# Scalar lumps with one and two horizons

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at I.Javakhishvili Tbilisi State University

RAFP, Sept 28, 2022

# Plan

- Motivation
- Introduction
- The setup
- Analytical & Numerical results
- Concluding remarks

# (Personal) Motivation

## Black holes, oscillating instantons and the Hawking-Moss transition

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ABSTRACT: Static oscillating bounces in Schwarzschild de Sitter spacetime are investigated. The oscillating bounce with many oscillations gives a super-thick bubble wall, for which the total vacuum energy increases while the mass of the black hole decreases due to the conservation of Arnowitt-Deser-Misner (ADM) mass. We show that the transition rate of such an "up-tunneling" consuming the seed black hole is higher than that of the Hawking-Moss transition. The correspondence of analyses in the static and global coordinates in the Euclidean de Sitter space is also investigated.

JHEP07(2020)024

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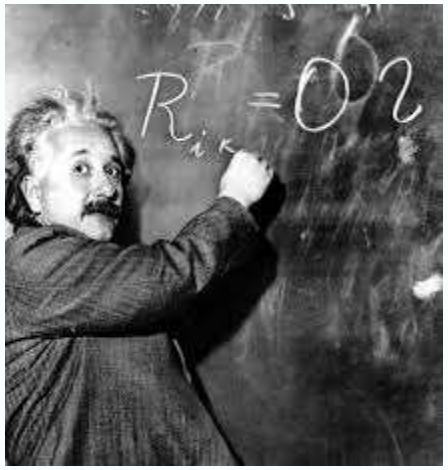
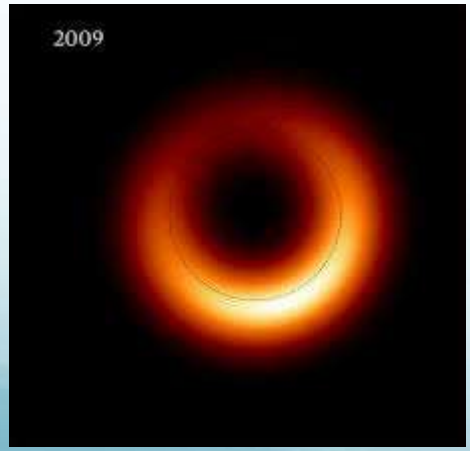
# Introduction: solitons and black holes

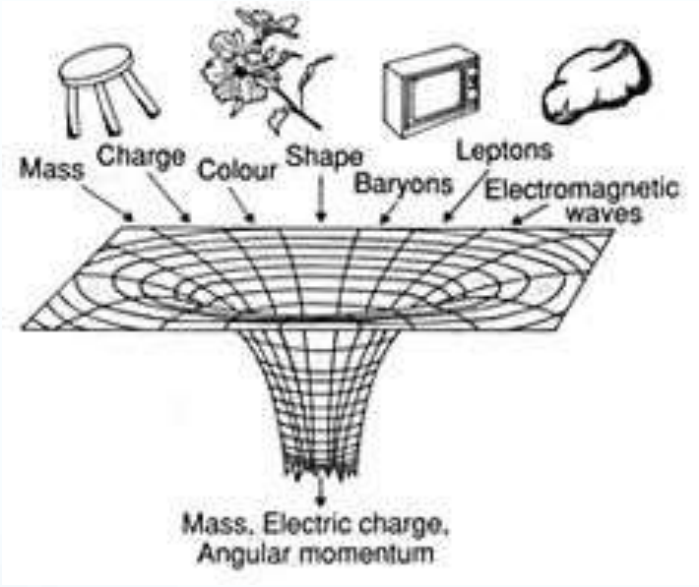


- $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda g_{\mu\nu}$
- Matter field equations = 0




- $ds^2 = -A\, dt^2 + \frac{dr^2}{B} + r^2\, d\,\Omega^2$
- A(r), B(r) - metric
- φ (r), W(r), … - matter

# Colored black holes :





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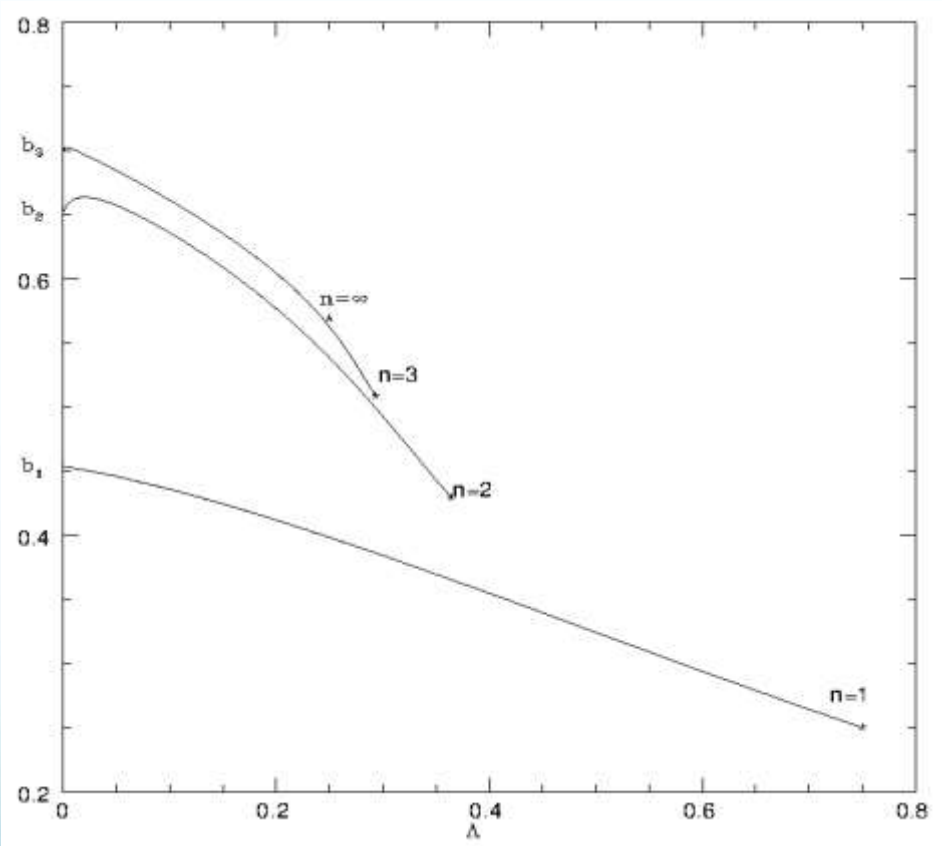
# Non-Abelian solitons and black holes in (anti) de Sitter space:




| $n$ | $\Lambda_{crit}(n)$ | $\Lambda_\circ(n)$ | $\Lambda_\star(n)$ |
|---|---|---|---|
| 1 | 0.330 | 0.334 | 0.75 |
| 2 | 0.239 | 0.250 | 0.364 |
| 3 | 0.237 | 0.247 | 0.293 |

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# The setup

Let us consider a self-interacting scalar field theory minimally coupled to gravity, with action

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa}\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right], \quad (1)$$

where $\kappa = 8\pi G$.

