AXION ELECTRODYNAMICS

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BASIC FORMALISM IN A DIELECTRIC ENVIRONMENT

Consider a pseudoscalar axion $a = a(\mathbf{r}, t)$ present in the entire universe, making a two-photon interaction with the electromagnetic field.
Assume a dielectric environment where the permittivity is ε and the permeability is μ, where these material parameters are constants. The constitutive relations are $\mathbf{D} = \varepsilon\mathbf{E}, \mathbf{B} = \mu\mathbf{H}$.
There are two field tensors, $F_{\alpha\beta}$ and $H_{\alpha\beta}$, where α and β run from 0 to 3. We assume the standard Minkowski space with the convention $g_{00} = -1$. The dual is defined as $\tilde{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$, with $\varepsilon^{0123} = 1$.
The field tensors are

$$F_{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix},$$

$$H^{\alpha\beta} = \begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & H_z & -H_y \\ -D_y & -H_z & 0 & H_x \\ -D_z & H_y & -H_x & 0 \end{pmatrix}.$$

The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\beta} H^{\alpha\beta} + \mathbf{A} \cdot \mathbf{J} - \rho\Phi - \frac{1}{2} \partial_\mu a \partial^\mu a - \frac{1}{2} m_a^2 a^2 - \frac{1}{4} g_\gamma \frac{\alpha}{\pi} \frac{1}{f_a} a(x) F_{\alpha\beta} \tilde{F}^{\alpha\beta}.$$

Here, ρ and $\mathbf{A}$ are the usual electromagnetic charge and current densities;
g_γ is a model-dependent constant for which we adopt the value 0.36;
α is the fine structure constant, and f_a is the axion decay constant whose value is insufficiently known. Often assumed that $f_a \sim 10^{12}$ GeV.

Defining the combined axion-two-photon coupling constant as

$$g_{a\gamma\gamma} = g_\gamma \frac{\alpha}{\pi} \frac{1}{f_a},$$

we see that the last term in the Lagrangian can be written as $\mathcal{L}_{a\gamma\gamma} = g_{a\gamma\gamma} a(x)\, \mathbf{E} \cdot \mathbf{B}$. It is convenient to define the quantity $\theta(x)$,

$$\theta(x) = g_{a\gamma\gamma} a(x).$$

The extended Maxwell equations can then be written as

$$\nabla \cdot \mathbf{D} = \rho - \mathbf{B} \cdot \nabla\theta,$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \dot{\mathbf{D}} + \dot{\theta}\mathbf{B} + \nabla\theta \times \mathbf{E},$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}.$$

These equations are general, i.e., there are no restrictions so far on the spacetime variation of $a(x)$. The equations are moreover relativistic covariant, with respect to shift of the inertial system.

The governing equations for the fields can correspondingly be written as

$$\nabla^2\mathbf{E} - \varepsilon\mu\ddot{\mathbf{E}} = \nabla(\nabla\cdot\mathbf{E}) + \mu\dot{\mathbf{J}} + \mu\frac{\partial}{\partial t}\left[\dot{\theta}\mathbf{B} + \nabla\theta\times\mathbf{E}\right],$$

$$\nabla^2\mathbf{H} - \varepsilon\mu\ddot{\mathbf{H}} = -\nabla\times\mathbf{J} - \nabla\times[\dot{\theta}\mathbf{B} + \nabla\theta\times\mathbf{E}].$$

Assume now a strong static magnetic field $\mathbf{B}_e = B_e\hat{\mathbf{z}}$ acts in a region where ρ and $\mathbf{J}$ are zero and the axion field is spatially uniform but varies harmonically in time,

$$a(t) = a_0\cos\omega_a t.$$

This is the situation usually found in the inner region of a haloscope. Then it is convenient to separate out the part of $\mathbf{E}$ that is caused by the uniformly fluctuating axions. Calling this contribution $\mathbf{E}_a(t)$, we see that it is connected by the $\ddot{\theta}$ term. From the governing equation for $\mathbf{E}_a(t)$,

$$\nabla^2\mathbf{E}_a - \varepsilon\mu\ddot{\mathbf{E}}_a = \mu\ddot{\theta}\mathbf{B}_e$$

we obtain, after omitting the ∇^2 term,

$$\mathbf{E}_a(t) = -\frac{1}{\varepsilon} E_0 \cos \omega_a t \, \hat{\mathbf{z}},$$

where

$$E_0 = \theta_0 B_e.$$

After the separation of the component $\mathbf{E}_a$, the field equation (4) takes the reduced form

$$\nabla^2 \mathbf{E} - \varepsilon\mu \ddot{\mathbf{E}} = \nabla(\nabla \cdot \mathbf{E}) + \mu \dot{\mathbf{J}} + \mu[\dot{\theta}\dot{\mathbf{B}} + \nabla\theta \times \dot{\mathbf{E}}].$$

Likewise for the magnetic field

$$\nabla^2 \mathbf{H} - \varepsilon\mu \ddot{\mathbf{H}} = -\nabla \times \mathbf{J} - \left[\dot{\theta}\nabla \times \mathbf{B} + (\nabla\theta)\nabla \cdot \mathbf{E} - (\nabla\theta \cdot \nabla)\mathbf{E}\right].$$

Antenna-like behavior: One dielectric surface

Considering one single planar dielectric surface, placed at $x = 0$, separating the left region 1 (refractive index n_1) from the right region 2 (refractive index n_2).

