# Noncompact CP$^N$ as a phase space of superintegrable systems

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# Content

Based on

E.Khastyan, A.Nersessian, H. Shmavonyan *"Noncompact* $\mathrm{CP}^N$ *as a phase space of superintegrable systems"*

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- Basic facts on Kähler geometry.
  - Preliminary
  - Compact and non-compact projective spaces $\mathrm{CP}^N/\widetilde{\mathrm{CP}}^N$
- N-dimensional Klain model.
- Superintegrable models of conformal mechanics and of the oscillator- and Coulomb-like systems.
- Canonical coordinates.
  - Superintegrable systems
- Conclusion.

# Kähler Manifolds

The symplectic manifold $(M, \omega)$ is the even-dimensional manifold equipped with closed non-degenerate two-form

$$\omega = \frac{1}{2}\omega_{ij}(x)dx^i \wedge dx^j \ : d\omega = 0, \quad \det \omega_{ij} \neq 0. \tag{1}$$

This two-form defines the non-degenerate Poisson brackets

$$\{f, g\} = \omega^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} \ , \quad \text{with} \quad \omega^{ij}\omega_{jk} = \delta^i_k. \tag{2}$$

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Kähler manifold is the manifold with Hermitian metrics $ds^2 = g_{a\bar{b}} dz^a d\bar{z}^b$ whose imaginary part defines the symplectic structure

$$\omega_M = \imath g_{a\bar{b}} dz^a \wedge d\bar{z}^b, \; d\omega_M = 0 \quad \Rightarrow \quad g_{a\bar{b}} dz^a d\bar{z}^b = \frac{\partial^2 \mathcal{K}}{\partial z^a \partial \bar{z}^b} dz^a d\bar{z}^b, \tag{3}$$

where $\mathcal{K}(z, \bar{z})$ is a real function (Kähler potential) defined up to holomorphic and antiholomorphic functions:
$\mathcal{K}(z, \bar{z}) \to \mathcal{K}(z, \bar{z}) + U(z) + \bar{U}(\bar{z})$.

# Kähler Manifolds

Hence, Kähler manifold can be equipped with the Poisson brackets

$$\{f,g\}_M = \imath g^{\bar{a}b}\Big(\frac{\partial f}{\partial \bar{z}^a}\frac{\partial g}{\partial z^b} - \frac{\partial g}{\partial \bar{z}^a}\frac{\partial f}{\partial z^b}\Big), \quad g^{\bar{a}b}g_{b\bar{c}} = \delta^{\bar{a}}_{\bar{c}} \,. \qquad (4)$$

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Therefore, the isometries of Kähler structure should preserve both complex and symplectic structures, i.e. they are defined by the holomorphic Hamiltonian vector fields,

$$\mathrm{V}_\mu = \{h_\mu, \}_M = V^a_\mu(z)\frac{\partial}{\partial z^a} + \bar{V}^{\bar{a}}_\mu(\bar{z})\frac{\partial}{\partial \bar{z}^a}, \quad V^a_\mu = \imath g^{\bar{b}a}\partial_{\bar{b}}h_\mu(z, \bar{z}) \,. \tag{5}$$

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The real function $h_\mu(z,\bar{z})$ (sometimes called Killing potential) obeys the equation

$$\frac{\partial^2 h_\mu}{\partial z^a \partial z^b} - \Gamma^c_{ab}\frac{\partial h_\mu}{\partial z^c} = 0, \tag{6}$$

with $\Gamma^c_{ab} = g^{c\bar{d}}\partial_a g_{b\bar{d}}$ being the non-vanishing components of the Christoffel symbols.

# $\mathrm{CP}^N / \widetilde{\mathrm{CP}}^N$

$N$-dimensional complex projective space $\mathrm{CP}^N$
its non-compact analog $\widetilde{\mathrm{CP}}^N$.

They can be equipped with the $su(N+1)$-invariant (for the compact case) and the $su(N.1)$ invariant (for the non-compact case) Kähler metrics, known as the Fubini-Study ones.

