$\mathcal{N} = 2$ Calogero models within superfields

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New $\mathcal{N}=2$ superspace Calogero models, JHEP **05** *(2020) 132, arXiv:1912.05989*

Plan

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I. Motivation

Recent progress in the construction of supersymmetric extensions of Calogero models was achieved by adding to the system more fermions as compared to the standard supersymmetrization (see, e.g., for a review)

- A. Polychronakos, *Physics and mathematics of Calogero particles,* J. Phys. A **39** (2006) 12793, `arXiv:hep-th/0607033`.

It was inspired by supersymmetric extensions of the matrix models which, upon reduction or gauge fixing, give rise to the familiar bosonic systems. The approach developed in

- S. Fedoruk, E. Ivanov, O. Lechtenfeld, *Supersymmetric Calogero models by gauging,* Phys. Rev. D **79** (2009) 105015, `arXiv:0812.4276[hep-th]`.
- S. Fedoruk, E. Ivanov, *Gauged spinning models with deformed supersymmetry,* JHEP **1611** (2016) 103, `arXiv:1610.04202[hep-th]`.
- S. Fedoruk, E. Ivanov, O. Lechtenfeld, S. Sidorov, *Quantum* $SU(2|1)$ *supersymmetric Calogero–Moser spinning systems,* JHEP **1804** (2018) 043, `arXiv:1801.00206[hep-th]`.

for the rational spin-Calogero models with $\mathcal{N}=2,4$ supersymmetry was recently extended to $\mathcal{N}=2,4$ supersymmetric hyperbolic Calogero models

- S. Fedoruk, E. Ivanov, O. Lechtenfeld, *Supersymmetric hyperbolic Calogero–Sutherland models by gauging,* Nucl. Phys. B **944** (2019) 114633, `arXiv:1902.08023[hep-th]`.
- S. Fedoruk, $\mathcal{N}=2$ *supersymmetric hyperbolic Calogero-Sutherland model*, `arXiv:1910.07348 [hep-th]`.
- S. Fedoruk, $\mathcal{N}=4$supersymmetric U(2)-spin hyperbolic Calogero-Sutherland model `arXiv:arXiv:2007.11424 [hep-th]`.

In the series of papers

- S. Krivonos, O. Lechtenfeld, A. Sutulin, *$\mathcal{N}$-extended supersymmetric Calogero model*,
  Phys. Lett. B **784** (2018) 137, `arXiv:1804.10825[hep-th]`
- S. Krivonos, O. Lechtenfeld, A. Provorov, A. Sutulin, *Extended supersymmetric Calogero model*,
  Phys. Lett. B **791** (2019) 385, `arXiv:1812.10168[hep-th]`
- S. Krivonos, O. Lechtenfeld, *$\mathcal{N}=4$ supersymmetric Calogero-Sutherland models*,
  Phys. Rev. D **101**, (2020) 086010, `arXiv:2002.03929[hep-th]`

we developed a different approach. Mainly working in the Hamiltonian formulation, we worked out an ansatz for the supercharges which accommodates all Calogero models associated with the classical A_n, B_n, C_n and D_n Lie algebras and their trigonometric/hyperbolic extensions.

Having at hands the Hamiltonian description of $\mathcal{N}$-extended supersymmetric Calogero models, it is of interest to gain also its superfield formulation, at least in the simplest case of $\mathcal{N}=2$ supersymmetry. Note that a superspace description of the $\mathcal{N}$-extended supersymmetric Euler-Calogero-Moser system has been provided in

- S. Krivonos, O. Lechtenfeld, A. Sutulin,
  Supersymmetric many-body Euler-Calogero-Moser model, Phys. Lett. B **790** (2019) 191.

The superspace picture will help in understanding the general supersymmetry structure and clarify the role played by the additional matrix fermions. This is the main goal of this work.

In the beginning, Section 2, we review the Hamiltonian description of the supersymmetric Calogero models.

Then their superfield treatment will be performed in Section 3 ($A_1 \oplus A_{n-1}$ models) and in Section 4 (B_n, C_n, D_n models).

The most remarkable result here is a universal nonlinear fermionic chiral supermultiplet which collects all matrix fermions occurring in all super-extended Calogero models.

In Section 5 we present more general supercharges (and a superspace Lagrangian), which provide an $\mathcal{N}=2$ supersymmetrization for a bosonic potential $\frac{g^2}{2}\sum_{i<j} f(x_i - x_j)^2$ with an arbitrary function f.

We conclude with a short summary and possible extensions.

II. Hamiltonian description of $\mathcal{N}=2$ supersymmetric Calogero models

In the Hamiltonian approach the n-particle supersymmetric Calogero model with $\mathcal{N}=2$-extended supersymmetry features the following degrees of freedom:

- n bosonic coordinates x_i and momenta p_i with $i=1,\ldots,n$,
- $2\,n$ fermions ψ_i and $\bar{\psi}_i$,
- $2\,n(n-1)$ fermions ξ_{ij} and $\bar{\xi}_{ij}$ with $\xi_{ii}=\bar{\xi}_{ii}=0$.

