New Integrable Coset Sigma Models

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Affine Gaudin models

Many classical integrable field theories can be viewed as specific realizations of dihedral affine Gaudin models associated to an untwisted affine Kac-Moody algebra supplied with an action of the dihedral group

Vicedo, 2017

Examples

- *Principle Chiral Model*
- *Deformations of the Principle Chiral Model*
- *Affine Toda field theories*
- *…*

Idea: *Use affine Gaudin models to construct new integrable field theories*

Delduc, Lacroix, Magro and Vicedo, 1811.12316, 1903.00368

Outline

Results

- *New coset integrable sigma models - Lagrangian*
- *Integrable sigma models on* $T^{1,1}$

Derivation

- *Canonical fields on* T^*G
- *Affine Gaudin models and their realization*
- *Lagrangian formulation*

Future directions

with Cristian Bassi & Sylvain Lacroix, 2010.05573 [hep-th]

Results

Set up

G is a real semi-simple Lie group, $\mathfrak{g}$ is the Lie algebra

κ is a non-degenerate ad-invariant bilinear form on $\mathfrak{g}$, opposite of the Killing form

σ is an automorphism of order T of G and $G^{(0)} \subset G$ is a subgroup of fixed points

$$\mathfrak{g} = \oplus_{k=0}^{T-1} \mathfrak{g}^{(k)}, \qquad \mathfrak{g}^{(k)} = \pi^{(k)} \mathfrak{g}$$

Classical r-matrix

split Casimir

$$\mathcal{R}^0_{\underline{\mathbf{12}}}(w,z) = \sum_{k=0}^{T-1} \frac{w^k z^{T-1-k}}{z^T - w^T} \pi^{(k)}_{\underline{\mathbf{1}}} C_{\underline{\mathbf{12}}}$$

Coset space

$$G^N \Big/ G^{(0)}_{\text{diag}} = \overbrace{G \times G \times \ldots \times G}^{N} \Big/ G^{(0)}_{\text{diag}}$$

The coset is formed by acting on $(g_1, \cdots, g_N) \in G^N$ by right translation of the diagonal subgroup

$$G^{(0)}_{\text{diag}} = \{(h, \cdots, h),\ h \in G^{(0)}\}$$

New integrable gauge coset sigma models

$$\boxed{S=\sum_{r=1}^{N} S_{\mathrm{WZW},\,\Bbbk_r}[g_r]-\frac{KT^3}{2}\iint \mathrm{d}x\,\mathrm{d}t\sum_{r,s=1}^{N}\operatorname*{res}_{w=z_s}\operatorname*{res}_{z=z_r}\kappa_{\underline{\mathbf{12}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(w,z)\varphi_+(z)\varphi_-(w),\,j_{+,r\underline{\mathbf{1}}}\,j_{-,s\underline{\mathbf{2}}}\Big)}$$

Wess – Zumino action $\quad S_{\mathrm{WZW},\,\Bbbk}[g]=\frac{\Bbbk}{2}\iint \mathrm{d}x\,\mathrm{d}t\;\kappa\left(g^{-1}\partial_+g,g^{-1}\partial_-g\right)+\Bbbk\,I_{\mathrm{WZ}}[g]$

Currents $\quad j_{\pm,r}(x)=g_r^{-1}(x)\partial_\pm g_r(x) \qquad r=1,\ldots,N$

Functions containing parameters

$$\varphi_+(z)=\frac{\prod_{i=N}^{2N-2}\left(z^T-\zeta_i^T\right)}{\prod_{r=1}^{N}\left(z^T-z_r^T\right)} \qquad \text{and} \qquad \varphi_-(z)=\frac{z^{T-1}\prod_{i=1}^{N-1}\left(z^T-\zeta_i^T\right)}{\prod_{r=1}^{N}\left(z^T-z_r^T\right)}$$

$$\varphi(z)=TK\varphi_+(z)\varphi_-(z) \longleftarrow \text{twist function}$$

$$\Bbbk_r = -\tfrac{T}{2}\operatorname{res}_{z=z_r}\varphi(z)\mathrm{d}z$$

The model depends on $3N-2$ continuous free parameters $(2N-2\ \zeta's,\, N-1\ z's$ and $K)$

For $N=1$ one gets the well-known gauge sigma model on $G/G^{(0)}$ with the Wess-Zumino term

Limiting case

$N = 2$

$$z_2 = \frac{1}{\alpha}, \qquad K = \frac{\lambda_2^2}{\alpha^2}, \qquad \zeta_+ = \frac{\lambda_1}{\lambda}, \qquad \zeta_- = \frac{\lambda}{\lambda_2 \alpha} \qquad \alpha \to 0 \qquad \lambda_1,\, \lambda_2 \text{ and } \lambda \text{ fixed}$$

$$S[g_1, g_2] = \iint \mathrm{d}x\,\mathrm{d}t \sum_{r=1}^{2} \left(\frac{\lambda^2}{2}\, \kappa\left(j^{(0)}_{+,r}, j^{(0)}_{-,r}\right) + \frac{\lambda_r^2}{2}\, \kappa\left(j^{(1)}_{+,r}, j^{(1)}_{-,r}\right)\right) - \lambda^2\, \kappa\left(j^{(0)}_{+,2}, j^{(0)}_{-,1}\right)$$
$$+ \lambda^2\, I_{\mathrm{WZ}}[g_1] - \lambda^2\, I_{\mathrm{WZ}}[g_2].$$

