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Split Octonions and Triality in (4+4)-Space

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Outline

- Review, aims and objectives
- Overview of mathematical framework
  - Hypercomplex numbers $\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O} \to \cdots$
  - Clifford algebras $\mathcal{C}\ell_{p,q}$
  - Vectors and spinors in space with (p, q) signature
- Orthogonal group reps $SO\,(4,4)$, $Spin\,(4,4)$
- Split octonions and triality
- Summary and outlook

Review, aims and objectives

- Octonions
  - Color symmetry[Gunaydin, Gursey 1973; Morita 1981], GUT[Sudbery 1984, Dixon 1990; Castro 2007], Quantum mechanics[Gunaydin, Piron, Ruegg 1978], associator quantization[Lohmus, Paal, Sorgsepp 1998], M theory[Lukierski, Toppan 2002].
- Split octonions
  - Particle generations[Gunaydin, Gursey 1974, Silagadze 1995], electrodynamics[Nash 1989], gravity[Nash 2010], geometry[Gogberashvili 2009, 2015].
- Aims
  - Split octonions as universal object for physics to replace vectors, spinors, matrices etc.
  - Split octonions as a possible way of uniting internal and extra dim geometric symmetries.
- Objectives
  - Equivalence of vector and chiral spinors in (4+4) space.
  - Triality symmetry in (4+4) space with split octonions.

Cayley-Dickson constructions and Hurwitz algebras

- Given an algebra $(\mathbb{A}_n, +, \times)$ we can define $(\mathbb{A}_{n+1}, +, \times)$ for $\mathbb{A}_{n+1} = \mathbb{A}_n^2$ where

$$
\begin{aligned}
(a, b) + (c, d) &= (a + c, b + d) \;, && \text{(with } a, b, c, d \in \mathbb{A}_n) \\
(a, b) \times (c, d) &= (ac - \gamma d^* b, da + bc^*) \;, && \text{(with } \gamma = \pm 1) \\
(a, b)^* &= (a^*, -b) \;.
\end{aligned}
$$

- For example starting from $\mathbb{A}_0 = \mathbb{R}$ and $\gamma = 1$ we get $\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O} \to \mathbb{S} \to \cdots$.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$	$\mathbb{S}$	$\cdots$	$\mathbb{A}_n$
dimension:	1	2	4	8	16	$\cdots$	2^n
order: $a < b$	✓	×	×	×	×	$\cdots$	×
commutative: $ab = ba$	✓	✓	×	×	×	$\cdots$	×
associative: $a\,(bc) = (ab)\,c$	✓	✓	✓	×	×	$\cdots$	×
division: $\forall\,(a \neq 0)\,\exists a^{-1}$ such that $aa^{-1} = 1$	✓	✓	✓	✓	×	$\cdots$	×

- Only first 3 have matrix representations, e.g. $i = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

- Motivating example: $(xe_1 + ye_2)^2 = x^2 e_1^2 + y^2 e_2^2 + xy(e_1 e_2 + e_2 e_1) = x^2 + y^2$.
- Defining relations: $e_m e_n + e_n e_m = 2g_{mn} = 2\mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{p}, \underbrace{-1, -1, \ldots, -1}_{q})$.
- Familiar examples: Pauli σ matrices - $\mathcal{C}\ell_{3,0}$, Dirac γ matrices $\mathcal{C}_{1,3}$.
- In $\mathcal{C}\ell_{3,0}$ we correspond $e_1 \leftrightarrow \sigma_1, \quad e_2 \leftrightarrow \sigma_2, \quad e_3 \leftrightarrow \sigma_3$. Then $\mathcal{C}\ell_{3,0}$ Multivector is:

$$x = \underbrace{x_0}_{\text{scalar}} + \underbrace{x_1 e_1 + x_2 e_2 + x_3 e_3}_{\text{vector}} + \underbrace{x_{23} e_{23} + x_{31} e_{31} + x_{12} e_{12}}_{\text{bivector}} + \underbrace{x_{123} e_{123}}_{\text{volume}}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$x_0 = \begin{pmatrix} x_0 & 0 \\ 0 & x_0 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} \quad \tilde{\mathbf{x}} = \ldots$$

- Reflection: $\mathbf{x}' = \mathbf{axa}^{-1}$, rotation $\mathbf{x}' = \exp\left(-\frac{1}{2}\tilde{\mathbf{a}}\right)\mathbf{x}\exp\left(\frac{1}{2}\tilde{\mathbf{a}}\right)$ which is a $SO(3)$ rep.