In what follows, we will be interested in static, spherically symmetric fields and parametrize the metric as

$$ds^2 = -f(r)\mathrm{e}^{2\delta(r)}dt^2 + \frac{dr^2}{f(r)} + R^2(r)d\Omega_2^2. \quad (4)$$

# Schwarzschild gauge

As long as $R'(r) \neq 0$, we can choose Schwarzschild gauge $R(r) \equiv r$.

$$\varphi'' = -\left(\frac{2}{r} + \frac{f'}{f} + \delta'\right)\varphi' + \frac{1}{f}\frac{\partial V}{\partial \varphi}, \tag{6}$$

$$rf' = 1 - f - \kappa r^2\left(\frac{1}{2}f\varphi'^2 + V\right), \tag{7}$$

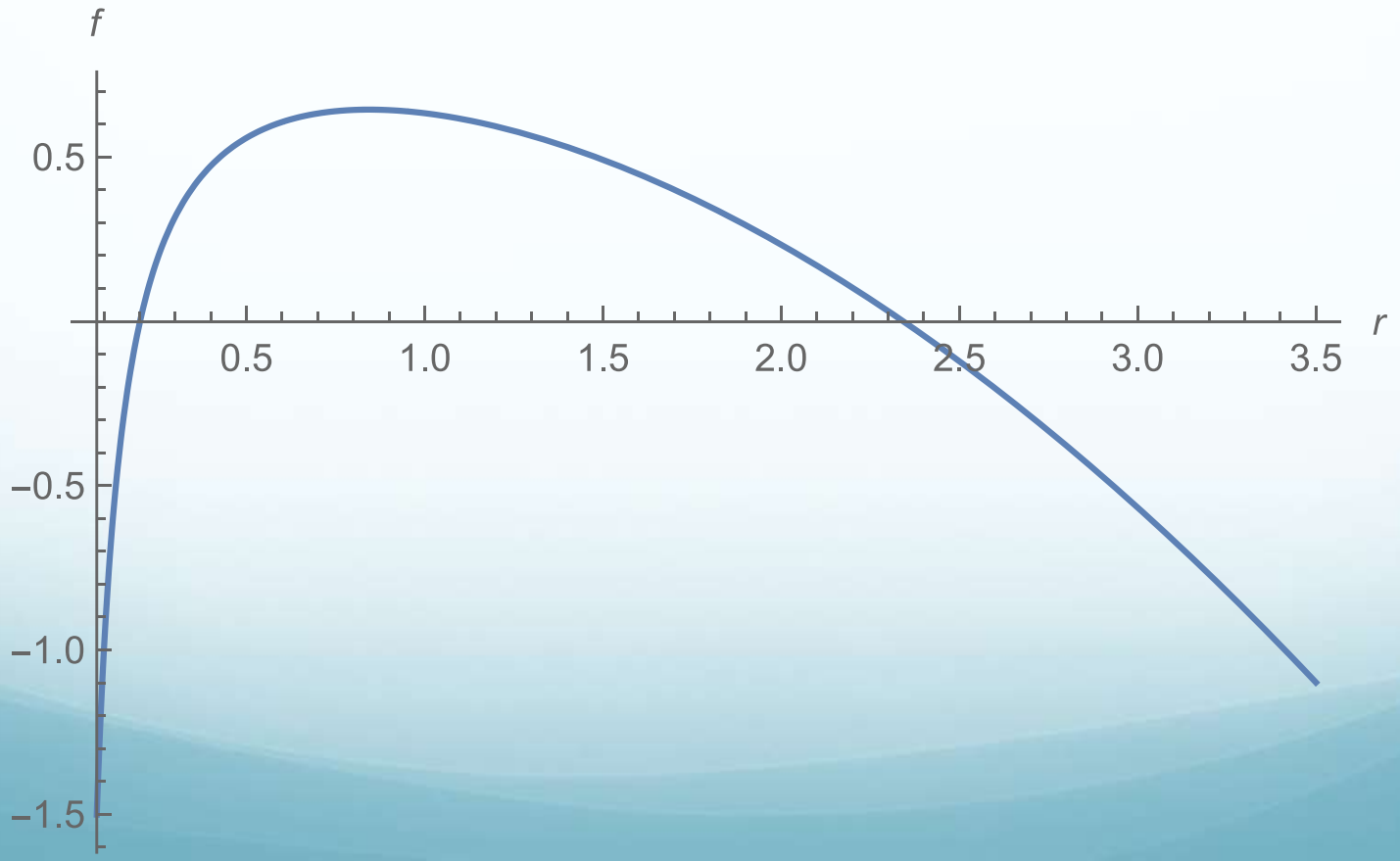
$$\delta' = \frac{\kappa}{2} r\varphi'^2. \tag{8}$$

Schwarzschild-de Sitter solution

$$f_0(r) = 1 - \frac{2\kappa M_0}{r} - \frac{\kappa V(\varphi_m)}{3} r^2$$

# Schwarzschild-de Sitter space-time

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \, , R(r) = r \, , \phi = 0$$



# General gauge

If at some point $R'(r) = 0$, the Schwarzschild gauge becomes inappropriate. In such a case, we may set $\delta \equiv 0$ while leaving $R(r)$ general, and we call this choice "general" gauge. Varying the reduced action (5) with respect to $\varphi$, $R$, and $f$, we obtain the corresponding field equations

$$\varphi'' = -\left(2\frac{R'}{R} + \frac{f'}{f}\right)\varphi' + \frac{1}{f}\frac{\partial V}{\partial \varphi}, \tag{18}$$

$$f'' = -2\frac{R'}{R}f' - 2\kappa V, \tag{19}$$

$$R'' = -\frac{\kappa}{2}R\varphi'^2, \tag{20}$$

while variation with respect to $\delta$, prior to setting it to zero, gives the constraint equation

$$f\frac{R'^2}{R^2} + f'\frac{R'}{R} = \frac{1}{R^2} + \kappa\left(\frac{1}{2}f\varphi'^2 - V\right). \tag{21}$$

# Regular Singular Point of ordinary differential equation

**Definition of Regular Singular Points**

Let

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

be a linear second order differential equation. Then $x_0$ is called a *regular singular* point if

$$\lim_{x \to x_0} (x - x_0) \frac{Q(x)}{P(x)}$$

and

$$\lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)}$$

are both finite.