Their non-vanishing Poisson brackets are

$$\{x_i,p_j\}=\delta_{ij}, \qquad \left\{\psi_i,\bar{\psi}_j\right\}=-\mathrm{i}\,\delta_{ij}, \qquad \left\{\xi_{ij},\bar{\xi}_{km}\right\}=-\mathrm{i}\,(1-\delta_{ij})\,(1-\delta_{km})\,\delta_{im}\delta_{jk}. \tag{1}$$

A central role of our construction take the composite objects Π_{ij} and $\widetilde{\Pi}_{ij}$ defined as

$$\Pi_{ij} = (\psi_i-\psi_j)\,\bar{\xi}_{ij}+\left(\bar{\psi}_i-\bar{\psi}_j\right)\xi_{ij}+\sum_{k=1}^{n}\left(\xi_{ik}\bar{\xi}_{kj}+\bar{\xi}_{ik}\xi_{kj}\right), \tag{2}$$

$$\widetilde{\Pi}_{ij} = 2\delta_{ij}\psi_i\bar{\psi}_i+(\psi_i+\psi_j)\,\bar{\xi}_{ij}-\left(\bar{\psi}_i+\bar{\psi}_j\right)\xi_{ij}+\sum_{k=1}^{n}\left(\xi_{ik}\bar{\xi}_{kj}-\bar{\xi}_{ik}\xi_{kj}\right). \tag{3}$$

One may check that Π_{ij} and $\widetilde{\Pi}_{ij}$ together form with respect to Poisson brackets (1) an $s(u(n)\oplus u(n))$ algebra (note that $\sum_i \Pi_{ii}=0$).

It is known that an $\mathcal{N}=2$ supersymmetric Calogero models of A-type are defined by a generic form of their supercharges,

$$Q = \sum_{i=1}^{n} p_i \psi_i - \mathrm{i} \sum_{i \neq j}^{n} \left[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \xi_{ji}, \tag{4}$$

$$\overline{Q} = \sum_{i=1}^{n} p_i \bar{\psi}_i - \mathrm{i} \sum_{i \neq j}^{n} \left[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \bar{\xi}_{ji},$$

with some function $f(z_{ij})$, to be specified in a moment. Note that $\widetilde{\Pi}_{ij}$ does not appear here. These supercharges form an $\mathcal{N}=2$ super-Poincaré algebra,

$$\{Q, \overline{Q}\} = -2\mathrm{i}\,\mathcal{H} \qquad \text{and} \qquad \{Q, Q\} = \{\overline{Q}, \overline{Q}\} = 0\,, \tag{5}$$

together with the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + \frac{1}{2} \sum_{i \neq j}^{n} \left[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \tag{6}$$

$$\times \left[(g + \Pi_{ii}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ji} \right] + \frac{\alpha}{2} \sum_{i,j}^{n} \Pi_{ij} \Pi_{ji}. \tag{7}$$

Here, we abbreviated $z_{ij} = x_i - x_j$, and the constant parameter α and the function $f(z_{ij})$ are given as follows,

$$\begin{aligned} \text{rational Calogero model} \quad & \alpha = 0, \quad f(z_{ij}) = \frac{1}{z_{ij}} = \frac{1}{x_i - x_j}, \\ \text{hyperbolic Calogero-Moser model} \quad & \alpha = -1, \quad f(z_{ij}) = \frac{1}{\sinh(z_{ij})} = \frac{1}{\sinh(x_i - x_j)}, \\ \text{trigonometric Calogero-Moser model} \quad & \alpha = 1, \quad f(z_{ij}) = \frac{1}{\sin(z_{ij})} = \frac{1}{\sin(x_i - x_j)}. \end{aligned} \tag{8}$$

For the B, C and D-type models, the supercharges take a more complicated generic form (including $\widetilde{\Pi}_{ij}$),

$$
\begin{aligned}
\mathcal{Q} &= \sum_{i=1}^{n} p_i \psi_i - \mathrm{i} \sum_{i \neq j}^{n} \left[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \xi_{ji} \\
&+ \mathrm{i} \sum_{i \neq j}^{n} \left[(g + \Pi_{jj}) f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})} \widetilde{\Pi}_{ij} \right] \xi_{ji} + \mathrm{i} \sum_{i}^{n} \left[(g' + \Pi_{ii}) f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})} \widetilde{\Pi}_{ii} \right] \psi_i , \\
\overline{\mathcal{Q}} &= \sum_{i=1}^{n} p_i \bar{\psi}_i - \mathrm{i} \sum_{i \neq j}^{n} \left[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} \Pi_{ij} \right] \bar{\xi}_{ji} \\
&- \mathrm{i} \sum_{i \neq j}^{n} \left[(g + \Pi_{jj}) f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})} \widetilde{\Pi}_{ij} \right] \bar{\xi}_{ji} - \mathrm{i} \sum_{i}^{n} \left[(g' + \Pi_{ii}) f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})} \widetilde{\Pi}_{ii} \right] \bar{\psi}_i .
\end{aligned}
\tag{9}
$$

Here, $y_{ij} = x_i + x_j$, and the function f is the same as in (8).

The supercharges (9) form the same $\mathcal{N}=2$ super-Poincaré algebra (5) together with the Hamiltonian

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i\neq j}^{n}\Big[(g+\Pi_{jj})f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})}\Pi_{ij}\Big]\Big[(g+\Pi_{ii})f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})}\Pi_{ji}\Big] \\ &+ \frac{1}{2}\sum_{i\neq j}^{n}\Big[(g+\Pi_{jj})f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})}\widetilde{\Pi}_{ij}\Big]\Big[(g+\Pi_{ii})f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})}\widetilde{\Pi}_{ji}\Big] \\ &+ \frac{1}{2}\sum_{i}^{n}\Big[(g'+\Pi_{ii})f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})}\widetilde{\Pi}_{ii}\Big]\Big[(g'+\Pi_{ii})f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})}\widetilde{\Pi}_{ii}\Big].\end{aligned} \tag{10}$$

Its bosonic sector reads

$$\mathcal{H}_{bos} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{g^2}{2}\sum_{i\neq j}^{n}\Big(f^2(z_{ij}) + f^2(y_{ij})\Big) + \frac{g'^2}{2}\sum_{i}^{n} f^2(y_{ii}). \tag{11}$$

Due to the presence of only two coupling constants, g and g', we may describe B, C and D-type models in the rational case and C and D (but not B)-type models in the hyperbolic/trigonometric case.