Matrix rep of $\mathcal{C}\ell_{p,q}$

- Notation: ${}^m\mathbb{A}\,(N)$ block diagonal matrix with m blocks of Mat$(N, \mathbb{A})$

$\mathcal{C}\ell_{p,q} \simeq$

$q\backslash p$	0	1	2	3	4	5	6	7
0	$\mathbb{R}$	${}^2\mathbb{R}$	$\mathbb{R}\,(2)$	$\mathbb{C}\,(2)$	$\mathbb{H}\,(2)$	${}^2\mathbb{H}\,(2)$	$\mathbb{H}\,(4)$	$\mathbb{C}\,(8)$
1	$\mathbb{C}$	$\mathbb{R}\,(2)$	${}^2\mathbb{R}\,(2)$	$\mathbb{R}\,(4)$	$\mathbb{C}\,(4)$	$\mathbb{H}\,(4)$	${}^2\mathbb{H}\,(4)$	
2	$\mathbb{H}$	$\mathbb{C}\,(2)$	$\mathbb{R}\,(4)$	${}^2\mathbb{R}\,(4)$	$\mathbb{R}\,(8)$	$\mathbb{C}\,(8)$		
3	${}^2\mathbb{H}$	$\mathbb{H}\,(2)$	$\mathbb{C}\,(4)$	$\mathbb{R}\,(8)$	${}^2\mathbb{R}\,(8)$			
4	$\mathbb{H}\,(2)$	${}^2\mathbb{H}\,(2)$	$\mathbb{H}\,(4)$	$\mathbb{C}\,(8)$				
5	$\mathbb{C}\,(4)$	$\mathbb{H}\,(4)$	${}^2\mathbb{H}\,(4)$					
6	$\mathbb{R}\,(8)$	$\mathbb{C}\,(8)$						
7	${}^2\mathbb{R}\,(8)$							

- The rest can be recovered using rules $\mathcal{C}\ell_{p,q} \simeq \mathcal{C}\ell_{p-4,q+4}$ and $\mathcal{C}\ell_{p+8,q} = \text{Mat}\,(16, \mathcal{C}\ell_{p,q})$

$Spin\,(N)$ group example

- Spinor space is a minimal left ideal.
- Left ideal: subspace which is closed under left action.
- Procedure $\mathcal{C}\ell_{3,0} \simeq \mathrm{Mat}\,(2,\mathbb{C})$: find an idempotent $f^2 = f$ e.g. $f = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$. and act on f with $\mathrm{Mat}\,(2,\mathbb{C})$ from the left

$$\psi = \mathrm{Mat}\,(2,\mathbb{C})\, f = \begin{pmatrix} \psi_1 + i\psi_2 & 0 \\ \psi_3 + i\psi_4 & 0 \end{pmatrix}, \qquad \text{(where } \psi_1, \psi_2, \psi_3, \psi_4 \in \mathbb{R}\text{)}$$

- So spinor space is spanned by $\frac{1}{2}(1+\sigma_3)$, $\frac{i}{2}(1+\sigma_3)$, $\frac{1}{2}(\sigma_1 - i\sigma_2)$, and $\frac{i}{2}(\sigma_1 - i\sigma_2)$
- $Spin\,(3)$ is represented by $\psi' = \exp\left(-\frac{1}{2}\tilde{\mathbf{a}}\right)\psi$

Dimension of spinors and vectors in different spaces

- Number of free components in the underlying field ($\mathbb{R}$ in this case for $\mathcal{C}\ell_{p,0} = \mathcal{C}\ell_{p,0}(\mathbb{R})$)

Vector:	0	1	2	3	4	5	6	7	8	...	16	...
Spinor:	1	1	2	4	8	8	8	16	16	...	256	...
		$\uparrow$	$\uparrow$		$\uparrow$				$\uparrow$		$\not\uparrow$	