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Metrics and respective Kähler potentials

$$g_{a\bar{b}}dz^a d\bar{z}^b = \frac{gdzd\bar{z}}{1\pm z\bar{z}} \mp \frac{g(\bar{z}dz)(zd\bar{z})}{(1\pm z\bar{z})^2}, \quad \mathcal{K} = \pm g\log(1\pm z\bar{z}), \qquad g>0. \tag{7}$$

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Notice: In the non-compact case the range of validity of the coordinates $z^a$ is as follows

$$|z^a| < 1, \quad \sum_{a=1}^{N} z^a \bar{z}^a < 1 \tag{8}$$

The inverse metrics defining Poisson brackets is given by the expression

$$g^{\bar{a}b} = \frac{1}{g}(1 \pm z\bar{z})(\delta^{\bar{a}b} \pm \bar{z}^a z^b). \tag{9}$$

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Killing potentials: $h_{a\bar{b}} = g\frac{\bar{z}^a z^b \mp \delta_{a\bar{b}}}{1\pm z\bar{z}}$, $h_a = g\frac{2\bar{z}^a}{1\pm z\bar{z}}$, $h_{\bar{a}} = g\frac{2z^a}{1\pm z\bar{z}}$ .

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These generators form the $su(N+1)$ algebra for the upper sign, and the $su(N.1)$ for the lower one(the generators $h_{a\bar{b}}$ form $u(N)$ algebra):

$$\{h_a, h_b\} = 0, \quad \{h_a, h_{\bar{b}}\} = -4\imath h_{a\bar{b}}, \tag{10}$$

$$\{h_a, h_{b\bar{c}}\} = \pm\imath\left(\delta_{a\bar{c}} h_b + \delta_{b\bar{c}} h_a\right), \tag{11}$$

$$\{h_{a\bar{b}}, h_{c\bar{d}}\} = \pm\imath(\delta_{a\bar{d}} h_{c\bar{b}} - \delta_{\bar{b}c} h_{a\bar{d}}). \tag{12}$$

# N-dimensional Klain model

Noncompact complex projective space: Fubini-Study structure of $\widetilde{\mathrm{CP}}^N$

$$g_{a\bar{b}}dz^a d\bar{z}^b = \frac{g dz d\bar{z}}{1 - z\bar{z}} + \frac{g(\bar{z}dz)(zd\bar{z})}{(1 - z\bar{z})^2}, \quad \mathcal{K} = -g\log(1 - z\bar{z}) \tag{13}$$

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To construct $N$-dimensional analog of the Klein model we perform the transformation

$$z^N = \frac{1 - \imath w}{1 + \imath w}, \quad z^\alpha = \sqrt{2}\frac{\tilde{z}^\alpha}{1 + \imath w}, \tag{14}$$

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*here and further instead of $\tilde{z}^\alpha$ we use the former notation $z^\alpha$.

# N-dimensional Klain model

This yields the following expressions for the Kähler structure and potential

$$ds^2 = \frac{g[dw + \imath\bar{z}^\alpha dz^\alpha][d\bar{w} - \imath z^\beta d\bar{z}^\beta)]}{[\imath(w - \bar{w}) - z^\gamma \bar{z}^\gamma]^2} + \frac{g dz^\alpha d\bar{z}^\alpha}{\imath(w - \bar{w}) - z^\gamma \bar{z}^\gamma}, \qquad (15)$$

$$\mathcal{K} = -g \log\left[\imath(w - \bar{w}) - z^\gamma \bar{z}^\gamma\right], \qquad \alpha, \beta, \gamma = 1, \ldots N - 1, \qquad (16)$$

with the following range of validity of the coordinates $w, z^\alpha$

$$\mathrm{Im}\, w < 0, \qquad \sum_{\alpha=1}^{N-1} z^\alpha \bar{z}^\alpha < -2\, \mathrm{Im}\, w. \qquad (17)$$