# Close to regular origin we have 1 free parameter:

$$\varphi(r) = \varphi_0 + \frac{V'(\varphi_0)}{6} r^2 + V'(\varphi_0)\left(\frac{\kappa V(\varphi_0)}{36} + \frac{V''(\varphi_0)}{120}\right) r^4 + O(r^6), \tag{9}$$

$$f(r) = 1 - \frac{\kappa V(\varphi_0)}{3} r^2 - \frac{\kappa V'(\varphi_0)^2}{60} r^4 + O(r^6), \tag{10}$$

$$R(r) = r - \frac{\kappa V'(\varphi_0)^2}{360} r^5 + O(r^7), \tag{11}$$

where $\varphi_0$ is a free parameter.

# Close to horizon, f(r_h)=0, we find 4 free parameters:

$$\varphi(\rho) = \varphi_h + \frac{V'(\varphi_h)}{f'_h}\rho + \frac{V'(\varphi_h)[2\kappa V(\varphi_h) + V''(\varphi_h)]}{4f'^2_h}\rho^2 + O(\rho^3), \tag{23}$$

$$f(\rho) = f'_h\rho - \frac{1}{R_h^2}\rho^2 + O(\rho^3), \tag{24}$$

$$R(\rho) = R_h + \frac{1 - \kappa R_h^2 V(\varphi_h)}{f'_h R_h}\rho - \frac{\kappa R_h V'(\varphi_h)^2}{4f'^2_h}\rho^2 + O(\rho^3), \tag{25}$$

where $\rho \equiv r - r_h$ and $r_h$, $f'_h$, $\varphi_h$, and $R_h$ are free parameters.

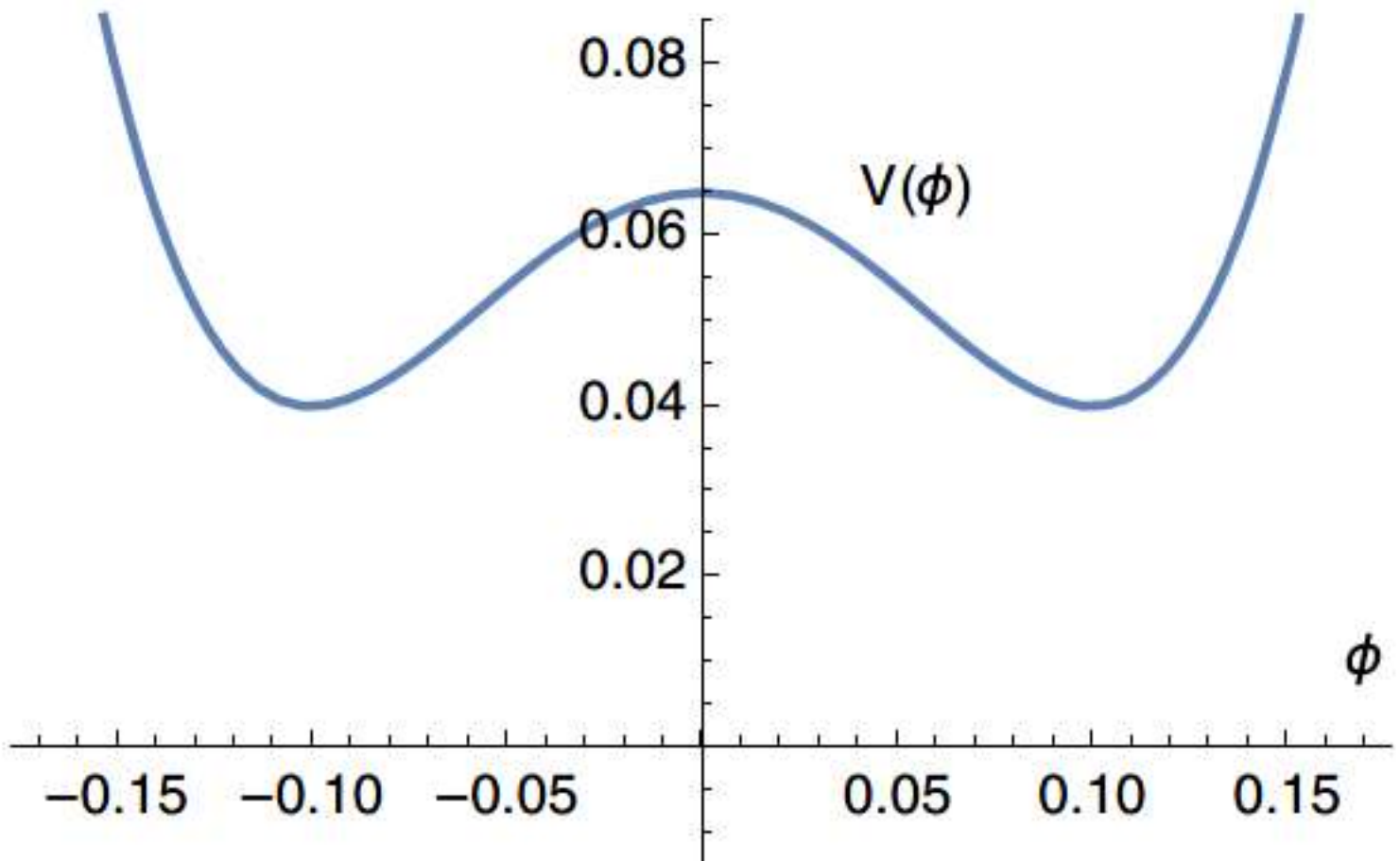
The scalar potential that we will consider is of double-well form,

$$V(\varphi) = \frac{\Lambda}{\kappa} + \frac{\lambda}{4}(\varphi^2 - v^2)^2, \tag{26}$$

where we have also included a cosmological constant $\Lambda$. This potential contains a local maximum at $\varphi = 0$, of magnitude $V(0) = \frac{\Lambda}{\kappa} + \frac{\lambda}{4}v^4$, and two minima at $\varphi = \pm v$. In what follows, we will use $G = 1$ units (implying $\kappa = 8\pi$). Furthermore, as shown in Ref. [2], the fields may be redefined so as to effectively set $\Lambda = 1$. This implies that we do not have to consider different values for the cosmological constant in searching for solutions, and specifying only the parameters $\lambda$ and $v$ of the potential is sufficient.