- In higher dimensions

Using the general formula: $\mathcal{C}\ell_{p+8,q} = \text{Mat}\,(16, \mathcal{C}\ell_{p,q})$

$$\mathcal{C}\ell_{8,0} = \text{Mat}\,(16, \mathcal{C}\ell_{0,0}) = \text{Mat}\,(16, \mathbb{R})$$

and $\mathcal{C}\ell_{16,0} = \text{Mat}\,(16, \mathcal{C}\ell_{8,0}) = \text{Mat}\,(16, \text{Mat}\,(16, \mathbb{R})) = \text{Mat}\,(256, \mathbb{R})$

Vectors of (4+4) space and $SO(4,4)$

- If $\mathcal{C}\ell_{8,0}$ is generated by Γ_μ $(\mu = 0, 1, \ldots, 7)$ matrices then we can obtain $\mathcal{C}\ell_{4,4}$ generators as $\gamma_\mu = \Gamma_\mu$ (for $\mu = 0, 1, 2, 3$) and $\gamma_\mu = i\Gamma_\mu$ (for $\mu = 4, 5, 6, 7$)
- Real $\mathbb{R}^8$ vectors in (4+4) space: $\mathcal{X} = \sum_{\beta=0}^{7} x_\beta \gamma_\beta$ (where $x_0, x_1, \ldots \in \mathbb{R}$)
- Quadratic form $\mathbb{R}^8 \mapsto \mathbb{R}$ is: $\mathcal{X}^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2$
- $SO(4,4)$ under which$\mathcal{X}^2$ is invariant is represented as

$$\mathcal{X}' = L_{\mu\nu}(\vartheta)\,\mathcal{X}\,L_{\mu\nu}^{-1}(\vartheta) \qquad \text{where} \qquad L_{\mu\nu}(\vartheta) = \exp\left(-\frac{1}{2}\vartheta\gamma_\mu\gamma_\nu\right)$$

- Tangential space of $SO(4,4)$ is spanned by $L_{\mu\nu}(\vartheta) \simeq 1 - \frac{1}{2}\vartheta\gamma_\mu\gamma_\nu$
- Element $B = -\gamma_1\gamma_3\gamma_5\gamma_7$ is special because of the property $\mathcal{X}^T = B\mathcal{X}B$.

Spinors of (4+4) space and $Spin\,(4,4)$

- Since $\mathcal{C}\ell_{4,4} \simeq \mathrm{Mat}\,(16,\mathbb{R})$ spinor $\eta = \phi + \psi$ is 16 dimensional where chiral spinors are $\phi = (\phi_0, \phi_1, \ldots, \phi_7, 0, 0, \ldots, 0)^T$ and $\psi = (0, 0, \ldots, 0, \psi_0, \psi_1, \ldots, \psi_7)^T$
- Vectors and chiral spinors are of the same dim $\mathcal{X}, \phi, \psi \in \mathbb{R}^8$, this only occurs in 4D and 8D spaces.
- $Spin\,(4,4)$ is represented as $\eta' = L_{\mu\nu}\,(\vartheta)\,\eta$ invariant under which is $\eta^T B \eta = \phi^T B \phi + \psi^T B \psi$.

Change of spinor basis to reveal triality

- Spinors after basis change are $\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} -\phi_2+i\phi_3 \\ \phi_0-i\phi_1 \\ -\phi_7-i\phi_6 \\ -\phi_5+i\phi_4 \\ -\phi_5-i\phi_4 \\ \phi_7-i\phi_6 \\ -\phi_0-i\phi_1 \\ -\phi_2-i\phi_3 \end{pmatrix}$, $\quad \psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_2-i\psi_3 \\ -\psi_0-i\psi_1 \\ -\psi_7-i\psi_6 \\ -\psi_5+i\psi_4 \\ \psi_5+i\psi_4 \\ -\psi_7+i\psi_6 \\ -\psi_0+i\psi_1 \\ -\psi_2-i\psi_3 \end{pmatrix}$.
- In this basis they have the property that

$$\phi^T B\phi = \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2 - \phi_4^2 - \phi_5^2 - \phi_6^2 - \phi_7^2$$

$$\psi^T B\psi = \psi_0^2 + \psi_1^2 + \psi_2^2 + \psi_3^2 - \psi_4^2 - \psi_5^2 - \psi_6^2 - \psi_7^2$$

which mirrors

$$\mathcal{X}^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2$$

- Trilinear form $\mathcal{F} : \mathbb{R}^8 \times \mathbb{R}^8 \times \mathbb{R}^8 \mapsto \mathbb{R}$ exists, it is $\mathcal{F}(\phi, \mathcal{X}, \psi) = \phi^T B\mathcal{X}\psi$.
- $\mathcal{F}$ is invariant under $L_{\mu\nu}(\vartheta)$ i.e. $\mathcal{F}(\phi', \mathcal{X}', \psi') = \mathcal{F}(\phi, \mathcal{X}, \psi)$.