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The respective Poisson brackets are defined by the relations

$$\{w, \bar{w}\} = -A(w - \bar{w}), \quad \{w, \bar{z}^\alpha\} = A\bar{z}^\alpha, \quad \{z_\alpha, \bar{z}_\beta\} = \imath A\delta^{\bar{\beta}\alpha}, \qquad (18)$$

where

$$A := \frac{\imath(w - \bar{w}) - z^\gamma\bar{z}^\gamma}{g}. \qquad (19)$$

# N-dimensional Klain model: Killing potentials

The Killing potentials of the Kähler structure above are defined by

$$h_{N\bar{N}} = \frac{w\bar{w}+1}{A}, \quad h_{\alpha\bar{N}} = \frac{1}{\sqrt{2}}\frac{\bar{z}^\alpha(1-\imath w)}{A}, \tag{20}$$

$$h_{\alpha\bar{\beta}} = \frac{\bar{z}^\alpha z^\beta + \frac{1}{2}\delta_{\alpha\bar{\beta}}(1+\imath w)(1-\imath\bar{w})}{A}, \tag{21}$$

$$h_N = \frac{(1+\imath w)(1+\imath\bar{w})}{A}, \quad h_\alpha = \sqrt{2}\frac{\bar{z}^\alpha(1+\imath w)}{A} \tag{22}$$

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These potentials form $su(N.1)$ algebra, which reads the same

$$\begin{gathered}\{h_a, h_b\} = 0, \quad \{h_a, h_{\bar{b}}\} = -4\imath h_{a\bar{b}}, \\ \{h_a, h_{b\bar{c}}\} = -\imath\left(\delta_{a\bar{c}}h_b + \delta_{b\bar{c}}h_a\right), \\ \{h_{a\bar{b}}, h_{c\bar{d}}\} = -\imath(\delta_{a\bar{d}}h_{c\bar{b}} - \delta_{\bar{b}c}h_{a\bar{d}}).\end{gathered} \qquad \begin{gathered}a, b, c, d = N, \alpha \\ \alpha = 1, ..., N-1.\end{gathered} \tag{23}$$

# Conformal mechanics

For our purposes, instead of Killing potentials above, it is more convenient to use the following ones

$$H = \frac{w\bar{w}}{A}, \quad K = \frac{1}{A}, \quad D = \frac{w+\bar{w}}{A}, \tag{24}$$

$$H_{\alpha\bar{N}} = \frac{\bar{z}^\alpha w}{A}, \quad H_\alpha = \frac{\bar{z}^\alpha}{A}, \quad H_{\alpha\bar{\beta}} = \frac{\bar{z}^\alpha z^\beta}{A}. \tag{25}$$

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Remember:

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Certainly, these functions are not independent, for there are many obvious relations between them, e.g.

$$H = \frac{1}{2}\sum_{\alpha=1}^{N-1}\frac{H_{\alpha\bar{N}}\bar{H}_{N\bar{\alpha}}}{H_{\alpha\bar{\alpha}}}, \quad H_{\alpha\bar{\beta}} = \frac{H_{\alpha}H_{\bar{\beta}}}{K}, \quad \text{etc.} \tag{27}$$

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In these terms the $su(1.N)$ algebra relations read

$$\{H,K\}=-D,\quad \{H,D\}=-2H,\quad \{K,D\}=2K, \tag{28}$$

$$\{H,H_\alpha\}=-H_{\alpha\bar{N}},\quad \{H,H_{\alpha\bar{N}}\}=\{H,H_{\alpha\bar{\beta}}\}=0, \tag{29}$$

$$\{K,H_{\alpha\bar{N}}\}=H_\alpha,\quad \{K,H_\alpha\}=\{K,H_{\alpha\bar{\beta}}\}=0, \tag{30}$$

$$\{D,H_\alpha\}=-H_\alpha,\quad \{D,H_{\alpha\bar{N}}\}=H_{\alpha\bar{N}},\quad \{D,H_{\alpha\bar{\beta}}\}=0, \tag{31}$$

$$\{H_\alpha,H_\beta\}=\{H_{\alpha\bar{N}},H_{\beta\bar{N}}\}=\{H_\alpha,H_{\beta\bar{N}}\}=0, \tag{32}$$