III. $\mathcal{N}=2$ superspace $A_1 \oplus A_{n-1}$ Calogero models

To provide a superspace description of $\mathcal{N}=2$ supersymmetric Calogero models one has, firstly, to assemble the physical components x_i, ψ_i, $\bar\psi_i$, ξ_{ij} and $\bar\xi_{ij}$ into $\mathcal{N}=2$ superfields.
It immediately follows from the structure of the supercharges Q and $\overline{Q}$ (4) that under $\mathcal{N}=2$ supersymmetry the coordinates x_i transform into the fermions ψ_i and $\bar\psi_i$:

$$\delta x_i \equiv \left\{ x_i, \mathrm{i}\,\bar\epsilon Q + \mathrm{i}\,\epsilon \overline{Q} \right\} = \mathrm{i}\,\bar\epsilon \psi_i + \mathrm{i}\,\epsilon \bar\psi_i \ . \qquad (12)$$

Thus, one is let to n bosonic $\mathcal{N}=2$ superfields $\boldsymbol{x}_i$ with the components

$$x_i = \boldsymbol{x}_i|, \quad \psi_i = -\mathrm{i}D\boldsymbol{x}_i|, \quad \bar\psi_i = -\mathrm{i}\overline{D}\boldsymbol{x}_i|, \quad A_i = \tfrac{1}{2}\left[\overline{D}, D\right]\boldsymbol{x}_i|. \qquad (13)$$

We use here the $\mathcal{N}=2$ spinor covariant derivatives D and $\overline{D}$ obeying $\{D, \overline{D}\} = 2\mathrm{i}\partial_t$ and $\{D, D\} = \{\overline{D}, \overline{D}\} = 0$ and denote by $\mathcal{A}|$ the $\theta = \bar\theta = 0$ limit of a superspace expression $\mathcal{A}$.
Concerning the fermionic components $\xi_{ij}, \bar\xi_{ij}$, we have no other possibility than to put them into $2n(n-1)$ new fermionic superfields $\boldsymbol{\xi}_{ij}$ and $\bar{\boldsymbol{\xi}}_{ij}$ with vanishing diagonal parts

$$\boldsymbol{\xi}_{ii} = \bar{\boldsymbol{\xi}}_{ii} = 0 \quad \forall\, i. \qquad (14)$$

As $\mathcal{N}=2$ superfields the $\boldsymbol{\xi}_{ij}$ and $\bar{\boldsymbol{\xi}}_{ij}$ contain a lot of components. Hence, they have to be constrained somehow. The appropriate constraints derive from the explicit form of the supercharges Q and $\overline{Q}$ (4), which leads to the following supersymmetry transformations of the leading components ξ_{ij} and $\bar{\xi}_{ij}$ of these superfields,

$$
\begin{aligned}
\delta_Q \xi_{ij} \quad &\sim \quad \mathrm{i}\bar{\epsilon}\Big[-\frac{f'(z_{ij})}{f(z_{ij})}\,(\psi_i - \psi_j)\;\xi_{ij} + \xi_{ij}\Big(\sum_{k\neq i}^{n} f(z_{ik})\xi_{ik} - \sum_{k\neq j}^{n} f(z_{jk})\xi_{jk}\Big) \\
&\qquad - \sum_{k\neq i,j}^{n}\Big(\frac{f'(z_{ik})}{f(z_{ik})} + \frac{f'(z_{kj})}{f(z_{kj})}\Big)\,\xi_{ik}\xi_{kj}\Big]\,, \\
\delta_{\overline{Q}} \bar{\xi}_{ij} \quad &\sim \quad \mathrm{i}\epsilon\Big[-\frac{f'(z_{ij})}{f(z_{ij})}\,(\bar{\psi}_i - \bar{\psi}_j)\;\bar{\xi}_{ij} + \bar{\xi}_{ij}\Big(\sum_{k\neq i}^{n} f(z_{ik})\bar{\xi}_{ik} - \sum_{k\neq j}^{n} f(z_{jk})\bar{\xi}_{jk}\Big) \\
&\qquad - \sum_{k\neq i,j}^{n}\Big(\frac{f'(z_{ik})}{f(z_{ik})} + \frac{f'(z_{kj})}{f(z_{kj})}\Big)\,\bar{\xi}_{ik}\bar{\xi}_{kj}\Big]\,. \qquad (15)
\end{aligned}
$$

