aligned}
$$

$$
\boxed{\det(\mathbb{1} + S) = \sum_{n=0}^{4} e_n(S)}
$$

Extra: Interaction potential:

$$\det(\mathbb{1} + S) = \sum_{n=0}^{4} e_n(S)$$

$$V(S) = \sum_{n=0}^{4} \beta_n \, e_n(S)$$

Where:

$$S^\mu_{\ \nu} = \left(\sqrt{g^{-1} f}\right)^\mu_{\ \nu}$$

(“a” square root of the matrix $g^{\mu\lambda} f_{\lambda\nu}$. More on this later $\cdots$)

[de Rham, Gabadadze, Tolley (2010)]

[SFH, Rosen (2011); SFH, Rosen, Schmidt-May (2011)]

Extra: The matrix square root $\sqrt{A}$

See refs in [SFH, M. Kocic (arXiv:1706.07806)]

Put matrix A in *Jordan normal form*

$$A = Z\,\mathrm{diag}(J_1, \ldots, \ldots, J_s)\,Z^{-1}\,, \qquad J_i \equiv \begin{pmatrix} \lambda_i & 1 & 0 & \cdots \\ 0 & \lambda_i & 1 & \cdots \\ 0 & 0 & \ddots & \ddots \end{pmatrix}.$$

Then, the matrix function $F(A) = \sqrt{A}$ is

$$F(A) = Z\,\mathrm{diag}\left(F(J_1), \ldots, \ldots, F(J_p)\right)\,Z^{-1}\,,$$

$$F(J_k) \equiv \begin{pmatrix} F(\lambda_k) & F'(\lambda_k) & \cdots & \frac{1}{(n_k-1)!}F^{(n_k-1)}(\lambda_k) \\ & F(\lambda_k) & \ddots & \vdots \\ & & \ddots & F'(\lambda_k) \\ & & & F(\lambda_k) \end{pmatrix}.$$

Many roots: The same branch of $F(x) = \sqrt{x}$ must be chosen within each block, but it can vary between blocks.

Solution to the uniqueness problem of $V(S)$

Matrix square roots:

- Primary roots: Max 16 distinct roots, generic
- Nonprimary roots: Infinitely many, non-generic
  (when eigenvalues in different Jordan blocks coincide)

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General Covariance: $A^{\mu}_{\ \nu} = g^{\mu\rho} f_{\rho\nu}$ is a (1,1) tensor,

$$x^{\mu} \to \tilde{x}^{\mu} \Rightarrow A \to Q^{-1} A Q\,, \quad \text{for} \quad Q^{\mu}_{\ \nu} = \frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}}$$

Uniqueness and the local structure of spacetime

Potential consistency problems and their solutions *[SFH, M. Kocic (arXiv:1706.07806)]*

The bimetric theory given above is not yet well defined.

(2) The matrix $\sqrt{g^{-1}f}$ is non-unique, potentially complex & non-tensorial. Does a unique good choice exist?

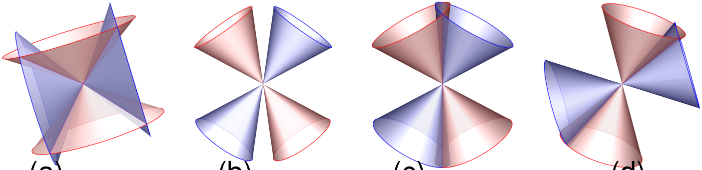
(1) The notions of "space" and "time" for $g_{\mu\nu}$ may not be consistent with that of $f_{\mu\nu}$.

Both problems have a common resolution

Potential consistency problems

Potential problem 1: Incompatible spacetimes

$g_{\mu\nu}$ & $f_{\mu\nu}$ may not admit compatible notions of *space* and *time* (3+1 splits)



(a) (b) (c) (d)

Implications: no consistent time evolution equations,
no Hamiltonian formulation.

Potential consistency problems

Problem 2: Uniqueness, reality, covariance

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Recall: General Covariance:

Since $A^{\mu}_{\ \nu} = g^{\mu\rho} f_{\rho\nu}$ is a (1,1) tensor, for $x^{\mu} \to \tilde{x}^{\mu}$ one has

$$A \to Q^{-1} A Q \,, \qquad \text{where} \qquad Q^{\mu}_{\ \nu} = \frac{\partial x^{\mu}}{\partial \tilde{x}^{\nu}}$$

What about $\left(\sqrt{A}\right)^{\mu}_{\ \nu}$?