Assume that the media are nonmagnetic and n_1, n_2 constants and real. A strong static magnetic field $\mathbf{B}_e = B_e\hat{\mathbf{z}}$ is imposed in the z direction.
An incoming wave polarized in the z direction comes in from the left, propagates in the x direction, and becomes partly reflected by the surface. The components of $\mathbf{E}$ and $\mathbf{H}$ parallel to the surface have to be continuous (as in ordinary electrodynamics) at $x = 0$.

Millar *et al.* (2017): Because of the axions there will be produced two outgoing electromagnetic waves, one going to the left and one going to the right. In this sense we can consider the dielectric surface to have "antenna-like" properties.
Distinguish the produced traveling fields by an extra index γ. Continuity of $\mathbf{E}_{||}$ gives

$$E_1^\gamma + E_1^a = E_2^\gamma + E_2^a.$$

Take into account the relationship

$$H^{\gamma} = \pm \frac{1}{n} E^{\gamma},$$

which implies

$$-n_1 E_1^{\gamma} = n_2 E_2^{\gamma},$$

expressing that the two wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ are antiparallel. As $E^a = -(1/\varepsilon)E_0$, we can then solve for the produced fields to get

$$E_1^{\gamma} = -\frac{E_0}{n_1}\left(\frac{1}{n_2} - \frac{1}{n_1}\right), \quad E_2^{\gamma} = \frac{E_0}{n_2}\left(\frac{1}{n_2} - \frac{1}{n_1}\right),$$

(recall that $E_0 = \theta_0 B_e$).

Two dielectric surfaces

One dielectric slab of thickness d,surrounded by vacuum. The refractive index is $n = \sqrt{\varepsilon}$.
Energy transmission coefficient

$$T = \left|\frac{E_T}{E_I}\right|^2$$

in the presence of axions, E_T and E_I referring to the transmitted and incident wave amplitudes. Then

$$T = \frac{4n^2}{4n^2 + (n^2-1)^2 \sin^2 kd},$$

where $k = 2\pi/\lambda = n\omega$. This expression does not contain E_0.

Special cases:
If $d = \lambda/4$ the transmission is at minimum, $T = T_{\mathrm{min}} = 4n^2/(n^2+1)^2$,
If $d = \lambda/2$, the transmission is at maximum, $T = T_{\mathrm{max}} = 1$.

Compare this transmission coefficient from that occurring in ordinary electrodynamics:

$$T_{\mathrm{elmag}} = \frac{4n^2}{(n^2+1)^2 - (n^2-1)^2 \sin^2 kd}.$$

It is seen that T and T_{elmag} are different, what could be expected since their derivations are different. Now, if $d = \lambda/4$, $T_{\mathrm{elmag}} = 1$, while if $d = \lambda/2$, $T_{\mathrm{elmag}} = 4n^2/(n^2+1)^2$, its minimum value.

The closed string geometry

Consider dielectric systems containing two interfaces separating media of refractive indices n_1 and n_2, but assume that the media are elastic so that medium 2 can be "turned back" and glued to the left side of medium 1. Therewith one gets a ring-formed system. Does such a system allow stationary oscillations to occur when axions are present?

Let σ denote the length coordinate along the string, such that the two dielectric junctions are at $\sigma = 0$ and $\sigma = L_I$. The total length of the string is $L = l_I + L_{II}$, so that the junctions $\sigma = 0$ and $\sigma = L$ are overlapping. We will be interested in the fields in the interior regions of the string. The string is lying in the xy plane, and a strong uniform magnetic field $\mathbf{B}_e$ is applied in the z direction.
Seek for stationary oscillations of the electromagnetic oscillations in the ring. If $E_I(\sigma, t)$ and $E_{II}(\sigma, t)$ are the electric fields in the two regions, one has

$$E_I(\sigma, t) = \xi_I e^{in_1\omega\sigma - i\omega t} + \eta_I e^{-in_1\omega\sigma - i\omega t},$$

$$E_{II}(\sigma, t) = \xi_{II} e^{in_2\omega(\sigma - L_I) - i\omega t} + \eta_{II} e^{-in_2\omega(\sigma - L_I) - i\omega t},$$

, where $\xi_I, \eta_I, \xi_{II}, \eta_{II}$ are constants. Analogously, using the same relationship $H = \pm nE$ as previously, we have for the magnetic field

$$H_I(\sigma, t) = n_1 \left[\xi_I e^{in_1\omega\sigma - i\omega t} - \eta_I e^{-in_1\omega\sigma - i\omega t}\right],$$

$$H_{II}(\sigma, t) = n_2 \left[\xi_{II} e^{in_2(\sigma - L_I) - i\omega t} - \eta_{II} e^{-in_2\omega(\sigma - L_I) - i\omega t}\right].$$