To realize this transformation property we are forced to impose a nonlinear chirality condition on the superfields $\boldsymbol{\xi}_{ij}$ and $\bar{\boldsymbol{\xi}}_{ij}$,

$$\begin{aligned} D\boldsymbol{\xi}_{ij} &= \mathrm{i}\Big[-\frac{f'(\boldsymbol{z}_{ij})}{f(\boldsymbol{z}_{ij})}\left(\boldsymbol{\psi}_i-\boldsymbol{\psi}_j\right)\boldsymbol{\xi}_{ij}+\boldsymbol{\xi}_{ij}\Big(\sum_{k\neq i}^{n} f(\boldsymbol{z}_{ik})\boldsymbol{\xi}_{ik}-\sum_{k\neq j}^{n} f(\boldsymbol{z}_{jk})\boldsymbol{\xi}_{jk}\Big) \\ &- \sum_{k\neq i,j}^{n}\Big(\frac{f'(\boldsymbol{z}_{ik})}{f(\boldsymbol{z}_{ik})}+\frac{f'(\boldsymbol{z}_{kj})}{f(\boldsymbol{z}_{kj})}\Big)\boldsymbol{\xi}_{ik}\boldsymbol{\xi}_{kj}\Big], \qquad (16) \\ \overline{D}\bar{\boldsymbol{\xi}}_{ij} &= \mathrm{i}\Big[-\frac{f'(\boldsymbol{z}_{ij})}{f(\boldsymbol{z}_{ij})}\left(\bar{\boldsymbol{\psi}}_i-\bar{\boldsymbol{\psi}}_j\right)\bar{\boldsymbol{\xi}}_{ij}+\bar{\boldsymbol{\xi}}_{ij}\Big(\sum_{k\neq i}^{n} f(\boldsymbol{z}_{ik})\bar{\boldsymbol{\xi}}_{ik}-\sum_{k\neq j}^{n} f(\boldsymbol{z}_{jk})\bar{\boldsymbol{\xi}}_{jk}\Big) \\ &- \sum_{k\neq i,j}^{n}\Big(\frac{f'(\boldsymbol{z}_{ik})}{f(\boldsymbol{z}_{ik})}+\frac{f'(\boldsymbol{z}_{kj})}{f(\boldsymbol{z}_{kj})}\Big)\bar{\boldsymbol{\xi}}_{ik}\bar{\boldsymbol{\xi}}_{kj}\Big]. \end{aligned}$$

This condition leaves in the superfields $\boldsymbol{\xi}_{ij}$ and $\bar{\boldsymbol{\xi}}_{ij}$ only the components

$$\xi_{ij} = \boldsymbol{\xi}_{ij}|, \quad B_{ij} = \overline{D}\boldsymbol{\xi}_{ij}|, \qquad \bar{\xi}_{ij} = \bar{\boldsymbol{\xi}}_{ij}|, \quad \overline{B}_{ij} = D\bar{\boldsymbol{\xi}}_{ij}|\,. \qquad (17)$$

Finally, to obtain the correct brackets (1) for $(\psi_i, \bar\psi_i)$ and $(\xi_{ij}, \bar\xi_{ij})$ after passing to the Hamiltonian formalism, the kinetic terms for these fermionic components must have the form

$$\mathcal{L}^{\psi}_{kin} = \frac{\mathrm{i}}{2}\sum_{i=1}^{n}\left(\dot\psi_i\bar\psi_i - \psi_i\dot{\bar\psi}_i\right) \qquad \text{and} \qquad \mathcal{L}^{\xi}_{kin} = \frac{\mathrm{i}}{2}\sum_{i,j}^{n}\left(\dot\xi_{ij}\bar\xi_{ji} - \xi_{ij}\dot{\bar\xi}_{ji}\right). \tag{18}$$

In $\mathcal{N}=2$ superspace, this amounts to the free action ($g=0$)

$$S_0 = \int dt\, d^2\theta \left[-\frac{1}{2}\sum_{i=1}^{n} D\boldsymbol{x}_i\, \overline{D}\boldsymbol{x}_i + \frac{1}{2}\sum_{i,j}^{n} \boldsymbol{\xi}_{ij}\bar{\boldsymbol{\xi}}_{ji}\right] \qquad \text{with} \quad d^2\theta \equiv D\overline{D}. \tag{19}$$

More interesting is the construction of the interaction terms. Again, some hints come from the transformation properties of the fermions ξ_{ij} and $\bar\xi_{ij}$ under $\overline{Q}$ and Q supersymmetry, respectively,

$$\delta_{\overline{Q}}\xi_{ij} \sim \mathrm{i}\epsilon\, g\, f(z_{ij}) + \ldots, \quad \delta_{Q}\bar\xi_{ij} \sim \mathrm{i}\bar\epsilon\, g\, f(z_{ij}) + \ldots . \tag{20}$$

To reproduce such terms in superspace, the unique possibility is to add to the action S_0 (19) a term

$$S_{int} = \mathrm{i}\,\frac{g}{2}\int dt\, d\bar{\theta}\, \sum_{i\neq j}^{n} f(\boldsymbol{z}_{ij})\xi_{ij} + \mathrm{i}\,\frac{g}{2}\int\, dt\, d\theta\, \sum_{i\neq j}^{n} f(\boldsymbol{z}_{ij})\,\bar{\xi}_{ij}. \tag{21}$$

To be supersymmetrically invariant, the integrands in (21) must be chiral and antichiral, respectively. It is not too hard to check that this is indeed so: the nonlinear chirality constraint (16) implies that

$$D\Big(\sum_{i\neq j}^{n} f(\boldsymbol{z}_{ij})\,\xi_{ij}\Big) = 0 \qquad \text{and} \qquad \overline{D}\Big(\sum_{i\neq j}^{n} f(\boldsymbol{z}_{ij})\,\bar{\xi}_{ij}\Big) = 0. \tag{22}$$

Combining all these facts together, we conclude that the superfield action reads

$$\begin{aligned} S \;\; &= \;\; \int dt\, d^2\theta\, \Big[-\frac{1}{2}\sum_{i=1}^{n} D\boldsymbol{x}_i\, \overline{D}\boldsymbol{x}_i + \frac{1}{2}\sum_{i,j}^{n} \xi_{ij}\bar{\xi}_{ji} \Big] \\ &+ \;\; \mathrm{i}\,\frac{g}{2}\int\, dt\, d\bar{\theta}\sum_{i\neq j}^{n} f(\boldsymbol{z}_{ij})\xi_{ij} + \mathrm{i}\,\frac{g}{2}\int dtd\theta\sum_{i\neq j}^{n} f(\boldsymbol{z}_{ij})\,\bar{\xi}_{ij}\,, \end{aligned} \tag{23}$$

where the superfields ξ_{ij} and $\bar{\xi}_{ij}$ are subject to the nonlinear chirality constraint (16).