Closer look at orthogonal transformations

- ϕ, $\mathcal{X}$ and ψ transform under $L_{01}(\vartheta)$ as

$$\begin{cases} x_0' = x_0 - \vartheta x_1 \\ x_1' = x_1 + \vartheta x_0 \\ x_2' = x_2 \\ x_3' = x_3 \\ x_4' = x_4 \\ x_5' = x_5 \\ x_6' = x_6 \\ x_7' = x_7 \end{cases} \qquad \begin{cases} \phi_0' = \phi_0 + \frac{1}{2}\vartheta\phi_1 \\ \phi_1' = \phi_1 - \frac{1}{2}\vartheta\phi_0 \\ \phi_2' = \phi_2 - \frac{1}{2}\vartheta\phi_3 \\ \phi_3' = \phi_3 + \frac{1}{2}\vartheta\phi_2 \\ \phi_4' = \phi_4 - \frac{1}{2}\vartheta\phi_5 \\ \phi_5' = \phi_5 + \frac{1}{2}\vartheta\phi_4 \\ \phi_6' = \phi_6 + \frac{1}{2}\vartheta\phi_7 \\ \phi_7' = \phi_7 - \frac{1}{2}\vartheta\phi_6 \end{cases} \qquad \begin{cases} \psi_0' = \psi_0 + \frac{1}{2}\vartheta\psi_1 \\ \psi_1' = \psi_1 - \frac{1}{2}\vartheta\psi_0 \\ \psi_2' = \psi_2 + \frac{1}{2}\vartheta\psi_3 \\ \psi_3' = \psi_3 - \frac{1}{2}\vartheta\psi_2 \\ \psi_4' = \psi_4 + \frac{1}{2}\vartheta\psi_5 \\ \psi_5' = \psi_5 - \frac{1}{2}\vartheta\psi_4 \\ \psi_6' = \psi_6 - \frac{1}{2}\vartheta\psi_7 \\ \psi_7' = \psi_7 + \frac{1}{2}\vartheta\psi_6 \end{cases}$$

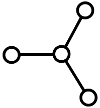
- We can construct such transformation for $\mathcal{X}$ such that it mimics ϕ
$L_{10}\left(\frac{\vartheta}{2}\right) L_{23}\left(\frac{\vartheta}{2}\right) L_{54}\left(\frac{\vartheta}{2}\right) L_{67}\left(\frac{\vartheta}{2}\right) \simeq 1 - \frac{1}{4}\vartheta\left(\gamma_1\gamma_0 + \gamma_2\gamma_3 + \gamma_5\gamma_4 + \gamma_6\gamma_7\right)$

Thriality symmetry

- The three objects $\mathcal{X}, \phi$ and ψ swapped places

$$\begin{cases} x'_0 = x_0 + \frac{1}{2}\vartheta x_1 \\ x'_1 = x_1 - \frac{1}{2}\vartheta x_0 \\ x'_2 = x_2 - \frac{1}{2}\vartheta x_3 \\ x'_3 = x_3 + \frac{1}{2}\vartheta x_2 \\ x'_4 = x_4 - \frac{1}{2}\vartheta x_5 \\ x'_5 = x_5 + \frac{1}{2}\vartheta x_4 \\ x'_6 = x_6 + \frac{1}{2}\vartheta x_7 \\ x'_7 = x_7 - \frac{1}{2}\vartheta x_6 \end{cases}, \quad \begin{cases} \phi'_0 = \phi_0 + \frac{1}{2}\vartheta \phi_1 \\ \phi'_1 = \phi_1 - \frac{1}{2}\vartheta \phi_0 \\ \phi'_2 = \phi_2 + \frac{1}{2}\vartheta \phi_3 \\ \phi'_3 = \phi_3 - \frac{1}{2}\vartheta \phi_2 \\ \phi'_4 = \phi_4 + \frac{1}{2}\vartheta \phi_5 \\ \phi'_5 = \phi_5 - \frac{1}{2}\vartheta \phi_4 \\ \phi'_6 = \phi_6 - \frac{1}{2}\vartheta \phi_7 \\ \phi'_7 = \phi_7 + \frac{1}{2}\vartheta \phi_6 \end{cases}, \quad \begin{cases} \psi'_0 = \psi_0 - \vartheta \psi_1 \\ \psi'_1 = \psi_1 + \vartheta \psi_0 \\ \psi'_2 = \psi_2 \\ \psi'_3 = \psi_3 \\ \psi'_4 = \psi_4 \\ \psi'_5 = \psi_5 \\ \psi'_6 = \psi_6 \\ \psi'_7 = \psi_7 \end{cases}.$$