We omit the time factor $e^{-i\omega t}$ and introduce the shorthand notation

$$\delta_1 = n_1 \omega L_I, \quad \delta_2 = n_2 \omega L_{II}.$$

The boundary conditions at the junctions are, for the electric field,

$$-\frac{E_0}{\varepsilon_1} + \xi_I e^{i\delta_1} + \eta_I e^{-i\delta_1} = -\frac{E_0}{\varepsilon_2} + \xi_{II} + \eta_{II}, \quad \sigma = L_I,$$

$$-\frac{E_0}{\varepsilon_2} + \xi_{II} e^{i\delta_2} + \eta_{II} e^{-i\delta_2} = -\frac{E_0}{\varepsilon_1} + \xi_I + \eta_I, \quad \sigma = L,$$

and for the magnetic field,

$$n_1(\xi_1 e^{i\delta_1} - \eta_I e^{-i\delta_1}) = n_2(\xi_{II} - \eta_{II}), \quad \sigma = L_I,$$

$$n_1(\xi_I - \eta_I) = n_2(\xi_{II} e^{i\delta_2} - \eta_{II} e^{-i\delta_2}), \quad \sigma = L,$$

where $E_0 = \theta_0 B_e = g_{a\gamma\gamma} a_0 B_e$ as before.
We introduce the symbol x for the refractive index ratio,

$$x = \frac{n_1}{n_2},$$

and consider the scheme

$$\begin{pmatrix} e^{i\delta_1} & e^{-i\delta_1} & -1 & -1 \\ 1 & 1 & -e^{i\delta_2} & -e^{-i\delta_2} \\ xe^{i\delta_1} & -xe^{-i\delta_1} & -1 & 1 \\ x & -x & -e^{i\delta_2} & e^{-i\delta_2} \end{pmatrix} \begin{pmatrix} \xi_I \\ \eta_I \\ \xi_{II} \\ \eta_{II} \end{pmatrix} = \begin{pmatrix} E_0\left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}\right) \\ E_0\left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}\right) \\ 0 \\ 0 \end{pmatrix}.$$

Here the determinant D of the system matrix M_{ik} can be calculated to be

$$D = \det M_{ik} = -8x + 2(1+x)^2\cos(\delta_1+\delta_2) - 2(1-x)^2\cos(\delta_1-\delta_2).$$

This is a real quantity.

We can now calculate explicit expressions for the field amplitudes $\xi_I, \eta_I, \xi_{II}, \eta_{II}$,in the two regions of the string.

One particular case of interest:

$$x = \frac{n_1}{n_2} \to 0.$$

The lengths L_I and L_{II} are assumed arbitrary. Then

$$D(x \to 0) = 2\cos(\delta_1 + \delta_2) - 2\cos(\delta_1 - \delta_2) = -4\sin\delta_1 \sin\delta_2,$$

which leads to

$$\xi_I = \frac{E_0}{2\varepsilon_1}\left[1 - i\frac{1-\cos\delta_1}{\sin\delta_1}\right], \quad x \to 0.$$

This expression does not contain the phase δ_2 related to the length L_{II}.

It is of further interest to consider $L_I \to 0$, corresponding to a kind of point defect sitting on an otherwise uniform string. As $\delta_1 \to 0$ in this case, it follows that

$$\xi_I = \frac{E_0}{2\varepsilon_1}, \quad x \to 0, L_I \to 0,$$

which is a real quantity. If $E_0 = 0$, the axion-induced forced oscillations vanish.
In conclusion, the axionic electrodynamic scheme is flexible enough to upheld stationary oscillations in the closed string geometry.

Casimir effect for the closed string

Now put $E_0 = 0$, so that the forced axion-induced oscillations vanish, and those remaining possible are only the free oscillations. They correspond to the system determinant D being zero. Striking similarity to the Casimir teory for a relativistic piecewise uniform string.
Uniform string, $x = 1$, corresponds to $n_1 = n_2 = n$. Then,

$$D(x = 1) = -8(1 - \cos \omega n L),$$

so that the eigenfrequencies become

$$\omega_N = \frac{2\pi N}{nL},$$

with $N = 1, 2, 3, ...$ After applying regularization (for instance cutoff), we obtain the Casimir energy

$$E_{\rm uniform} = 2 \times \frac{1}{2} \sum_{N=1}^{\infty} \omega_N,$$

where the factor 2 in front accounts for the degeneracy of the left-right running modes. The result is (Brevik - Nielsen 1990)

$$E_{\rm uniform} = -\frac{\pi}{6L}.$$

The case of arbitrary x

Can make use of the argument principle: Any meromorphic function $g(\omega)$ satisfies the equation

$$\frac{1}{2\pi i}\oint \omega \frac{d}{d\omega} \ln g(\omega) d\omega = \sum \omega_0 - \sum \omega_\infty,$$

where ω_0 are the zeros and ω_∞ are the poles of $g(\omega)$ inside the contour of integration. Contour is taken to be a semicircle of large radius R in the right half plane. A definite advantage of this method is that the multiplicities of zeros and poles are automatically included. (van Kampen *et al.*, 1968).