It is important to note that, after passing to new fermionic superfields

$$\boldsymbol{\lambda}_{ij} \equiv f(\boldsymbol{z}_{ij})\,\boldsymbol{\xi}_{ij} \qquad \text{and} \qquad \bar{\boldsymbol{\lambda}}_{ij} \equiv f(\boldsymbol{z}_{ij})\,\bar{\boldsymbol{\xi}}_{ij}\,, \tag{24}$$

the nonlinear constraint (16) is slightly simplified to

$$D\boldsymbol{\lambda}_{ij} = \mathrm{i}\left[\boldsymbol{\lambda}_{ij}\sum_{k\neq i}^{n}\boldsymbol{\lambda}_{ik} - \boldsymbol{\lambda}_{ij}\sum_{k\neq j}^{n}\boldsymbol{\lambda}_{jk} + (1-\delta_{ij})\sum_{k\neq i,j}^{n}\boldsymbol{\lambda}_{ik}\boldsymbol{\lambda}_{kj}\right], \tag{25}$$

$$\overline{D}\bar{\boldsymbol{\lambda}}_{ij} = \mathrm{i}\left[\bar{\boldsymbol{\lambda}}_{ij}\sum_{k\neq i}^{n}\bar{\boldsymbol{\lambda}}_{ik} - \bar{\boldsymbol{\lambda}}_{ij}\sum_{k\neq j}^{n}\bar{\boldsymbol{\lambda}}_{jk} + (1-\delta_{ij})\sum_{k\neq i,j}^{n}\bar{\boldsymbol{\lambda}}_{ik}\bar{\boldsymbol{\lambda}}_{kj}\right].$$

In this form, the constraint has lost any f-dependence, which however will reappear in the action,

$$\begin{aligned} S \;\; &= \;\; \int dt\, d^2\theta \left[-\frac{1}{2}\sum_{i=1}^{n} D\boldsymbol{x}_i\,\overline{D}\boldsymbol{x}_i + \frac{1}{2}\sum_{i,j}^{n}\frac{\boldsymbol{\lambda}_{ij}\bar{\boldsymbol{\lambda}}_{ji}}{f(\boldsymbol{z}_{ij})\,f(\boldsymbol{z}_{ji})}\right] \\ &+ \;\; \mathrm{i}\,\frac{g}{2}\int dt\, d\bar{\theta}\,\sum_{i\neq j}^{n}\boldsymbol{\lambda}_{ij} + \mathrm{i}\,\frac{g}{2}\int dt\, d\theta\,\sum_{i\neq j}^{n}\bar{\boldsymbol{\lambda}}_{ij}\,. \end{aligned} \tag{26}$$

Also, the component Lagrangian, Hamiltonian and Poisson brackets will be more complicated in terms of the composite superfields $\boldsymbol{\lambda}_{ij}$ and $\bar{\boldsymbol{\lambda}}_{ij}$.

Despite the extremely simple form of the superfield action (23), its component version looks quite complicated due to the constraint (16). Indeed, after integration over θ in (23) we get the off-shell Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\rm pot}$, where

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2}\sum_i^n \left(\dot{x}_i\dot{x}_i + A_iA_i\right) + \frac{\mathrm{i}}{2}\sum_i^n\left(\dot{\psi}_i\bar{\psi}_i - \psi_i\dot{\bar{\psi}}_i\right) + \frac{\mathrm{i}}{2}\sum_{i,j}^n\left(\dot{\xi}_{ij}\bar{\xi}_{ji} - \xi_{ij}\dot{\bar{\xi}}_{ji}\right) \\ &+ \frac{1}{2}\sum_{i,j}^n\left(\xi_{ij}D\left(\overline{D}\bar{\xi}_{ji}\right) - \overline{D}\left(D\xi_{ij}\right)\bar{\xi}_{ji} - D\xi_{ij}\overline{D}\bar{\xi}_{ji} + B_{ij}\overline{B}_{ji}\right), \qquad (27) \\ \mathcal{L}_{\rm pot} &= -\frac{g}{2}\sum_{i,j}^n f'(z_{ij})\Big(\left(\psi_i - \psi_j\right)\bar{\xi}_{ij} + \left(\bar{\psi}_i - \bar{\psi}_j\right)\xi_{ij}\Big) + \mathrm{i}\,\frac{g}{2}\sum_{i,j}^n f(z_{ij})\left(B_{ij} + \overline{B}_{ij}\right). \end{aligned}$$

To eliminate the auxiliary fields A_i and B_{ij} one firstly has to evaluate the terms in the second line of (27) by using the constraint (16). Then we finally obtain