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$$A \to \tilde{A} = Q^{-1} A\, Q \quad \Rightarrow \quad \sqrt{A} \to \sqrt{\tilde{A}} \stackrel{?}{=} Q^{-1} \sqrt{A}\, Q$$

Problem when *primary roots* degenerate to *non-primary roots*

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Problem when *primary roots* degenerate to *non-primary roots*

Can a **unique, real, covariant** *S be specified "meaningfully"?*

If not, the theory is ill defined

Solution: **General covariance + Reality**

Uniqueness and the local structure of spacetime

[SFH, M. Kocic (arXiv:1706.07806)]

General Covariance:

Only $S =$ *principal root* is always a $(1,1)$ tensor $\Rightarrow$ Uniqueness.

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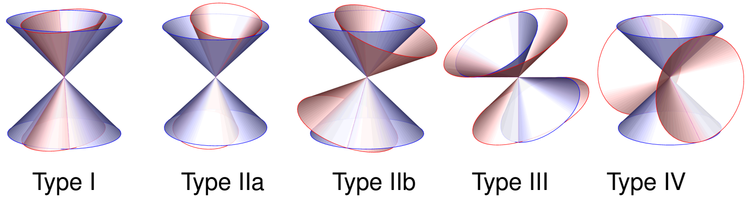
General Covariance:

Only S = *principal root* is always a $(1,1)$ tensor $\Rightarrow$ Uniqueness.

Reality (theorem):

S is real *iff* the null cones of $g_{\mu\nu}$ and $f_{\mu\nu}$ intersect as:



Types I-III: Allowed, proper 3+1 decompositions possible.

Type IV: Non-primary, excluded by general covariance

(Implication for accausality arguments in the literature)

Summary: Choice of the square root

Reality + **General Covariance** $\Rightarrow$

Real principal square root (unique) $\Rightarrow$

Intersecting null cone, Compatible 3+1 decompositions

Uniqueness of S

$S^{\mu}_{\ \nu} = (\sqrt{A})^{\mu}_{\ \nu}$:

- Primary roots: $\sqrt{A} \to \sqrt{Q^{-1}AQ} = Q^{-1}\sqrt{A}\,Q$
- Nonprimary roots: $\sqrt{Q^{-1}AQ} \neq Q^{-1}\sqrt{A}\,Q$

Step 1:
General covariance $\Rightarrow$ only primary roots are allowed.

A Consequence: Examples of backgrounds with local CTC's correspond to nonprimary roots and are excluded

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Step 1:
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A Consequence: Examples of backgrounds with local CTC's correspond to nonprimary roots and are excluded

Step 2:
Only the *principal root* is always primary. Hence, S must be a principle root.

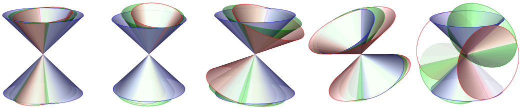
(Nonprinciple roots degenerate to nonprimary roots when some eigenvalues coincide).

The "mean" metric:

Consider the mean metric

$$h_{\mu\nu} = g_{\mu\rho}\left(\sqrt{g^{-1}f}\right)^{\rho}{}_{\nu}$$

h null-cones for the principal root always contain the intersections of *g* and *f* null-cones:



Useful for choosing "good" coordinate systems

Ghost in Massive Gravity

$$g_{\mu\nu}: \quad N, N_i(4), \gamma_{ij} = g_{ij}(6)$$

General Relativity:

$$\mathcal{L}_{gr} = \sqrt{g}R = \pi^{ij}\dot{\gamma}_{ij} - NR_0 - N^i R_i$$

$R_0 = R_i = 0$, GCT $\Rightarrow$ 2 polarizations

Massive Gravity:

$$\mathcal{L}_{mgr} = \sqrt{g}\left(R - V(g^{-1}f)\right) = \pi^{ij}\dot{\gamma}_{ij} - NR_0 - N^i R_i - \tilde{V}(N, N_i, \gamma, f)$$

No constraints, no GCT $\Rightarrow$ 5 polarizations + 1(BD ghost)

[Boulware, Deser (1972)]

Ghost free massive gravity: with a constraint.

[de Rham, Gabadadze, Tolley (2010)]

[SFH, Rosen (2011); SFH, Rosen, Schmidt-May (2011)]

Extra: Constraints in ghost-free "bi-metric" theory

[SFH, Rosen (1109.3515,1111.2070)]

[SFH, A. Lundkvist (arXiv:1802.07267)]

Let N, N_i (L, L_i): Lapse and shifts of $g_{\mu\nu}$ ($f_{\mu\nu}$)

$$\boxed{\mathcal{H} = LR^0 + L_i R^i + NC_1}$$

Constraints:

$R^0 = R^i = 0$, $C_1 = 0$, $C_2 \equiv \; dC_1/dt = \{\mathcal{H}, C_1\}_{PB} = 0$ +

Gauge fixing GCT:

$\Rightarrow 7 = 2 + 5$ nonlinear propagating modes, no BD ghost!