For the function $g(\omega)$ it is natural to start from the expression for D above, but normalized it in a convenient way. We first introduce a convenient parametrization which relates the pieces L_I and L_{II} to the total length L,

$$L_I = pL, \quad L_{II} = qL, \quad p + q = 1.$$

Then, define the quantity A as

$$A = \frac{1}{4}(1+x)^2 \cos[(n_1 p + n_2 q)\omega L] - \frac{1}{4}(1-x)^2 \cos[(n_1 p - n_2 q)\omega L].$$

For a uniform string, $x = 1$ ($n_1 = n_2 = n$), we have $A = \cos(n\omega L)$. We now define $g(\omega)$ as

$$g(\omega) = \left|\frac{1-A}{A}\right|.$$

With this form, the big semicircle does not contribute to the integration. Let $\omega = i\xi$, where ξ is the frequency along the imaginary axis. This Wick rotation implies that the quantity A goes into

$$A \to A(\xi) = \frac{1}{4}(1+x)^2 \cosh[(n_1 p + n_2 q)\xi L] - \frac{1}{4}(1-x)^2 \cosh[(n_1 p - n_2 q)\xi L].$$

By performing a partial integration along the imaginary axis, observing that the positive and negative frequencies contribute equally, we obtain for the Casimir energy

$$E = \frac{1}{2\pi}\int_0^\infty \ln\left|\frac{1-A(\xi)}{A(\xi)}\right| d\xi.$$

To simplify the formalism somewhat, choose $n_1 = 1$, the lowest possible value for n_1 in nondispersive theory. Then $n_2 = 1/x$. Also choose $L = 1$. This implies

$$A(\xi) = \frac{1}{4}(1+x)^2 \cosh\left[\left(p + \frac{q}{x}\right)\xi\right] - \frac{1}{4}(1-x)^2 \cosh\left[\left(p - \frac{q}{x}\right)\xi\right].$$

1. The case of a uniform string.
In this case $A(\xi)\big|_{x=1} = \cosh\xi$ for all p, and one gets

$$E_{\rm uniform} = \frac{1}{2\pi}\int_0^\infty \ln\left|\frac{1-\cosh\xi}{\cosh\xi}\right| d\xi = -\frac{3\pi}{16}.$$

2. The case when $p = 1/2$.
The string is then divided into two equal halves, the refractive index x being arbitrary. Now

$$A(\xi)\big|_{p=1/2} = \frac{1}{4}(1+x)^2\cosh\left[\left(1+\frac{1}{x}\right)\frac{\xi}{2}\right] - \frac{1}{4}(1-x)^2\cosh\left[\left(1-\frac{1}{x}\right)\frac{\xi}{2}\right],$$

and the Casimir energy is found by inserting this expression into the expression for E above.
3. The case when $p \to 0$.
This case is of interest since it corresponds to a "particle" sitting on a uniform string. One has now $A(\xi)\big|_{p\to 0} = x\cosh(\xi/x)$, and the Casimir energy becomes

$$E = \frac{1}{2\pi}\int_0^\infty \ln\left|\frac{1-x\cosh(\xi/x)}{x\cosh(\xi/x)}\right| d\xi.$$

Without loss of generality we can assume that x lies in the interval $0 < x < 1$. Figures 1 and 2 show how E varies with x in the respective two cases. Of special interest is when $x \to 0$ in case 3, as this corresponds to a particle of maximum refractive index contrast sitting on a uniform string.

Curvature-induced enhancement of the produced electric field at a dielectric boundary

Return to the axion field, and focus on a cylindrical haloscope. An enhancement of this field will occur near the centre of the cylinder. The enhancement is solely caused by the curvilinear geometry.
Assume a cylindrical vacuum region with radius R, surrounded by an exterior massive nonmagnetic environment. A strong uniform static magnetic field $\mathbf{B}_e$ is acting in the z direction. Assume only time-varying axions, together with the extra photons E^γ that they generate at the boundary. No propagation of electromagnetic modes are assumed to take place in the z direction.
Expansion, in ordinary electrodynamics,

$$E_z^{\rm ext} = H_0^{(1)}(k_2 r) a_{\rm ext} e^{-i\omega t}, \quad H_\theta^{\rm ext} = i n_2 H_0^{(1)'}(k_2 r) a_{\rm ext} e^{-i\omega t},$$

$$E_z^{\rm int} = J_0(\omega r) a_{\rm int} e^{-i\omega t}, \quad H_\theta^{\rm int} = i J_0'(\omega r) a_{\rm int} e^{-i\omega t}.$$

Azimuthal symmetry is assumed, so that only zeroth order $p = 0$ applies. Outgoing waves are assumed on the outside, and on the inside a stationary wave field is assumed, finite at the center.