# განვიხილოთ კონკრეტული პოტენციალი

$$V(\varphi) = \frac{\Lambda}{\kappa} + \frac{\lambda}{4}(\varphi^2 - v^2)^2$$



# Parameter count, numerical strategy

Our numerical strategy is as follows. In general gauge, we have three functions, $f(r)$, $R(r)$, and $\phi(r)$ satisfying second-order equations, which implies that one needs to specify six parameters to fix a solution. The constraint equation (21) reduces the number of free parameters by 1, leaving one with only five free parameters required for specifying a solution. As shown above in Eqs. (23), (24), (25), close to a horizon, we have four free parameters. At the first and second horizons, we will, respectively, call these parameters $r_{1,2}$, $f'_{1,2}$, $\varphi_{1,2}$, and $R_{1,2}$; i.e., we have eight parameters in total.

With our metric ansatz, there actually remains some residual gauge freedom; note that equations of motions are translationally invariant with respect to a shift of the variable $r$,

$$r \to r + \Delta, \tag{27}$$

and scale invariant with respect to the transformations

$$r \to \alpha r, \qquad f \to \alpha^2 f, \tag{28}$$

with arbitrary parameters $\Delta$ and $\alpha$. These transformations allow us to fix the coordinate location of the first horizon to be at the origin, $r_1 = 0$, and they also allow us to fix the derivative of $f$ at the location of the first horizon—we will set $f_1' = 1$. We are left with six parameters. Since a solution is specified by fixing five parameters, we are left with one parameter, which we can choose freely. We will choose this free parameter to be the first horizon size $R_1$. This freedom is equivalent to specifying the mass $M_0$ in the SdS solution

# Counting the number of parameters of the solutions

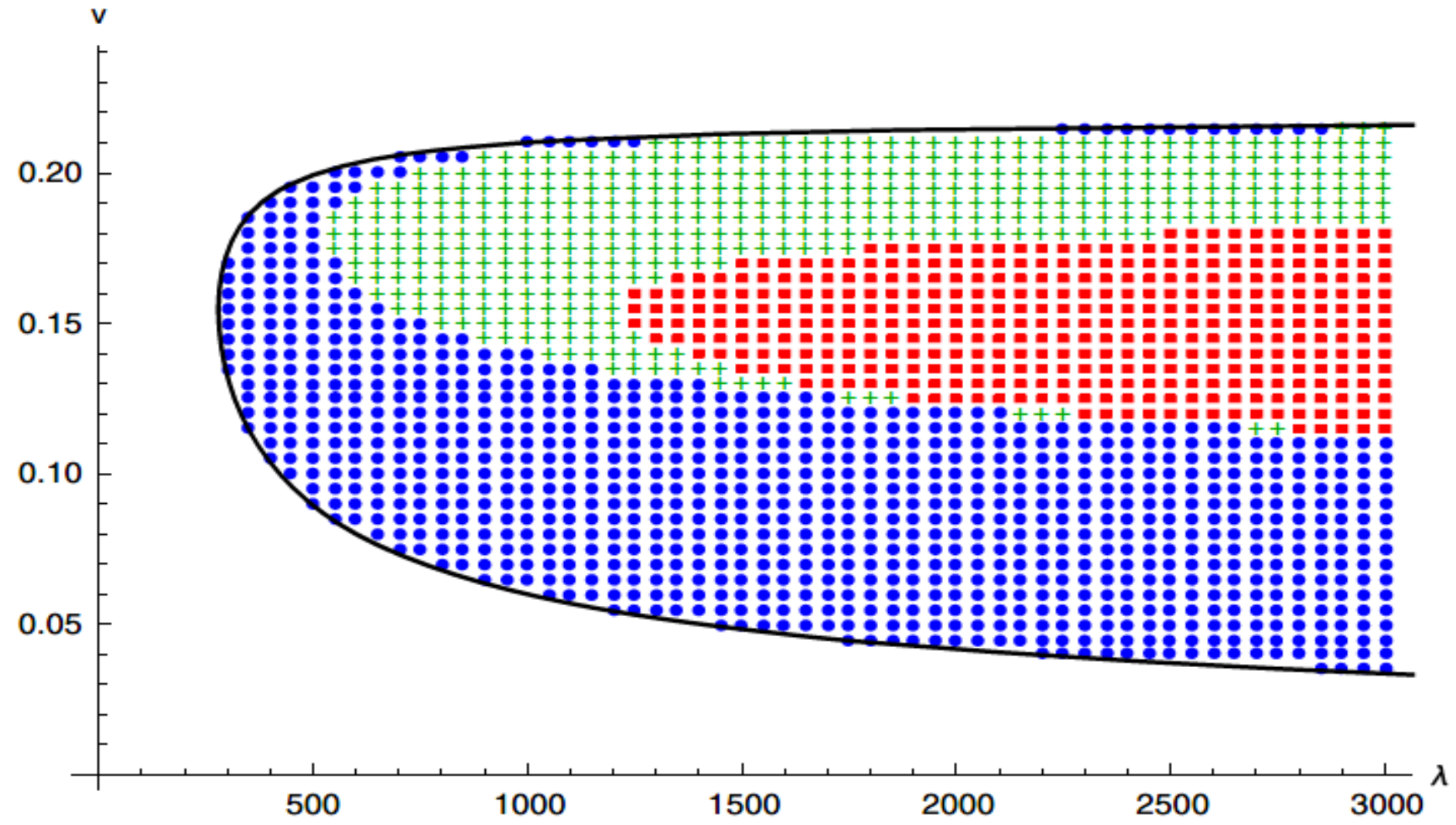
- 3 functions satisfy the second order differential equations - 1 constraint = 6 -1 = 5 parameters are needed to specify solution
- Regular origin: 1 parameter at the origin + 4 parameters at the horizon = 5 free parameters
- Two horizon case: 4 free parameters at the first horizon + 4 free parameters at the second horizon = 8 – 2 symmetry = 6, i.e. one parameter is free.