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\sum_{i=1}^n \dot{x}_i\dot{x}_i + \frac{\mathrm{i}}{2}\sum_{i=1}^n\left(\dot{\psi}_i\bar{\psi}_i - \psi_i\dot{\bar{\psi}}_i\right) + \frac{\mathrm{i}}{2}\sum_{i,j}^n\left(\dot{\xi}_{ij}\bar{\xi}_{ji} - \xi_{ij}\dot{\bar{\xi}}_{ji}\right) \qquad (28) \\ &- \frac{1}{2}\sum_{i\neq j}^n\left[(g+\Pi_{jj})f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})}\Pi_{ij}\right]\left[(g+\Pi_{ii})f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})}\Pi_{ji}\right] - \frac{\alpha}{2}\sum_{i,j}^n \Pi_{ij}\Pi_{ji}. \end{aligned}$$

Thus, the superfield action (23) with the superfields ξ_{ij} and $\bar{\xi}_{ij}$ subject to the constraint (16) indeed describes all $\mathcal{N}=2$ supersymmetric $A_1 \oplus A_{n-1}$ Calogero models.

IV. $\mathcal{N}=2$ superspace B_n, C_n and D_n Calogero models

The supercharges of the $\mathcal{N}=2$ supersymmetric B, C and D-type Calogero models (9) have a more complicated structure than those in (4). Therefore, it is expected that the nonlinear chirality constraint for the superfields ξ_{ij} and $\bar{\xi}_{ij}$ are more intricate as well. Indeed, the explicit structure of the supercharges (9) uniquely fixes this constraint to be

$$\begin{aligned} D\xi_{ij} \;=\; & \mathrm{i}\Big[-\frac{f'(\mathbf{z}_{ij})}{f(\mathbf{z}_{ij})}(\psi_i-\psi_j)\,\xi_{ij}-\frac{f'(\mathbf{y}_{ij})}{f(\mathbf{y}_{ij})}(\psi_i+\psi_j)\,\xi_{ij} \\ & +\Big\{\Big(\frac{f'(\mathbf{y}_{ii})}{f(\mathbf{y}_{ii})}-f(\mathbf{y}_{ii})\Big)\psi_i+\Big(\frac{f'(\mathbf{y}_{jj})}{f(\mathbf{y}_{jj})}+f(\mathbf{y}_{jj})\Big)\psi_j\Big\}\xi_{ij} \\ & +\xi_{ij}\Big(\sum_{k\neq i}^{n}\big(f(\mathbf{z}_{ik})+f(\mathbf{y}_{ik})\big)\xi_{ik}-\sum_{k\neq j}^{n}\big(f(\mathbf{z}_{jk})+f(\mathbf{y}_{jk})\big)\xi_{jk}\Big) \\ & -\sum_{k\neq i,j}^{n}\Big(\frac{f'(\mathbf{z}_{ik})}{f(\mathbf{z}_{ik})}+\frac{f'(\mathbf{z}_{kj})}{f(\mathbf{z}_{kj})}-\frac{f'(\mathbf{y}_{ik})}{f(\mathbf{y}_{ik})}+\frac{f'(\mathbf{y}_{kj})}{f(\mathbf{y}_{kj})}\Big)\xi_{ik}\xi_{kj}\Big], \end{aligned} \tag{29}$$

$$
\begin{aligned}
\overline{D}\bar{\xi}_{ij} \quad = \quad & \mathrm{i}\Big[-\frac{f'(\mathbf{z}_{ij})}{f(\mathbf{z}_{ij})}\,(\bar{\psi}_i - \bar{\psi}_j)\,\bar{\xi}_{ij} - \frac{f'(\mathbf{y}_{ij})}{f(\mathbf{y}_{ij})}\,(\bar{\psi}_i + \bar{\psi}_j)\,\bar{\xi}_{ij} \\
+ \quad & \Big\{ \Big(\frac{f'(\mathbf{y}_{ii})}{f(\mathbf{y}_{ii})} + f(\mathbf{y}_{ii}) \Big) \bar{\psi}_i + \Big(\frac{f'(\mathbf{y}_{jj})}{f(\mathbf{y}_{jj})} - f(\mathbf{y}_{jj}) \Big) \bar{\psi}_j \Big\} \bar{\xi}_{ij} \\
+ \quad & \bar{\xi}_{ij} \Big(\sum_{k\neq i}^{n} (f(\mathbf{z}_{ik}) - f(\mathbf{y}_{ik}))\bar{\xi}_{ik} - \sum_{k\neq j}^{n} (f(\mathbf{z}_{jk}) - f(\mathbf{y}_{jk}))\bar{\xi}_{jk} \Big) \\
- \quad & \sum_{k\neq i,j}^{n} \Big(\frac{f'(\mathbf{z}_{ik})}{f(\mathbf{z}_{ik})} + \frac{f'(\mathbf{z}_{kj})}{f(\mathbf{z}_{kj})} - \frac{f'(\mathbf{y}_{ik})}{f(\mathbf{y}_{ik})} + \frac{f'(\mathbf{y}_{kj})}{f(\mathbf{y}_{kj})} \Big)\, \bar{\xi}_{ik}\bar{\xi}_{kj} \Big] \,.
\end{aligned}
$$