In the present case: the antenna-like property of the boundary $r = R$ causes radiation to occur into the inward region also. That means, J_0 has to be replaced by the Hankel function $H_0^{(2)}$ of the second kind:

$$E_z^{\rm int} = H_0^{(2)}(\omega r) a_{\rm int} e^{-i\omega t}, \quad H_\theta^{\rm int} = iH_0^{(2)'}(\omega r) a_{\rm int} e^{-i\omega t}.$$

As before, $k_2 = n_2\omega$ with n_2 real. The coefficients $a_{\rm ext}$ and $a_{\rm int}$ are determined by the boundary conditions at $r = R$:

$$E_2^\gamma \to H_0^{(1)}(k_2 R) a_{\rm ext} e^{-i\omega t}, \quad E_1^\gamma \to H_0^{(2)}(\omega R) a_{\rm int} e^{-i\omega t}.$$

For simplicity, assume that the radius R is so large that

$$H_0^{(1)}(\rho) = \sqrt{\frac{2}{\pi\rho}}\, e^{i(\rho-\pi/4)}, \quad H_0^{(2)}(\rho) = \sqrt{\frac{2}{\pi\rho}}\, e^{-i(\rho-\pi/4)}, \quad \rho \gg 1.$$

As $H_0^{(1)'}(\rho) = iH_0^{(1)}(\rho)$, $H_0^{(2)'}(\rho) = -iH_0^{(2)}(\rho)$, we can write the boundary conditions as

$$H_0^{(1)}(k_2 R) a_{\rm ext} - \frac{E_0}{\varepsilon_2} = H_0^{(2)}(\omega R) a_{\rm int} - E_0,$$

$$n_2 H_0^{(1)}(k_2 R) a_{\rm ext} = -H_0^{(2)}(\omega R) a_{\rm int},$$

where $E_0 = \theta_0 B_e$, as before. The coefficients $a_{\rm ext}$ and $a_{\rm int}$ can then be found.

$$H_0^{(1)}(k_2 R) a_{\rm ext} = -\frac{E_0}{n_2}\left(1 - \frac{1}{n_2}\right),$$

$$H_0^{(2)}(\omega R) a_{\rm int} = E_0 \left(1 - \frac{1}{n_2}\right),$$

in agreement with plane geometry results.

Near the center of the cylinder: Logarithmic increase of $H_0^{(2)}(\rho)$. Thus curvature-induced enhancement of the generated electric field. To prevent instability, imagine a perfect cylindrical absorber of small radius $r = \delta$ centered at the z axis. Let the inward-generated field at $r = R$ be represented by $H_0^{(2)}(\omega R)$, and use the approximation

$$H_0^{(2)}(\rho) = J_0(\rho) - i N_0(\rho) = 1 + \frac{2i}{\pi} \ln \frac{2}{\gamma\rho}, \quad \rho \ll 1,$$

with $\gamma = 1.78107$.
Consider magnitudes only. Calculate the ratio

$$\Big|\frac{E_z^\gamma(\delta)}{E_z^\gamma(R)}\Big| = \Big|\frac{H_0^{(2)}(\omega\delta/c)}{H_0^{(2)}(\omega R/c)}\Big| = \Big|\frac{1 + \frac{2i}{\pi}\ln\frac{2c}{\gamma\omega\delta}}{H_0^{(2)}(\omega R/c)}\Big|.$$

Numerical estimates.

Take $R = 20$ cm, and $m_a c^2 = 10^{-4}$ eV for the axion energy, what corresponds to $\omega = 1.52 \times 10^{11}$ rad/s. Then $\omega R/c = 1.0 \times 10^2$, thus justifying use of the above approximation. One gets $|H_0^{(2)}(\omega R/c)| = 0.080$. With the minimum radius $\delta = 100\ \mu$m, corresponding to $\rho_{\rm min} = \omega\delta/c = 0.051$, it is seen that also the low-argument approximation (20) is justified. We obtain

$$\Big|\frac{E_z^\gamma(\delta)}{E_z^\gamma(R)}\Big| = 12.5 \times |1 + 2.0\, i| = 28.0.$$

There occurs thus a considerable enhancement of the amplitude near the cylinder center.
This enhancement is solely a geometrical focusing effect, being an extension of the theory worked out earlier for plane geometry. An observation of the increased signal near $r = 0$ may be of experimental interest. The idea may be looked upon as an alternative to the idea recently put forward by Liu et al. (PRL 2022), concerning the broadband solenoidal haloscope.

Collaboration with Amedeo Favitta and Masud Chaichian.