$$r \to r + \Delta,$$

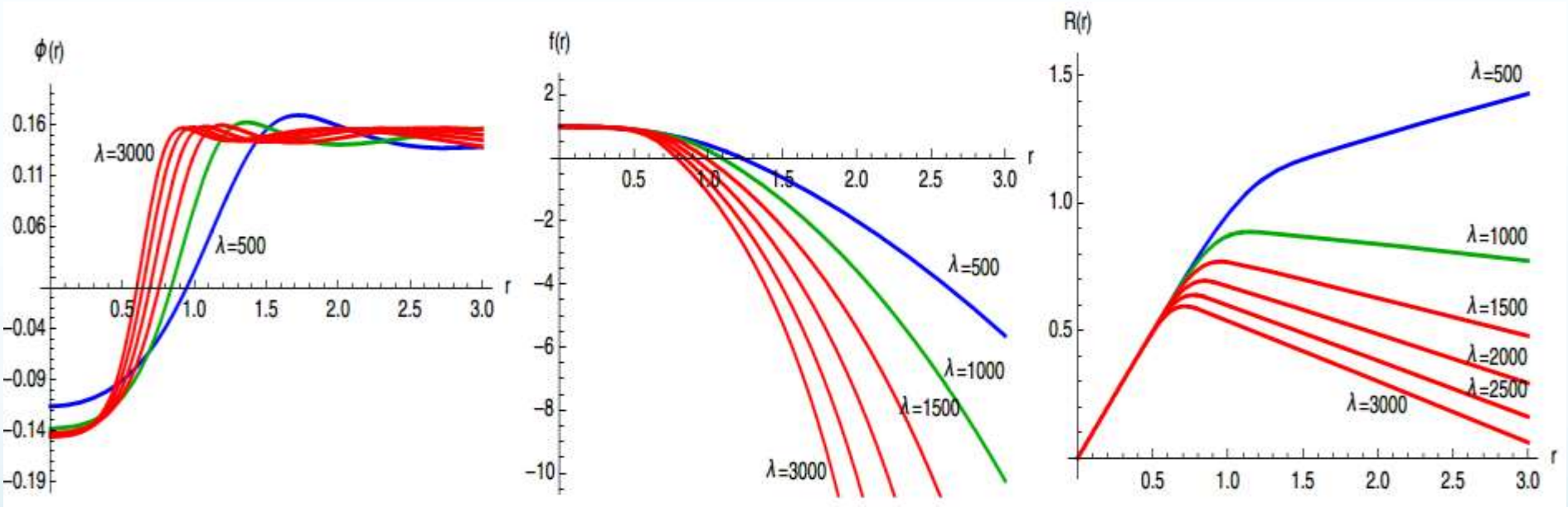
$$r \to \alpha r, \qquad f \to \alpha^2 f$$

In practical calculations, we thus first fix the first horizon size $R_1$. We then solve the field equations starting from some small $\epsilon$ away (typically of order $10^{-5}$) from the two horizons, with initial conditions given by the expansions in Eqs. (23), (24), and (25), in the direction of positive $f$. As explained above, at the first horizon, $r_1 = 0, f'_1 = 1$, and thus we only have a single free parameter there, namely, $\varphi_1$. From both horizons, we solve the equations of motion until $f(r)$ reaches a maximum. At this location, we match the two half-solutions together. This has two consequences: the first is that $f'$ is automatically continuous, and the second is that the coordinate location $r_2$ of the second horizon is implied. We then use a Newtonian algorithm to optimize all of the free parameters, until the fields and their derivatives are continuous at the matching location with sufficient precision.

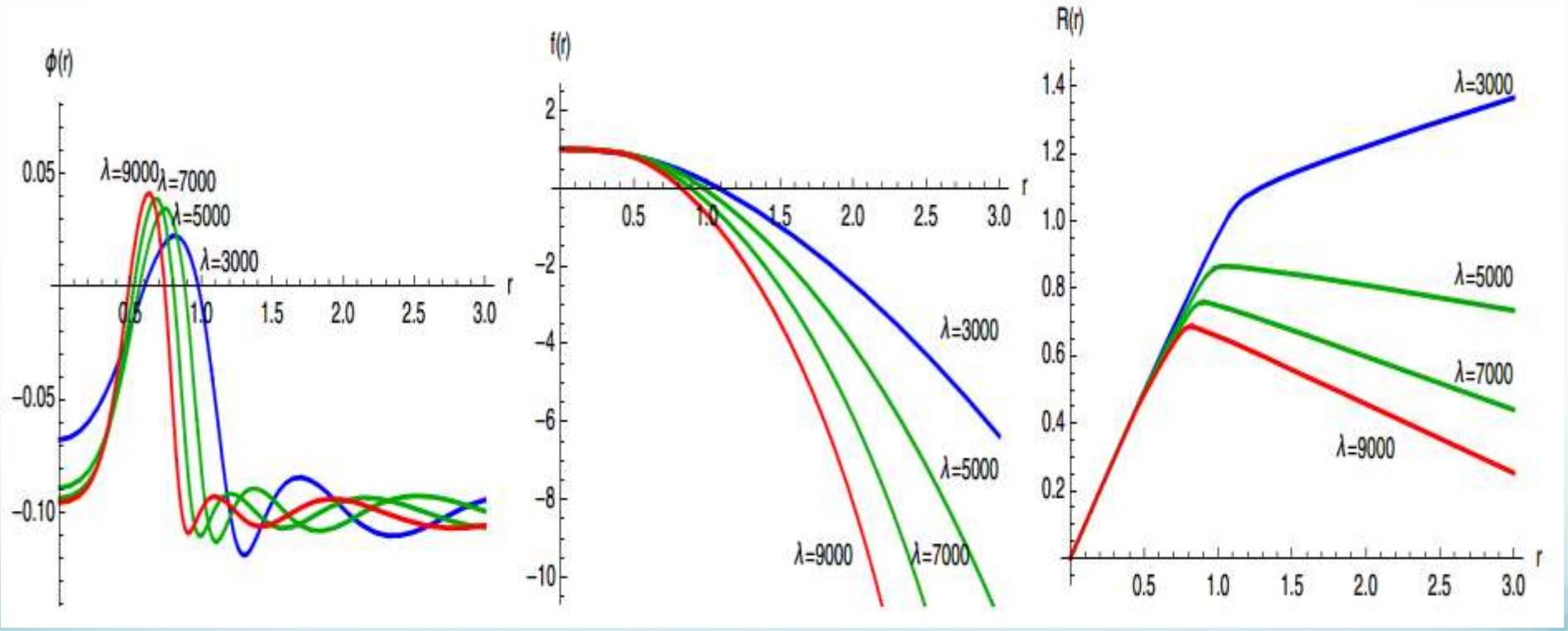
# Three classes of solutions: parameter space