The complicated form of this constraint disappears after passing to the composite superfields

$$\boldsymbol{\lambda}_{ij} = \big(f(\boldsymbol{z}_{ij}) + f(\boldsymbol{y}_{ij})\big)\boldsymbol{\xi}_{ij} \quad \text{and} \quad \bar{\boldsymbol{\lambda}}_{ij} = \big(f(\boldsymbol{z}_{ij}) - f(\boldsymbol{y}_{ij})\big)\bar{\boldsymbol{\xi}}_{ij}, \tag{30}$$

in which it acquires its familiar form (25),

$$D\boldsymbol{\lambda}_{ij} = \mathrm{i}\left[\boldsymbol{\lambda}_{ij}\sum_{k\neq i}^{n}\boldsymbol{\lambda}_{ik} - \boldsymbol{\lambda}_{ij}\sum_{k\neq j}^{n}\boldsymbol{\lambda}_{jk} + (1-\delta_{ij})\sum_{k\neq i,j}^{n}\boldsymbol{\lambda}_{ik}\boldsymbol{\lambda}_{kj}\right], \tag{31}$$

$$\overline{D}\bar{\boldsymbol{\lambda}}_{ij} = \mathrm{i}\left[\bar{\boldsymbol{\lambda}}_{ij}\sum_{k\neq i}^{n}\bar{\boldsymbol{\lambda}}_{ik} - \bar{\boldsymbol{\lambda}}_{ij}\sum_{k\neq j}^{n}\bar{\boldsymbol{\lambda}}_{jk} + (1-\delta_{ij})\sum_{k\neq i,j}^{n}\bar{\boldsymbol{\lambda}}_{ik}\bar{\boldsymbol{\lambda}}_{kj}\right].$$

Finally, the superfield action reads

$$\begin{aligned} S \;=\; & \int dt\, d^2\theta \left[-\frac{1}{2}\sum_{i=1}^{n} D\boldsymbol{x}_i\, \overline{D}\boldsymbol{x}_i + \frac{1}{2}\sum_{i,j}^{n}\boldsymbol{\xi}_{ij}\bar{\boldsymbol{\xi}}_{ji} - \frac{1}{2}g'\,h(\boldsymbol{y}_{ii})\right] \\ & + \mathrm{i}\frac{g}{2}\int dt\, d\bar{\theta}\sum_{i\neq j}^{n}\big(f(\boldsymbol{z}_{ij}) + f(\boldsymbol{y}_{ij})\big)\boldsymbol{\xi}_{ij} + \mathrm{i}\frac{g}{2}\int dt\, d\theta\sum_{i\neq j}^{n}\big(f(\boldsymbol{z}_{ij}) + f(\boldsymbol{y}_{ij})\big)\,\bar{\boldsymbol{\xi}}_{ij}, \end{aligned} \tag{32}$$

where $h'(\boldsymbol{y}_{ii}) = f(\boldsymbol{y}_{ii})$. Compared to the action of the $A_1 \oplus A_{n-1}$ Calogero models (26), only the term $\frac{1}{2}g'\int dt\, d^2\theta\, h(\boldsymbol{y}_{ii})$ carrying the new coupling constant g' appears in the action (32). All other terms just mimic those in (26).

It is a matter of straightforward but tedious calculations to check that, after excluding the auxiliary fields by their equations of motion, the final Lagrangian acquires the expected form

$$\begin{aligned}
\mathcal{L} \;=\;& \frac{1}{2}\sum_{i=1}^{n}\dot{x}_i\dot{x}_i + \frac{\mathrm{i}}{2}\sum_{i=1}^{n}\left(\dot{\psi}_i\bar{\psi}_i - \psi_i\dot{\bar{\psi}}_i\right) + \frac{\mathrm{i}}{2}\sum_{i,j}^{n}\left(\dot{\xi}_{ij}\bar{\xi}_{ji} - \xi_{ij}\dot{\bar{\xi}}_{ji}\right) \\
&- \frac{1}{2}\sum_{i\neq j}^{n}\left[(g+\Pi_{jj})f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})}\,\Pi_{ij}\right]\left[(g+\Pi_{ii})f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})}\,\Pi_{ji}\right] \\
&- \frac{1}{2}\sum_{i\neq j}^{n}\left[(g+\Pi_{jj})f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})}\,\widetilde{\Pi}_{ij}\right]\left[(g+\Pi_{ii})f(y_{ij}) - \frac{f'(y_{ij})}{f(y_{ij})}\,\widetilde{\Pi}_{ji}\right] \\
&- \frac{1}{2}\sum_{i}^{n}\left[(g'+\Pi_{ii})f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})}\,\widetilde{\Pi}_{ii}\right]\left[(g'+\Pi_{ii})f(y_{ii}) - \frac{f'(y_{ii})}{f(y_{ii})}\,\widetilde{\Pi}_{ii}\right].
\end{aligned} \tag{33}$$

Thus, the superfield action (32) with the superfields $\boldsymbol{\xi}_{ij}$ and $\bar{\boldsymbol{\xi}}_{ij}$ subject to the nonlinear chirality constraint (29) indeed describes the $\mathcal{N}=2$ supersymmetric B, C and D-type Calogero models.[1]

[1] Except again for B-type models in the trigonometric/hyperbolic case.

V. New $\mathcal{N}=2$ supersymmetric n-particle models

Our point of departure was the explicit form of the $\mathcal{N}=2$ supercharges for the Calogero models (4) and (9) constructed in our previous papers. In all cases we considered the function f to be not arbitrary but to be chosen from the list (8). Indeed, for generic f the supercharges (4) do not form the closed algebra (5).

As a consequence, the nonlinear chirality condition (16) ($\boldsymbol{\xi}$-constraints) is not self-consistent for an arbitrary function f (i.e. D acting on the r.h.s. of (16) does not vanish). On the other hand, the universal chirality constraint (25) ($\boldsymbol{\lambda}$-constraints) is self-consistent.