Electromagnetic energy-momentum tensor in a dielectric environment

As before, assume that ε and permeability μ are constants.
Start from the Poynting vector,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}.$$

In ordinary electrodynamics when a wave falls from vacuum normally onto a dielectric surface, the same expression for $\mathbf{S}$ has to hold in the interior also, as the field is unable to do work when passing a dielectric surface at rest.
Energy conservation equation

$$\nabla \cdot \mathbf{S} + \dot{W} = -\mathbf{E} \cdot \mathbf{J} - g_{a\gamma\gamma}(\mathbf{E} \cdot \mathbf{B})\dot{a},$$

where

$$W = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})$$

is the electromagnetic energy density. There is an exchange of electromagnetic energy with the axion "medium", if $\mathbf{E}$ and $\mathbf{B}$ are different from zero and $a(t)$ is time-varying, even if $\mathbf{J} = 0$.
Balance equation for electromagnetic momentum: What the momentum density $\mathbf{g}$?
Planck's principle of inertia of energy says that

$$\mathbf{g} = \mathbf{S}/c^2$$

(in physical units). That would mean the Abraham momentum density:

$$\mathbf{g}^{\mathrm{A}} = \mathbf{E} \times \mathbf{H}.$$

Consider the usual Maxwell stress tensor,

$$T_{ik} = E_i D_k + H_i B_k - \frac{1}{2}\delta_{ik}(\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}).$$

This expression is common for the Abraham and Minkowski alternatives, $T_{ik}^{\mathrm{A}} = T_{ik}^{\mathrm{M}} \equiv T_{ik}$.
The momentum conservation equation takes the form

$$\partial_k T_{ik} - \dot{g}_i^{\mathrm{A}} = f_i^{\mathrm{A}},$$

where f_i^{A} are the components of Abraham's force density

$$\mathbf{f}^{\mathrm{A}} = \rho\mathbf{E} + (\mathbf{J}\times\mathbf{B}) + (\varepsilon\mu - 1)\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{H}) - g_{a\gamma\gamma}(\mathbf{E}\cdot\mathbf{B})\nabla a.$$

This agrees with Landau and Lifshitz (1984), Chaichian *et al.* (2016), Møller (1972), Brevik (1979) and others (electrostriction omitted). Electrostriction can be omitted because it does not contribute to the total force on the axion cloud.

The third term on the right hand side, the Abraham term, has experimentally turned up only in a few experiments, mainly at low frequencies where the mechanical oscillations of a test body are directly detectable.
The Walker-Lahoz experiment from 1975 tested the oscillations of a high-permittivity disk acting as a torsional pendulum.
In optics, the Abraham force will fluctuate out. It is therefore mathematically simpler, and in accordance with all observational experience in optics, to include the Abraham momentum (physically, a mechanical accompanying momentum) in the effective field momentum. Therewith, the momentum density becomes simply the Minkowski momentum $\mathbf{g}^{\mathrm{M}}$, given by

$$\mathbf{g}^{\mathrm{M}} = \mathbf{D} \times \mathbf{B}.$$

Momentum conservation equation in the Minkowski case

$$\partial_k T_{ik} - \dot{g}_i^{\mathrm{M}} = f_i^{\mathrm{M}},$$

where

$$\mathbf{f}^{\mathrm{M}} = \rho\mathbf{E} + (\mathbf{J} \times \mathbf{B}) - g_{a\gamma\gamma}(\mathbf{E} \cdot \mathbf{B})\nabla a.$$

Relativistically covariant form for the energy-momentum balance: Minkowski's energy-momentum tensor

$$S_\mu^{\mathrm{M}\nu} = F_{\mu\alpha} H^{\nu\alpha} - \frac{1}{4} g_\mu^\nu F_{\alpha\beta} H^{\alpha\beta},$$

which has the same form in all inertial frames. Then the conservation equations for electromagnetic energy and momentum can be written as

$$-\partial_\nu S_\mu^{\mathrm{M}\nu} = f_\mu^{\mathrm{M}},$$

where $f_\mu^{\mathrm{M}} = (f_0, \mathbf{f}^{\mathrm{M}})$ is the four-force density. In the rest system,

$$f_0 = \mathbf{E} \cdot \mathbf{J} + g_{a\gamma\gamma}(\mathbf{E} \cdot \mathbf{B})\dot{a},$$

where $f_0^{\mathrm{M}} = f_0^{\mathrm{A}} \equiv f_0$.

Application 1: Space-dependent axions

Assume $\rho = \mathbf{J} = 0$, and $\dot{a} = 0$ so that $a = a(\mathbf{r})$ is a function of position only.
Of possible interest in the early universe: the Peccei-Quinn scalar field may vary in space (typically as a tanh function of the spatial coordinate).
In galactic haloes the axions may behave differently from what they do in the open space.
Generalized Maxwell equations reduce to

$$\nabla \cdot \mathbf{D} = -g_{a\gamma\gamma} \mathbf{B} \cdot \nabla a,$$

$$\nabla \times \mathbf{H} = g_{a\gamma\gamma} \nabla a \times \mathbf{E},$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = 0.$$

It is natural to define

$$\rho_a = -g_{a\gamma\gamma} \mathbf{B} \cdot \nabla a,$$

$$\mathbf{J}_a = g_{a\gamma\gamma} \nabla a \times \mathbf{E}.$$

Assume there is one homogeneous material plate, of infinite extent in the horizontal x and y directions, extending in the vertical direction from $z = 0$ to $z = L$. Assume that a strong magnetic field B_0 is applied in the z direction, and a strong electric field E_0 applied in the x direction,

$$\mathbf{B}_0 = B_0\hat{\mathbf{z}}, \quad \mathbf{E}_0 = E_0\hat{\mathbf{x}}.$$