# Three classes of solutions: : profiles of the functions for n=1



# Three classes of solutions: : profiles of the functions for n=2



# Solutions with two horizons

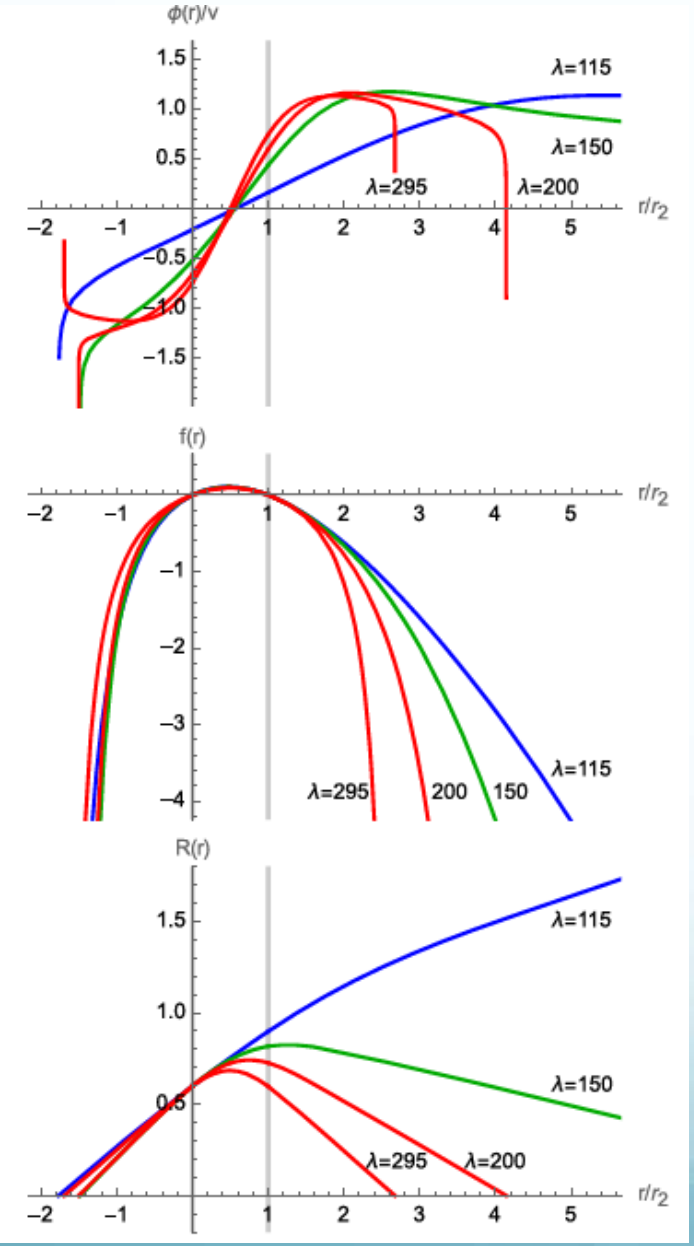
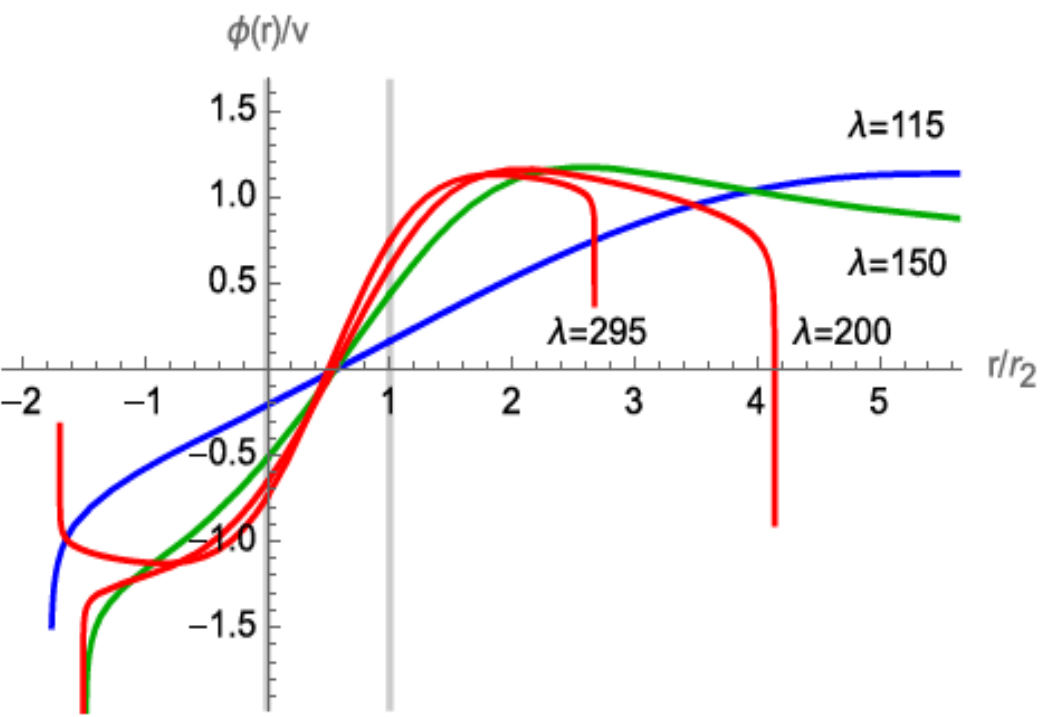
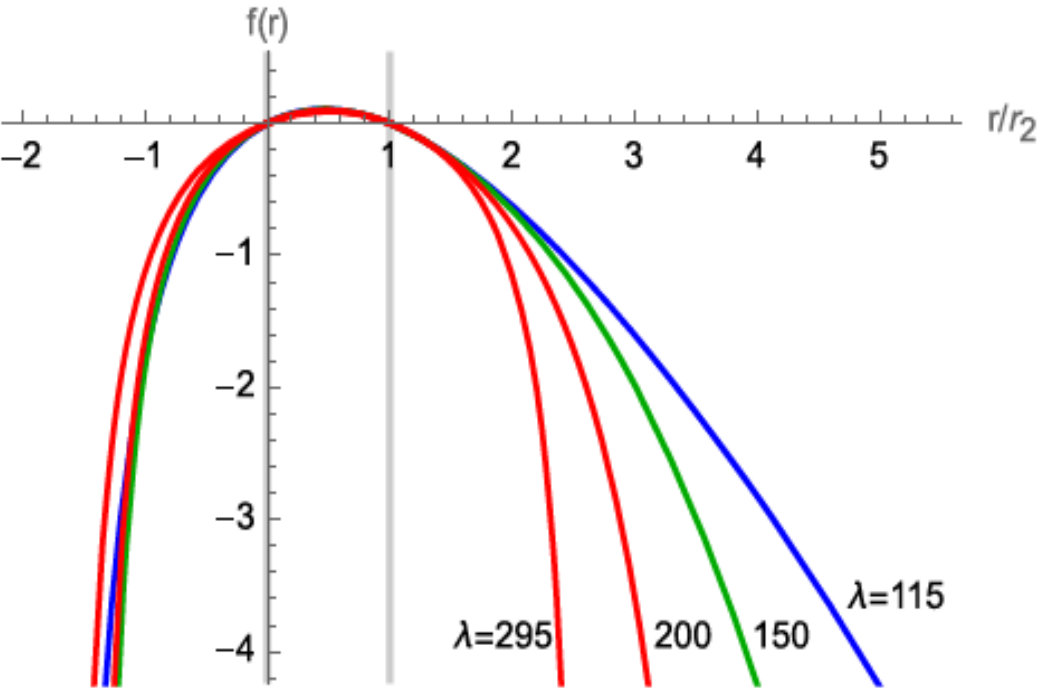