$$\{H_\alpha,H_{\bar{\beta}}\}=-\imath K\delta_{\alpha\bar{\beta}},\quad \{H_{\alpha\bar{N}},H_{N\bar{\beta}}\}=-\imath H\delta_{\alpha\bar{\beta}}, \tag{33}$$

$$\{H_{\alpha\bar{\beta}},H_{\gamma\bar{\delta}}\}=\imath(H_{\alpha\bar{\delta}}\delta_{\gamma\bar{\beta}}-H_{\gamma\bar{\beta}}\delta_{\alpha\bar{\delta}}), \tag{34}$$

$$\{H_\alpha,H_{N\bar{\beta}}\}=H_{\alpha\bar{\beta}}+\frac{1}{2}\left(g+\sum\nolimits_\gamma H_{\gamma\bar{\gamma}}-\imath D\right)\delta_{\alpha\bar{\beta}}, \tag{35}$$

$$\{H_\alpha,H_{\beta\bar{\gamma}}\}=-\imath H_\beta\delta_{\alpha\bar{\gamma}},\quad \{H_{\alpha\bar{N}},H_{\beta\bar{\gamma}}\}=-\imath H_{\beta\bar{N}}\delta_{\alpha\bar{\gamma}}. \tag{36}$$

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$$\{H_\alpha, H_{\bar{\beta}}\} = -\imath K\delta_{\alpha\bar{\beta}}, \quad \{H_{\alpha\bar{N}}, H_{N\bar{\beta}}\} = -\imath H\delta_{\alpha\bar{\beta}}, \tag{33}$$

$$\{H_{\alpha\bar{\beta}}, H_{\gamma\bar{\delta}}\} = \imath(H_{\alpha\bar{\delta}}\delta_{\gamma\bar{\beta}} - H_{\gamma\bar{\beta}}\delta_{\alpha\bar{\delta}}), \tag{34}$$

$$\{H_\alpha, H_{N\bar{\beta}}\} = H_{\alpha\bar{\beta}} + \frac{1}{2}\left(g + \sum_\gamma H_{\gamma\bar{\gamma}} - \imath D\right)\delta_{\alpha\bar{\beta}}, \tag{35}$$

$$\{H_\alpha, H_{\beta\bar{\gamma}}\} = -\imath H_\beta\delta_{\alpha\bar{\gamma}}, \quad \{H_{\alpha\bar{N}}, H_{\beta\bar{\gamma}}\} = -\imath H_{\beta\bar{N}}\delta_{\alpha\bar{\gamma}}. \tag{36}$$

The generators $H, K, D$ define the conformal algebra $su(1.1) = so(1.2)$, and the generators $H_{\alpha\bar{\beta}}$ define the algebra $u(N-1)$.

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- the triples $(H, H_{N\alpha}, H_{\alpha\bar{\beta}})$ and $(K, H_{\alpha}, H_{\alpha\bar{\beta}})$ transform into each other within discrete transformation

$$(w, z^{\alpha}) \to (-\frac{1}{w}, \frac{z^{\alpha}}{w}) \Rightarrow \begin{cases} D \to -D, \\ (H, H_{N\alpha}, H_{\alpha\bar{\beta}}) \to (K, -H_{\alpha}, H_{\alpha\beta}), \\ (K, H_{\alpha}, H_{\alpha\beta}) \to (H, H_{N\alpha}, H_{\alpha\bar{\beta}}) \end{cases}$$

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- the Hamiltonian $H$ has two sets of constants of motion $H_{N\alpha}$ and $H_{\alpha\bar{\beta}}$. Therefore it defines superintegrable system;
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Adding to the Hamiltonian $H$ the appropriate function of $K$, we get the superintegrable oscillator- and Coulomb-like systems.