Axion-generated charges and currents are expected to be small:

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_a,\ \mathbf{E} = \mathbf{E}_0 + \mathbf{E}_a,$$

Thus neglect second order terms $\mathbf{B}_a \cdot \nabla a$ and $\nabla a \times \mathbf{E}_a$.

$$\rho_a = -g_{a\gamma\gamma}\nabla \cdot (a\mathbf{B}_0),$$

$$\mathbf{J}_a = -g_{a\gamma\gamma}\nabla \times (a\mathbf{E}_0).$$

Assume that the axion field has a constant gradient in the z direction inside the plate,

$$a(z) = \alpha z, \quad 0 < z < L,$$

where $\alpha > 0$ is constant, not necessarily small. Express α as $\alpha = a_0/L$, where a_0 is the maximum value at $z = L$. In the outside regions, $a(z) = 0$.

Integration across the boundary $z = L$ yields the surface charge density,

$$\sigma_a = g_{a\gamma\gamma} a_0 B_0,$$

whereas in the interior region the volume charge density is

$$\rho_a = -g_{a\gamma\gamma} \alpha B_0.$$

Thus, the total charge $\rho_a L$ in the interior region per unit surface balances the surface charge density,

$$\rho_a L + \sigma_a = 0.$$

After the imposition of the strong external fields B_0 and E_0, the plate remains electrically neutral.
Correspondingly, for the current densities $\mathbf{J}_a$ one obtains first a surface current density,

$$\mathbf{K}_a = -g_{a\gamma\gamma} a_0 E_0 \hat{\mathbf{y}},$$

while in the interior

$$\mathbf{J}_a = g_{a\gamma\gamma} \alpha E_0 \hat{\mathbf{y}}.$$

Thus, the total current in the interior per unit surface, $\mathbf{J}_a L$, balances the contribution from the surface,

$$\mathbf{J}_a L + \mathbf{K}_a = 0.$$

There is no net current in the y direction. Note that ε and μ do not appear.

Consider next the axionic magnetic and electric fields. From Maxwell's equations

$$\mathbf{H}_a = g_{a\gamma\gamma}\alpha z\mathbf{E}_0,$$

$$\mathbf{D}_a = -g_{a\gamma\gamma}\alpha z\mathbf{B}_0.$$

The induced magnetic and electric fields are thus respectively horizontal and vertical, increasing linearly in the z direction. At the bottom of the plate, $z = 0$, the induced fields vanish.
As $\mathbf{B}_0$ and $\mathbf{E}_0$ are orthogonal, it follows that

$$\mathbf{f}^{\mathrm{A}} = \mathbf{f}^{\mathrm{M}} = 0.$$

What is the force $\mathbf{F}_{\mathrm{surface}}$ on the layer $z = L$? That force is zero:

$$\mathbf{F}_{\mathrm{surface}} = \sigma_a\mathbf{E}_0 + \mathbf{K}_a \times \mathbf{B}_0 = 0.$$

There is thus in this case no electromagnetic force on the plate due to the interaction with axions.
The fourth component

$$f_0 = 0.$$

Application 2: Time-dependent axions

Assume that the axion field depends only on time, $a = a(t)$. Assume hereafter $\varepsilon = \mu = 1$. Maxwell's equations become

$$\nabla \cdot \mathbf{E} = 0,$$

$$\nabla \times \mathbf{H} = \dot{\mathbf{E}} + g_{a\gamma\gamma}\dot{a}\mathbf{H},$$

$$\nabla \cdot \mathbf{H} = 0,$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{H}},$$

while the field equations reduce to

$$\nabla^2\mathbf{E} - \ddot{\mathbf{E}} = g_{a\gamma\gamma}\dot{a}\dot{\mathbf{B}},$$

$$\nabla^2\mathbf{H} - \ddot{\mathbf{H}} = -g_{a\gamma\gamma}\dot{a}\nabla \times \mathbf{H}.$$

The axion-generated field $\mathbf{E}_a$ has been split off.

Assume now that in the outer space

$$a(t) = a_0 \sin \omega_a t.$$

Put $\omega_a = m_a$, and take $m_a = 10$ μeV. In physical units, $\omega_a = 1.52 \times 10^{10}$ rad/s. which corresponds to an oscillation wavelength of $\lambda_a = 2\pi/\omega_a = 12.4$ cm. Associate axions with dark matter, whose energy density is estimated to be $\rho_{\rm DM} = 0.35$ GeV/cm^3.

$$\rho_a = \frac{1}{2} m_a^2 a_0^2.$$

Take the initial shape of an electromagnetic wave emitted from the Earth to be

$$\mathbf{A}_0(x, t) = c\, e^{-x^2/2D^2} \cos k_0 x\, \hat{\mathbf{y}},$$

where $\mathbf{A}_0$ is the vector potential, c is the wave amplitude. With

$$\mathbf{A}_0(x.t) = \frac{1}{2} \int_{-\infty}^{\infty} \left[\mathbf{A}_0(k) e^{i(kx - \omega t)} + \mathbf{A}_0^*(k) e^{-i(kx-\omega t)} \right] dk,$$

one obtains by inversion

$$\mathbf{A}_0(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \left[\mathbf{A}_0(x, 0) + \frac{i}{\omega} \frac{\partial \mathbf{A}_0}{\partial t}(x, 0) \right] dx.$$