EOM's:

$$R^g_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R^g + V^g_{\mu\nu} = T^g_{\mu\nu} \,, \quad R^f_{\mu\nu} - \tfrac{1}{2} f_{\mu\nu} R^f + V^f_{\mu\nu} = T^f_{\mu\nu}$$

The HKT metric

General Relativity in 3+1 decomposition ($g_{\mu\nu}$: γ_{ij}, N, N_i):

$$\sqrt{g}R \sim \pi^{ij}\partial_t\gamma_{ij} - NR^0 - N_i R^i$$

Constraints: $R^0 = 0\,, R^i = 0$.
Algebra of General Coordinate Transformations (GCT):

$$\left\{R^0(x), R^0(y)\right\} = -\left[R^i(x)\frac{\partial}{\partial x^i}\delta^3(x-y) - R^i(y)\frac{\partial}{\partial y^i}\delta^3(x-y)\right]$$

$$\left\{R^0(x), R_i(y)\right\} = -R^0(y)\frac{\partial}{\partial x^i}\delta^3(x-y)$$

$$\left\{R_i(x), R_j(y)\right\} = -\left[R_j(x)\frac{\partial}{\partial x^i}\delta^3(x-y) - R_i(y)\frac{\partial}{\partial y^j}\delta^3(x-y)\right]$$

$R_i = \gamma_{ij}R^j$, γ_{ij}: metric of spatial 3-surfaces.

- Any generally covariant theory contains such an algebra.
- HKT: The tensor that lowers the index on R^i is the physical metric of 3-surfaces.

The HKT metric in bimetric theory

Consider $g_{\mu\nu} = (\gamma_{ij}, N, N_i)$ and $f_{\mu\nu} = (\phi_{ij}, L, L_i)$,

$$\mathcal{L}_{g,f} \sim \pi^{ij}\gamma_{ij} + p^{ij}\phi_{ij} - M\tilde{R}^0 - M_i\tilde{R}^i$$

On the surface of second class Constraints.
GCT Algebra:

$$\left\{\tilde{R}^0(x), \tilde{R}^0(y)\right\} = -\left[\tilde{R}^i(x)\tfrac{\partial}{\partial x^i}\delta^3(x-y) - \tilde{R}^i(y)\tfrac{\partial}{\partial y^i}\delta^3(x-y)\right]$$

$$\left\{\tilde{R}^0(x), \tilde{R}_i(y)\right\} = -\tilde{R}^0(y)\tfrac{\partial}{\partial x^i}\delta^3(x-y)$$

$\tilde{R}_i = \phi_{ij}\tilde{R}^j$, ϕ_{ij}: the 3-metric of $f_{\mu\nu}$, or
$\tilde{R}_i = \gamma_{ij}\tilde{R}^j$, γ_{ij}: the 3-metric of $g_{\mu\nu}$.
The HKT metric of bimetric theory is $g_{\mu\nu}$ or $f_{\mu\nu}$, consistent with ghost-free matter couplings

[SFH, A. Lundkvist [arXiv:1802.07267]]

Mass spectrum & Limits

[SFH, Schmidt-May, von Strauss (arXiv:1208.1515)]

$$\bar{f} = c^2 \bar{g}\,, \quad g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,, \quad f_{\mu\nu} = \bar{f}_{\mu\nu} + \delta f_{\mu\nu}$$

Linear modes:

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Linear modes:

Massless spin-2: $$\delta G_{\mu\nu} = \left(\delta g_{\mu\nu} + \frac{m_f^2}{m_g^2}\,\delta f_{\mu\nu}\right) \quad (2)$$

Massive spin-2 : $$\delta M_{\mu\nu} = \left(\delta f_{\mu\nu} - c^2 \delta g_{\mu\nu}\right) \quad (5)$$

$g_{\mu\nu}$, $f_{\mu\nu}$ are mixtures of *massless* and *massive* modes

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The General Relativity limit: $m_g = M_P\,, \quad m_f/m_g \to 0$

Massive gravity limit: $m_g = M_P\,, \quad m_f/m_g \to \infty$

GR limit and cosmology

The General Relativity limit:

$$m_g = M_P\,, \quad \alpha = m_f/m_g \to 0$$

Cosmological solutions in the GR limi (e.g.):

$$3H^2 = \frac{\rho}{M_{Pl}^2} - \frac{2}{3}\frac{\beta_1^2}{\beta_2}m^2 - \alpha^2\frac{\beta_1^2}{3\beta_2^2}H^2 + \mathcal{O}(\alpha^4)$$

The GR approximation breaks down at sufficiently strong fields

[Akrami, SFH,Konnig,Schmidt-May,Solomon (arXiv:1503.07521)]

More on bimetric cosmology:

[Lüben, Mörtsell, Schmidt-May (arXiv:1812.08686)]

Massive spin-2 particle as dark matter (not discussed here).
Also local (blackhole solutions) not discussed here.