# Oscillator-like Hamiltonian

We define the oscillator-like Hamiltonian by the expression

$$H_{osc} = H + \omega^2 K \tag{37}$$

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and introduce the following generators

$$A_\alpha = H_{\alpha\bar{N}} + \imath\omega H_\alpha, \quad B_\alpha = H_{\alpha\bar{N}} - \imath\omega H_\alpha \ : \left\{ \begin{array}{c} \{H_{osc}, A_\alpha\} = -\imath\omega A_\alpha \\ \{H_{osc}, B_\alpha\} = \imath\omega B_\alpha \end{array} \right. \tag{38}$$

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These generators and their complex conjugates form the following algebra

$$\{A_\alpha, \bar{A}_\beta\} = -\imath\big(H_{osc} - \omega(g + \textstyle\sum_{\gamma=1}^{N-1} H_{\gamma\bar{\gamma}})\big)\delta_{\alpha\bar{\beta}} + 2\imath\omega H_{\alpha\bar{\beta}}, \tag{39}$$

$$\{B_\alpha, \bar{B}_\beta\} = -\imath\big(H_{osc} + \omega(g + \textstyle\sum_{\gamma=1}^{N-1} H_{\gamma\bar{\gamma}})\big)\delta_{\alpha\bar{\beta}} - 2\imath\omega H_{\alpha\bar{\beta}}, \tag{40}$$

$$\{A_\alpha, \bar{B}_\beta\} = -\imath\delta_{\alpha\bar{\beta}}\big(H_{osc} - 2\omega^2 K + \imath\omega D\big), \tag{41}$$

with their Poisson brackets with $H_{\alpha\bar{\beta}}$ reading

$$\{A_\alpha, H_{\beta\bar{\gamma}}\} = -\imath\delta_{\alpha\bar{\gamma}} A_\beta, \qquad \{B_\alpha, H_{\beta\bar{\gamma}}\} = -\imath\delta_{\alpha\bar{\gamma}} B_\beta \tag{42}$$

# Oscillator-like Hamiltonian

Then we immediately deduce that the Hamiltonian besides $H_{\alpha\bar{\beta}}$, has the additional constants of motion which provide the system by the maximal superintegrability property

$$M_{\alpha\beta} = A_\alpha B_\beta : \quad \{H_{osc}, M_{\alpha\beta}\} = 0. \tag{43}$$

$$M_{\alpha\beta} = H_{\alpha\bar{N}} H_{\beta\bar{N}} + \omega^2 H_\alpha H_\beta + \imath\omega(H_\alpha H_{\beta\bar{N}} - H_{\alpha\bar{N}} H_\beta) = \frac{\bar{z}^\alpha \bar{z}^\beta}{A^2}(w^2 + \omega^2)$$

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These constants of motion are functionally dependent, so that among them one can choose the $N-1$ integrals which guarantee superintegrability of the system.

# Oscillator-like Hamiltonian

Then we immediately deduce that the Hamiltonian besides $H_{\alpha\bar{\beta}}$, has the additional constants of motion which provide the system by the maximal superintegrability property

$$M_{\alpha\beta} = A_\alpha B_\beta : \quad \{H_{osc}, M_{\alpha\beta}\} = 0. \tag{43}$$

$$M_{\alpha\beta} = H_{\alpha\bar{N}} H_{\beta\bar{N}} + \omega^2 H_\alpha H_\beta + \imath\omega(H_\alpha H_{\beta\bar{N}} - H_{\alpha\bar{N}} H_\beta) = \frac{\bar{z}^\alpha \bar{z}^\beta}{A^2}(w^2 + \omega^2)$$

These constants of motion are functionally dependent, so that among them one can choose the $N-1$ integrals which guarantee superintegrability of the system.

These generators and the $H_{\alpha\beta}$ form the following symmetry algebra

$$\{H_{\alpha\bar{\beta}}, M_{\gamma\delta}\} = \imath\delta_{\bar{\beta}\gamma} M_{\alpha\delta} + \imath\delta_{\bar{\beta}\delta} M_{\gamma\alpha}, \quad \{M_{\alpha\beta}, M_{\gamma\delta}\} = 0, \tag{44}$$

$$\{M_{\alpha\beta}, \overline{M}_{\gamma\delta}\} = 4\imath\left(\left(g + \sum_{\sigma=1}^{N-1} H_{\sigma\bar{\sigma}}\right) H_{\alpha\bar{\gamma}} H_{\beta\bar{\delta}} - \frac{M_{\alpha\beta}\bar{M}_{\gamma\delta}}{\sum_{\sigma=1}^{N-1} H_{\sigma\bar{\sigma}}}\right). \tag{45}$$