FIG. 1. Examples of solutions with the potential "width" fixed at $v = 0.18$ for various coupling constants $\lambda$. For all of these solutions, the first horizon size is set at $R_1 = 0.6$ and resides at $r = 0$. The location of the horizons is indicated by the thick vertical gray lines (one being at the origin $r = 0$). The scalar field is rescaled such that the minima of the potential reside at $\pm 1$. Also, these solutions have been plotted as functions of $r/(r_2 - r_1) = r/r_2$ in order to facilitate a comparison; the second horizon then always resides at $r/r_2 = 1$. See the main text for a full description.

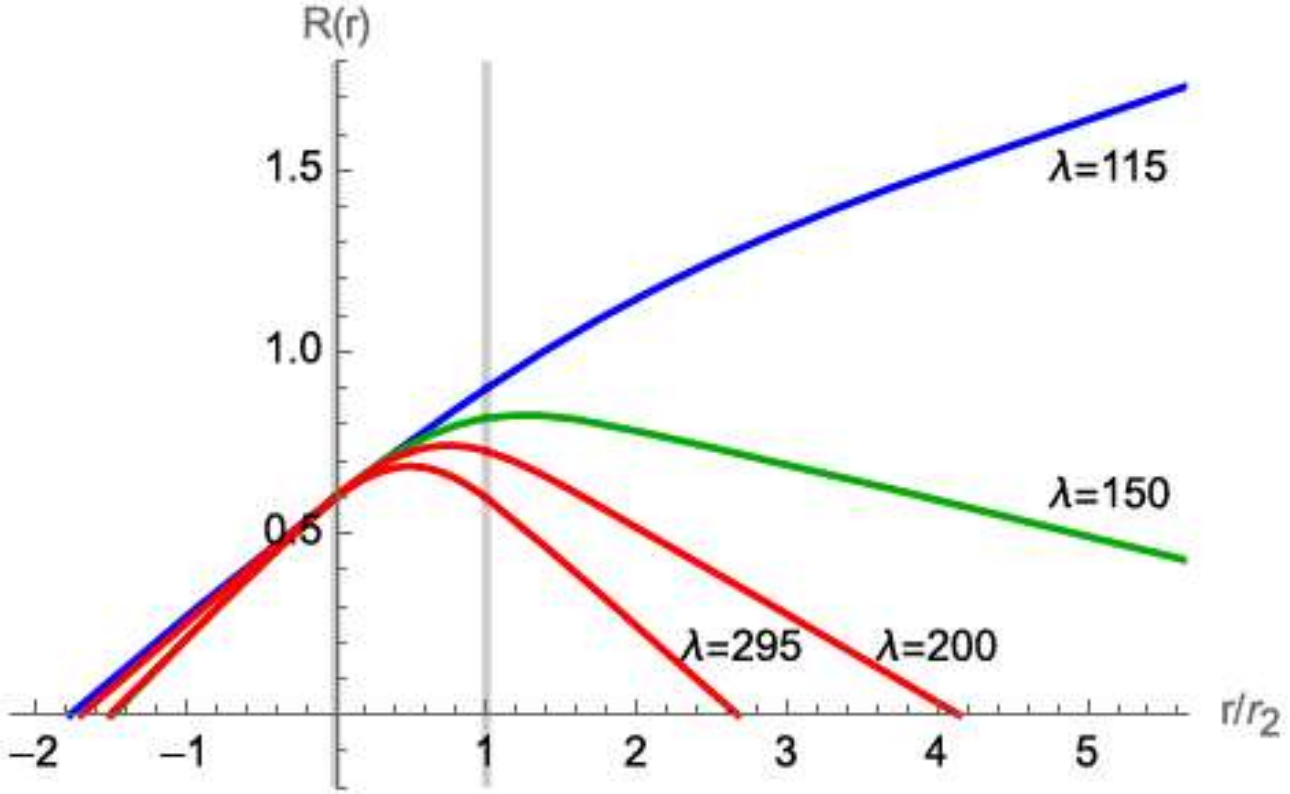
TABLE I. Optimized horizon values for the solutions in Fig. 1, given here to three significant digits.

| $\lambda$ | $\varphi_1$ | $\varphi_2$ | $f_2'$ | $R_2$ |
|---|---|---|---|---|
| 115 | −0.0376 | 0.0291 | 0.765 | 0.898 |
| 150 | −0.0935 | 0.0784 | 0.828 | 0.817 |
| 200 | −0.118 | 0.107 | 0.900 | 0.726 |
| 295 | −0.134 | 0.134 | 1.000 | 0.598 |