With the assumption

$$\frac{\partial \mathbf{A}_0}{\partial t}(x,0) = 0$$

one then gets

$$\mathbf{A}_0(k) = \mathbf{A}_0^+(k) + \mathbf{A}_0^-(k),$$

where

$$\mathbf{A}_0^+(k) = \frac{cD}{2\sqrt{2\pi}} \exp\left[-\frac{1}{2}D^2(k-k_0)^2\right] \hat{\mathbf{y}}.$$

Thus

$$\mathbf{A}_0^+(x,t) = \int_0^\infty \mathbf{A}_0^+(k)\cos(kx - \omega t)dk, \quad \omega = k > 0.$$

Omit the superscript plus. Incident wave components are

$$\mathbf{E}_0(x,t) = -\dot{\mathbf{A}}_0(x,t), \quad \mathbf{H}_0(x,t) = \nabla \times \mathbf{A}_0(x,t).$$

Force from the incident wave on the axions: $\mathbf{f} = 0$. Related to the axion cloud assumed homogeneous.

The dissipation component f_0 is also zero, since the incident fields $\mathbf{E}_0$ and $\mathbf{H}_0$ are orthogonal.

In order to calculate the interacting fields, go back to the field equations (Sikivie 2021). For the magnetic potential,

$$\nabla^2 \mathbf{A} - \ddot{\mathbf{A}} = -g_{a\gamma\gamma}\dot{a}\nabla \times \mathbf{A},$$

in which we let $\mathbf{A} \to \mathbf{A}_0$ on the right hand side.
Neglect $\nabla^2\mathbf{A}$ on the left hand side,

$$\ddot{\mathbf{A}}(x,t) = -g_{a\gamma\gamma}a_0\omega_a \int_0^\infty A_0(k)k\sin(kx-\omega t)\cos\omega_a t\, dk\, \hat{\mathbf{z}}.$$

Write the trigonometric product as a sum of two terms, and extract the term that leads to resonance. In complex representation,

$$\ddot{\mathbf{A}}(x,t) = \frac{1}{2}g_{a\gamma\gamma}a_0\omega_a \mathrm{Im}\int_0^\infty A_0(k)ke^{i(kx-\omega t+\omega_a t)}\,\hat{\mathbf{z}}.$$

Defining $\mathbf{A}(k,t)$ via

$$\mathbf{A}(x,t) = \mathrm{Im}\int_0^\infty e^{ikx}\mathbf{A}(k,t)dk,$$

one can write

$$\ddot{\mathbf{A}}(k,t) = -\frac{1}{2}g_{a\gamma\gamma}a_0\omega_a A_0(k)ke^{-i(\omega-\omega_a)t}\,\hat{\mathbf{z}}.$$

Defining

$$\mathcal{A}(k,t) = \mathbf{A}(k,t)e^{-i\omega t},$$

one gets

$$\ddot{\mathbf{A}}(k,t) = \ddot{\mathcal{A}}(k,t)e^{i\omega t} + 2i\omega\dot{\mathcal{A}}(k,t)e^{i\omega t} - \omega^2\mathcal{A}(k,t)e^{i\omega t},$$

Keep only the resonance producing term containing $\dot{\mathcal{A}}(k,t)$:

$$\dot{\mathcal{A}}(k,t) = \frac{i}{4}g_{a\gamma\gamma}a_0\omega_a A_0(k)e^{-i(2\omega-\omega_a)t}\,\hat{\mathbf{z}}.$$

After integration with respect to t:

$$\mathcal{A}(k,t) = -g_{a\gamma\gamma}a_0\omega_a\frac{cD}{8\sqrt{2\pi}}\exp\left[-\frac{1}{2}D^2(k-k_0)^2\right]\frac{e^{-i(2\omega-\omega_a)t}}{2\omega-\omega_a}\,\hat{\mathbf{z}}.$$

Resonance occurs when $\omega = \omega/2$. Imaginary part:

$$\lim\frac{\sin\alpha x}{\pi x}\Big|_{\alpha\to\infty} = \delta(x),$$

Result:

$$\mathrm{Im}\mathcal{A}(k,t) = g_{a\gamma\gamma} a_0 \omega_a \frac{cD}{16}\sqrt{\frac{\pi}{2}} \exp\left[-\frac{1}{2}D^2(k-k_0)^2\right] \delta(\omega - \frac{1}{2}\omega_a)\,\hat{\mathbf{z}}.$$

From this interaction term at resonance, one can calculate the axion echo. The Gaussian profile in the emitted wave from the Earth influences the strength of the effect. To receive a maximum echo, the center frequency $k_0 = \omega_0$ in the pulse should be chosen equal to the resonance value of k, which is $\omega_a/2$.

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