This implies a weaker chirality condition on ξ_{ij} and $\bar\xi_{ij}$:

$$\begin{aligned} D\xi_{ij} &= \mathrm{i}\Big[-\frac{f'(\boldsymbol{z}_{ij})}{f(\boldsymbol{z}_{ij})}\left(\psi_i-\psi_j\right)\,\xi_{ij}+\xi_{ij}\Big(\sum_{k\neq i}^{n} f(\boldsymbol{z}_{ik})\xi_{ik}-\sum_{k\neq j}^{n} f(\boldsymbol{z}_{jk})\xi_{jk}\Big) \\ &+ \sum_{k\neq i,j}^{n}\frac{f(\boldsymbol{z}_{ik})\,f(\boldsymbol{z}_{kj})}{f(\boldsymbol{z}_{ij})}\,\xi_{ik}\xi_{kj}\Big], \qquad (34) \\ \overline{D}\bar\xi_{ij} &= \mathrm{i}\Big[-\frac{f'(\boldsymbol{z}_{ij})}{f(\boldsymbol{z}_{ij})}\left(\bar\psi_i-\bar\psi_j\right)\,\bar\xi_{ij}+\bar\xi_{ij}\Big(\sum_{k\neq i}^{n} f(\boldsymbol{z}_{ik})\bar\xi_{ik}-\sum_{k\neq j}^{n} f(\boldsymbol{z}_{jk})\bar\xi_{jk}\Big) \\ &+ \sum_{k\neq i,j}^{n}\frac{f(\boldsymbol{z}_{ik})\,f(\boldsymbol{z}_{kj})}{f(\boldsymbol{z}_{ij})}\,\bar\xi_{ik}\bar\xi_{kj}\Big]. \end{aligned}$$

For the sake of simplicity we consider here the A-type case, in which the $(\boldsymbol{\lambda}_{ij},\bar{\boldsymbol{\lambda}}_{ij})$ and $(\xi_{ij},\bar\xi_{ij})$ are related as $\boldsymbol{\lambda}_{ij}\equiv f(\boldsymbol{z}_{ij})\,\xi_{ij},\ \ \bar{\boldsymbol{\lambda}}_{ij}\equiv f(\boldsymbol{z}_{ij})\,\bar\xi_{ij}$. One may check that this nonlinear chirality constraint is perfectly self-consistent for an *arbitrary* function f.

Now, let us again start from the superspace action (19), but where the fermionic superfields $\boldsymbol{\xi}_{ij}$ and $\bar{\boldsymbol{\xi}}_{ij}$ must obey the constraint (34). Passing to the components and eliminating the auxiliary components one arrives at

$$\begin{aligned} \mathcal{Q} &= \sum_{i=1}^{n} p_i \psi_i - \mathrm{i} \sum_{i \neq j}^{n} \Big[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} (\psi_i - \psi_j) \bar{\xi}_{ij} \\ &+ \sum_{k \neq i,j}^{n} \frac{f(z_{ik}) f(z_{ij})}{f(z_{kj})} \xi_{ik} \bar{\xi}_{kj} \Big] \xi_{ji}, \end{aligned} \tag{35}$$

$$\begin{aligned} \overline{\mathcal{Q}} &= \sum_{i=1}^{n} p_i \bar{\psi}_i - \mathrm{i} \sum_{i \neq j}^{n} \Big[(g + \Pi_{jj}) f(z_{ij}) + \frac{f'(z_{ij})}{f(z_{ij})} (\bar{\psi}_i - \bar{\psi}_j) \xi_{ij} \\ &+ \sum_{k \neq i,j}^{n} \frac{f(z_{ik}) f(z_{ij})}{f(z_{kj})} \bar{\xi}_{ik} \xi_{kj} \Big] \bar{\xi}_{ji}. \end{aligned} \tag{36}$$

For the functions listed in (8) these supercharges coincide with the ones in (4).

However, the supercharges (35) generate the $\mathcal{N}=2$ super-Poincaré algebra (5) for an *arbitrary* function f. It is not too hard to find the bosonic part of the new Hamiltonian,

$$\mathcal{H}_{\rm bos} = \frac{1}{2}\sum_{i}^{n} p_i^2 + \frac{g^2}{2}\sum_{i\neq j}^{n} f^2(z_{ij})\,. \qquad (37)$$

Thus, the supercharges (35) provide an $\mathcal{N}=2$ supersymmetrization of a wide class of multi-particle systems with a bosonic Hamiltonian of the type (37).

A detailed analysis of such models will be given elsewhere.

VI. Conclusion

In the present talk we have provided a superspace description of the $\mathcal{N}=2$ supersymmetric Calogero models, rational as well as trigonometric/hyperbolic, associated with the classical A_n, B_n, C_n and D_n Lie algebras.

We presented a minimal superfield content accomodating the $2n^2$ fermions for the $\mathcal{N}=2$ supersymmetric n-particle model. As $2n$ fermions accompany the n bosonic coordinates in general bosonic $\mathcal{N}=2$ superfields, the remaining $2n(n-1)$ fermions must be put into additional fermionic $\mathcal{N}=2$ superfields, which have to be constrained such as to describe those fermions alone.

The nonlinear chirality condition written in terms of commposite superfields solves this task. These composite fermionic $\mathcal{N}=2$ superfields make the constraint look simple and universal but complicate the Lagrangian.

Finally, we presented more general supercharges (and the superspace Lagrangian) which provides an $\mathcal{N}=2$ supersymmetrization of bosonic n-particle systems with an arbitrary repulsive two-body interaction.

We consider our results here as a preparation for attacking Calogero systems with more supersymmetry in a superspace setting.