- Examples of different classes of solutions:
- Blue (monotonic R)
- Green (R has maximum after second horizon)
- Red (R has maximum between horizons)

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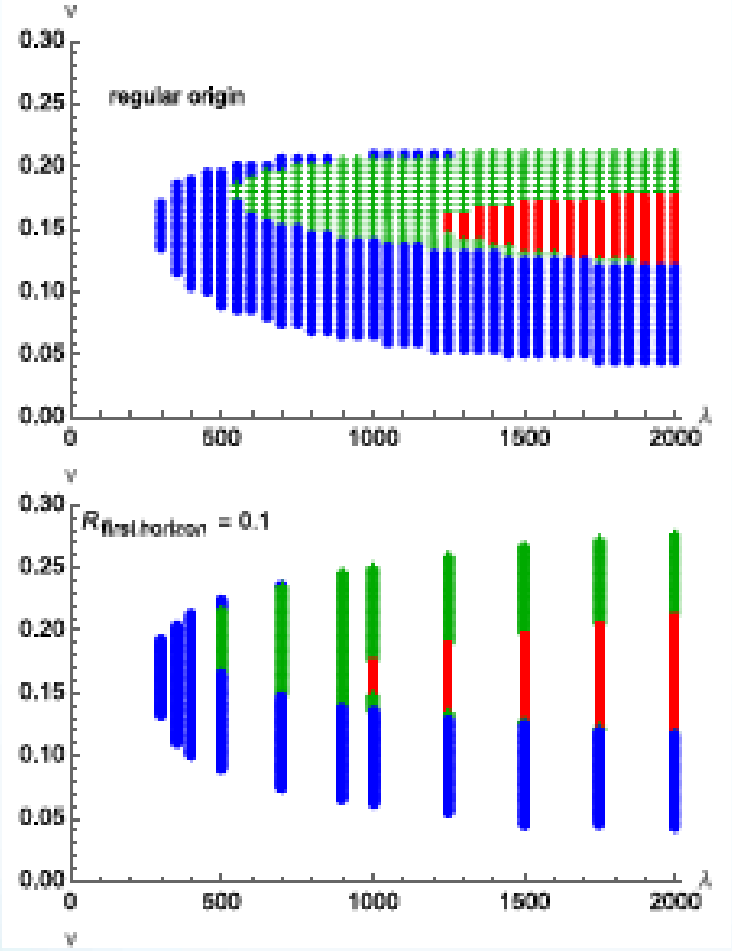


FIG. 3. Overview plots for different sizes $R_1$ of the first horizon, as a function of the parameters $\lambda$, $v$ in the potential. We also included a reference plot at the top of the existence of solutions with a regular origin and a single horizon. Blue dots indicate the existence of solutions with monotonic $R(r)$, green crosses indicate solutions in which $R(r)$ turns around outside of a horizon, and red squares indicate double-black-hole solutions in which $R(r)$ turns around in between the two horizons. A full description is given in the main text.

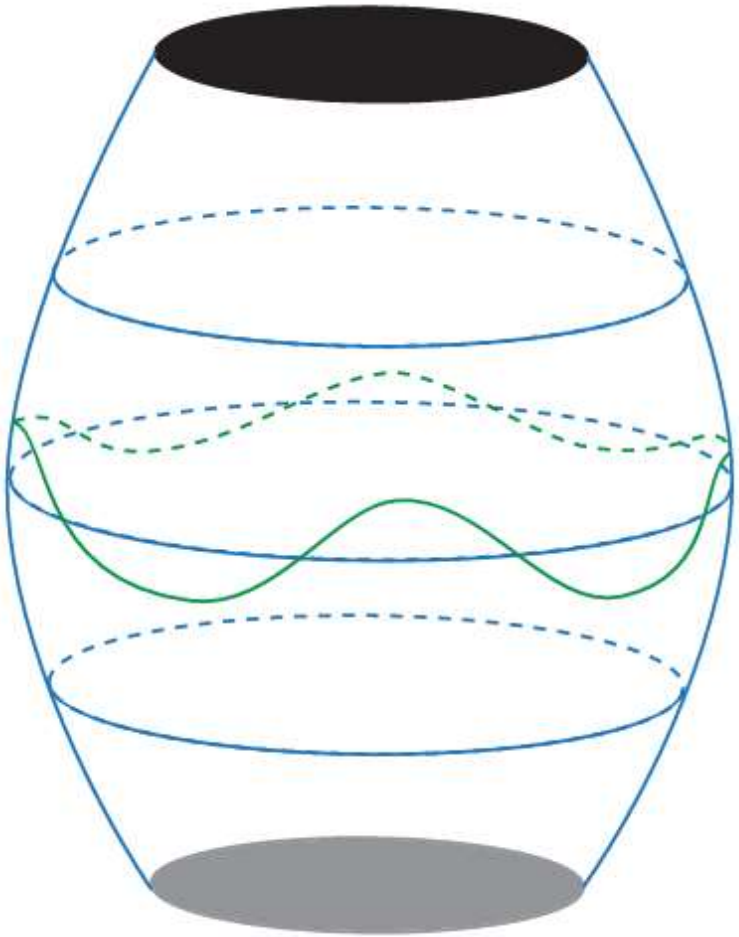

FIG. 10. A drawing of the solutions with nonmonotonic $R(r)$. Depicted here are only the $r$ direction and a transverse circle instead of transverse spheres. The spacetime is capped off by two black holes, one on each pole. In green, we show a particle undergoing accelerated motion near the equator, which itself represents an unstable orbit. If the particle travels too close to either black hole, it will inexorably be pulled in. The vacuum energy changes across the spacetime: it interpolates between the two sides of the scalar potential barrier as one traverses from one black hole to the other.

# Conclusions :

- We studied self-interacting scalar field theory coupled to gravity and found scalar lumps with regular origin surrounded by a horizon and scalar lumps with two horizons.
- New classes of solutions with non-monotonic function R(r) are found
- In particular we found static, spherically symmetric solutions with two black holes.
- It is interesting to search globally regular solutions of this system, analogues to the ones found in the EYMΛ theory.

# Thanks for attention!


Supported by

**Shota Rustaveli National Science Foundation of Georgia Grant FR 21-860**