# Coulomb-like Hamiltonian

We define the Coulomb-like Hamiltonian with the additional constants of motion which provide the system by the maximal superinetgrability property as follows

$$H_{Coul} = H - \frac{\gamma}{\sqrt{2K}}, \quad R_\alpha = H_{\alpha\bar{N}} + \imath\gamma \frac{H_\alpha}{(g + \sum_{\gamma=1}^{N-1} H_{\gamma\bar{\gamma}})\sqrt{2K}} \quad : \tag{46}$$

$$\{H_{Coul}, R_\alpha\} = \{H_{Coul}, H_{\alpha\bar{\beta}}\} = 0. \tag{47}$$

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The whole symmetry algebra is as follows

$$\begin{gathered}\{R_\alpha, R_{\bar{\beta}}\} = -\imath\delta_{\alpha\bar{\beta}}\left(H_{Coul} - \frac{\imath\gamma^2}{2(g+\sum_{\gamma=1}^{N-1} H_{\gamma\bar{\gamma}})^2}\right) + \frac{\imath\gamma^2 H_{\alpha\bar{\beta}}}{2(g+\sum_{\gamma=1}^{N-1} H_{\gamma\bar{\gamma}})^3}, \\ \{R_\alpha, R_\beta\} = 0, \qquad \{R_\alpha, H_{\beta\bar{\gamma}}\} = -\imath\delta_{\alpha\bar{\gamma}} R_\beta.\end{gathered} \tag{48}$$

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$$\{R_\alpha, R_\beta\} = 0, \qquad \{R_\alpha, H_{\beta\bar{\gamma}}\} = -\imath\delta_{\alpha\bar{\gamma}} R_\beta.$$

To clarify the origin of these models it is convenient to transit to the canonical coordinates.

# Canonical coordinates

We parameterize the complex coordinates $w, z^\alpha$ by real $p_r, r, \pi_\alpha, \phi_\alpha$ ones, using convention $\{r, p_r\} = 1, \quad \{\phi_\alpha, \pi_\beta\} = \delta_{\alpha\beta}$ in yhe following way

$$w = \frac{p_r}{r} - \imath\frac{\pi + g}{r^2}, \quad z^\alpha = \frac{\sqrt{2\pi_\alpha}}{r}\mathrm{e}^{\imath\varphi_\alpha}, \quad \text{with} \quad \begin{array}{c} \pi_\alpha \geq 0, \quad \varphi_\alpha \in [0, 2\pi), \\ r > 0. \end{array} \tag{49}$$

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In these terms the generators of conformal algebra take the form of conformal mechanics with separated "radial" and "angular" parts

$$H = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2}, \quad K = \frac{r^2}{2}, \quad D = p_r r, \tag{50}$$

where the angular part of Hamiltonian is given by the expression

$$\mathcal{I} = \frac{1}{2}\left(\sum_{\alpha=1}^{N-1}\pi_\alpha + g\right)^2 . \tag{51}$$

# Canonical coordinates

The rest generators of $su(1.N)$ algebra read

$$H_{\alpha\bar{N}} = \sqrt{2\pi_\alpha}\left(\frac{p_r}{2} - \imath\frac{\pi + g}{2r}\right)\mathrm{e}^{-\imath\varphi_\alpha}, \quad H_\alpha = r\sqrt{\frac{\pi_\alpha}{2}}\mathrm{e}^{-\imath\varphi_\alpha}, \tag{52}$$

$$H_{\alpha\bar{\beta}} = \sqrt{\pi_\alpha\pi_\beta}\mathrm{e}^{-\imath\left(\varphi_\alpha - \varphi_\beta\right)}, \tag{53}$$

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$$H_{\alpha\bar{\beta}} = \sqrt{\pi_\alpha\pi_\beta}\mathrm{e}^{-\imath\left(\varphi_\alpha-\varphi_\beta\right)}, \tag{53}$$