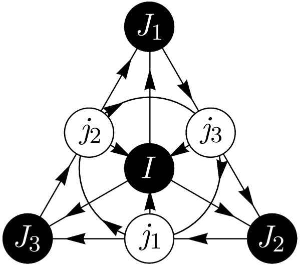
- Dynkin diagram of $SO(8)$ and $SO(4,4)$



Split octonions

- Algebraic relations and the Fano plane

$$
\begin{aligned}
&j_m j_n = -\delta_{mn} + \sum_k \epsilon_{mnk} j_k \ , \\
&I^2 = 1 \ , \\
&J_m J_n = \delta_{mn} + \sum_k \epsilon_{mnk} j_k \ , \qquad (m, n, k = 1, 2, 3) \\
&j_n I = J_n \ , \\
&J_m j_n = \delta_{mn} I - \sum_k \epsilon_{mnk} J_k \ .
\end{aligned}
$$



- For $\omega_0, \omega_1, \ldots \in \mathbb{R}$, real split octonion Ω is given by
  $\Omega = \omega_0 + \omega_1 j_1 + \omega_2 j_2 + \omega_3 j_3 + \omega_4 I + \omega_5 J_1 + \omega_6 J_2 + \omega_7 J_3$
- Quadratic form $\bar{\Omega}\Omega = \omega_0^2 + \omega_1^2 + \omega_2^2 + \omega_3^2 - \omega_4^2 - \omega_5^2 - \omega_6^2 - \omega_7^2$
- Dot product: $\cdot : \mathbb{R}^8 \times \mathbb{R}^8 \mapsto \mathbb{R}$ is represented as $\Omega_1 \cdot \Omega_2 = \frac{1}{2}\left(\bar{\Omega}_1\Omega_2 + \bar{\Omega}_2\Omega_1\right)$

Vector, chiral spinors and triality in split octonions

- Split octonion representation of (4+4) vector and spinors:

$$\mathrm{X} = x_0 + x_1 j_1 + x_2 j_2 + x_3 j_3 + x_4 I + x_5 J_1 + x_6 J_2 + x_7 J_3 \ ,$$

$$\Phi = \phi_0 + \phi_1 j_1 + \phi_2 j_2 + \phi_3 j_3 + \phi_4 I + \phi_5 J_1 + \phi_6 J_2 + \phi_7 J_3 \ ,$$

$$\Psi = \psi_0 + \psi_1 j_1 + \psi_2 j_2 + \psi_3 j_3 + \psi_4 I + \psi_5 J_1 + \psi_6 J_2 + \psi_7 J_3 \ .$$

- Unlike in matrix representation expressions for quadratic forms are identical:

$$\bar{\mathrm{X}}\mathrm{X} = \mathcal{X}^2 \ ,$$

$$\bar{\Phi}\Phi = \phi^T B \phi \ ,$$

$$\bar{\Psi}\Psi = \psi^T B \psi \ .$$

- Trilinear form with split octonions:

$$\mathcal{F}(\Phi, \mathrm{X}, \Psi) = -\bar{\Phi} \cdot (\mathrm{X}\Psi)$$

Summary and outlook

- Established so far:
  - Vector and chiral spinors are equivalent in (4+4) space.
  - There exists trilinear form in (4+4) space which is invariant under orthogonal transformation on vector and spinors.
  - We can parameterize vector and spinors with a single split octonion each.
  - Split octonion number system can be used to express trilinear form.
- Future work:
  - Representing $SO(4,4)$ with split octonions.
  - Applying split octonionic analyticity condition to trilinear form.
  - Getting split octonionic field equations.

References: *THANKS FOR YOUR ATTENTION!*

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