In these coordinates the oscillator- and Coulomb-like Hamiltonians take the form,

$$H_{osc} = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2} + \frac{\omega^2 r^2}{2}, \quad H_{Coul} = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2} - \frac{\gamma}{r}, \tag{54}$$

with angular part $\mathcal{I}$ given as above $\qquad \mathcal{I} = \frac{1}{2}\left(\sum_{\alpha=1}^{N-1}\pi_\alpha + g\right)^2$

# Superintegrable Systems

In accordance with Liouville theorem, the integrability of the system with $2N$-dimensional phase space means the existence $N$ functionally independent involutive integrals $F_1 = H, \ldots, F_N : \{F_a, F_b\} = 0$. This yields the existence of the so-called action-angle variables $(I_a(F), \Phi_a)$:

$$H = H(I), \quad \{I_a, \Phi_b\} = \delta_{ab}, \quad \{I_a, I_b\} = \{\Phi_a, \Phi_b\} = 0, \qquad \Phi_a \in [0, 2\pi), \tag{55}$$

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The system becomes maximally superintegrable when the Hamiltonian is expressed via action variables as follows

$$H = H\left(\sum_{a=1}^{N} n_a I_a\right), \quad n_a \in \mathcal{N} \tag{56}$$

where $n_a$ are integers (or rational numbers). Indeed, in that case the system possesses the additional (non-involutive) integrals $I_{ab} = \cos(n_a\Phi_b - n_b\Phi_a)$, among them $N-1$ integrals are functionally independent.

# Superintegrable Systems

Now, let us suppose that $\pi_\alpha, \varphi_\alpha$ are related with the action-angle variables $(I_\alpha, \Phi_\alpha)$ of some $(N-1)$-dimensional angular mechanics by the relations

$$\pi_\alpha = n_\alpha I_\alpha, \quad \varphi_\alpha = \frac{\Phi_\alpha}{n_\alpha}, \qquad \text{where} \quad n_\alpha \in \mathcal{N}. \tag{57}$$

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$$\mathcal{I} = \frac{1}{2}\left(\sum_{\alpha=1}^{N-1} n_\alpha I_\alpha + g\right)^2, \qquad \text{with} \quad n_\alpha \in \mathcal{N}, \tag{58}$$

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T. Hakobyan, O. Lechtenfeld and A. Nersessian, *Superintegrability of generalized Calogero models with oscillator or Coulomb potential*, Phys. Rev. D 90 (2014)

Though the algebraic relations hold upon this identification, the generators $H_\alpha, H_{\alpha\bar{N}}, H_{\alpha\bar{\beta}}$ become locally defined, $\varphi_\alpha \in [0, 2\pi/m_\alpha)$, so they fail to be constants of motion. However, taking their relevant powers we get the globally defined generators which form the nonlinear algebra

$$\widetilde{H}_\alpha := (H_\alpha)^{n_\alpha} = d_\alpha(I) r^{n_\alpha} \mathrm{e}^{-\imath\Phi_\alpha}, \tag{59}$$

$$\widetilde{H}_{\alpha\bar{N}} := (H_{\alpha\bar{N}})^{n_\alpha} = d_{\alpha\bar{N}}(I) \left( p_r - \imath \frac{\sum_{\gamma=1}^{N-1} n_\gamma I_\gamma + g}{r} \right)^{n_\alpha} \mathrm{e}^{-\imath\Phi_\alpha}, \tag{60}$$

$$\widetilde{H}_{\alpha\bar{\beta}} := (H_{\alpha\bar{\beta}})^{n_\alpha n_\beta} = d_{\alpha\bar{\beta}}(I) \mathrm{e}^{-\imath\left(n_\beta\Phi_\alpha - n_\alpha\Phi_\beta\right)}, \tag{61}$$

where

$$d_\alpha(I) = \left(\frac{n_\alpha I_\alpha}{2}\right)^{n_\alpha/2}, \quad d_{\alpha\bar{N}}(I) = \left(\frac{n_\alpha I_\alpha}{2}\right)^{n_\alpha/2}, \tag{62}$$

$$d_{\alpha\bar{\beta}}(I) = (n_\alpha n_\beta I_\alpha I_\beta)^{n_\alpha n_\beta/2}. \tag{63}$$

# Superintegrable Systems

Thus, we get

$$\{H, \widetilde{H}_{\alpha\bar{N}}\} = \{H, \widetilde{H}_{\alpha\beta}\} = 0, \qquad \{K, \widetilde{H}_{\alpha}\} = \{K, \widetilde{H}_{\alpha\beta}\} = 0, \qquad (64)$$

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In a similar way we construct the constants of motion of the oscillator- and Coulomb-like systems, respectively.
For the oscillator-like system the integrals take the form

$$\widetilde{M}_{\alpha\beta} := (A_\alpha B_\beta)^{n_\alpha n_\beta} = \frac{1}{2} d_{\alpha\bar{\beta}}(I) \mathrm{e}^{-\imath\left(n_\beta \Phi_\alpha - n_\alpha \Phi_\beta\right)} \left( \left( \imath p_r + \frac{\sum_{\gamma=1}^{N-1} n_\gamma I_\gamma + g}{r} \right)^2 - \omega^2 r^2 \right)^{n_\alpha n_\beta}, \tag{65}$$

For the Coulomb-like system the integrals take the form

$$\widetilde{R}_\alpha = (R_\alpha)^{n_\alpha} = d_\alpha(I) \mathrm{e}^{-\imath\Phi_\alpha} \left( p_r + \frac{\imath\gamma}{\sum_{\gamma=1}^{N-1} n_\gamma I_\gamma + g} - \frac{\imath\left(\sum_{\gamma=1}^{N-1} n_\gamma I_\gamma + g\right)}{r} \right)^{n_1} \tag{66}$$

# Conclusion

In this paper we have shown that the superintegrable generalizations of conformal mechanics, oscillator and Coulomb systems can be naturally described in terms of the noncompact complex projective space considered as a phase space.

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This observation yields some interesting directions for further studies, among them.

- the construction of the $\mathcal{N} = 2k$ superconformal mechanics associated with $su(1.N|k)$ superalgebra. For this purpose one should consider phase superspace equipped with the Kähler structure with the potential

$$\mathcal{K} = -g\log(\imath(w-\bar{w}) - z^{\alpha}\bar{z}^{\alpha} - \imath\eta_A\bar{\eta}_A), \quad A = 1,\dots k, \qquad (67)$$

where $\eta_A$ are Grassmann variables.

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  where $\eta_A$ are Grassmann variables.This should be the direct generalization of the one-dimensional system considered in T. Hakobyan and A. Nersessian, *Lobachevsky geometry of (super)conformal mechanics*, Phys. Lett. A 373 (2009)

# Conclusion

- We expect that it will be possible to construct, in a similar way, the $\mathcal{N} = 2k$ supersymmetric extensions of the considered oscillator- and (repulsive) Coulomb-like systems as well, in particular, the superextension of Smorodinsky-Winternitz system.

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- Performing the transformation to the higher-dimensional Poincare model, we expect to present the considered models in the Ruijsenaars-Schneider-like form and in this way to find, some superinegrable cases of the Ruijsenaars-Schneider systems, as well as their supersymmetric/superconformal extensions

# Conclusion

- describing the superintegrable deformations of the free particle on the spheres/hyperboloids, and the spherical/hyperbolic oscillators, in a similar way. For this purpose we expect to consider the "$\kappa$-deformation" of the Kähler structure of the Klein model, in the spirit of the so-called "$\kappa$-deformation approach" developed in M. F. Ranada, *The Tremblay-Turbiner-Winternitz system on spherical and hyperbolic spaces: Superintegrability, curvature- dependent formalism and complex factorization*, J. Phys. A 47 (2014) *A new approach to the higher order superintegrability of the Tremblay-Turbiner-Winternitz system, J. Phys. A 45 (2012)* *Higher order superintegrability of separable potentials with a new approach to the Post-Winternitz system, J. Phys. A 46 (2013)*

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- constructing spin-extensions of the above models, choosing the noncompact analogs of complex Grassmanians as phase spaces.

# The End

**Thank You!**