Define $U = g_1$ and $V = g_2^{-1}$, and use $I_{\mathrm{WZ}}[g^{-1}] = -I_{\mathrm{WZ}}[g]$

In the case $\lambda_1 = \lambda_2 = \lambda$ the action can be rewritten as

$$S[U,V] = S_{\mathrm{WZW},\,\lambda^2}[U] + S_{\mathrm{WZW},\,\lambda^2}[V] + \lambda^2 \iint \mathrm{d}x\,\mathrm{d}t\ \kappa\left((\partial_+ V V^{-1})^{(0)}, (U^{-1}\partial_- U)^{(0)} \right)$$

Guadagnini – Martellini – Mintchev model (GMM)

2dim CFT

$$\partial_+ \mathcal{L}_-(z) - \partial_- \mathcal{L}_+(z) + [\mathcal{L}_+(z), \mathcal{L}_-(z)] = 0,$$
$$\partial_+ \tilde{\mathcal{L}}_-(z) - \partial_- \tilde{\mathcal{L}}_+(z) + [\tilde{\mathcal{L}}_+(z), \tilde{\mathcal{L}}_-(z)] = 0$$

$$\mathcal{L}_+(z) = j^{(0)}_{+,2}, \qquad \mathcal{L}_-(z) = j^{(0)}_{-,1} + \frac{j^{(1)}_{-,1}}{z},$$
$$\tilde{\mathcal{L}}_+(z) = j^{(0)}_{+,2} + z\, j^{(1)}_{+,2}, \qquad \tilde{\mathcal{L}}_-(z) = j^{(0)}_{-,1}.$$

$\leftarrow$ Lax connection

Integrable sigma models on $T^{1,1}$

$G = SU(2)$ with $\mathfrak{g} = \mathfrak{su}(2)$ generated by $I_a = i\sigma_a/2$, where σ_a is the a-th Pauli matrix

$$\sigma : \quad \sigma(I_1) = -I_1,\ \sigma(I_2) = -I_2 \text{ and } \sigma(I_3) = I_3$$

$$\mathfrak{g}^{(0)} = \mathfrak{u}(1) = \text{span}\{I_3\}$$
$$G^{(0)} = U(1) = \exp(\mathbb{R}I_3)$$

Pick the following parametrisation for the fields $(g_1, g_2) \in SU(2) \times SU(2)$

$$g_1 = \exp(\phi_1 I_3) \exp(\theta_1 I_2) \exp(\psi I_3),$$
$$g_2 = \exp(-\phi_2 I_3) \exp(-\theta_2 I_2) \exp(-\tilde{\psi} I_3)$$

Action

$$S = \frac{1}{4}\iint \mathrm{d}x\,\mathrm{d}t\Big(\big(\lambda^2 + \lambda_1^2 + (\lambda^2 - \lambda_1^2)\cos(2\theta_1)\big)\partial_-\phi_1\partial_+\phi_1 + 2\lambda_1^2\,\partial_-\theta_1\partial_+\theta_1 + 2\lambda^2\,\partial_-\psi\partial_+\psi + 4\lambda^2\,\partial_-\phi_1\partial_+\psi\cos\theta_1$$
$$+ \big(\lambda^2 + \lambda_2^2 + (\lambda^2 - \lambda_2^2)\cos(2\theta_2)\big)\partial_-\phi_2\partial_+\phi_2 + 2\lambda_2^2\,\partial_-\theta_2\partial_+\theta_2 + 2\lambda^2\,\partial_-\tilde{\psi}\partial_+\tilde{\psi} + 4\lambda^2\,\partial_-\tilde{\psi}\partial_+\phi_2\cos\theta_2$$
$$+ 4\lambda^2\big(\cos\theta_1\,\partial_-\phi_1 + \partial_-\psi\big)\big(\cos\theta_2\,\partial_+\phi_2 + \partial_+\tilde{\psi}\big)\Big)$$

Gauge symmetry

$$g_r \mapsto g_r h,\ h \in U(1) \quad \Longrightarrow \quad (\psi, \tilde{\psi}) \mapsto (\psi + \eta, \tilde{\psi} - \eta) \quad \Longrightarrow \quad \text{set } \tilde{\psi} = 0.$$

Integrable sigma models on $T^{1,1}$

$$S = \frac{1}{2} \iint \mathrm{d}x\, \mathrm{d}t \ \left(G_{ij} + B_{ij}\right) \partial_- y^i \partial_+ y^j$$

Topologically $\quad SU(2) \times SU(2) \big/ U(1)$

$$ds^2 = G_{ij} \mathrm{d}y^i \mathrm{d}y^j = \lambda_1^2 (\mathrm{d}\theta_1^2 + \sin^2\theta_1\, \mathrm{d}\phi_1^2) + \lambda_2^2 (\mathrm{d}\theta_2^2 + \sin^2\theta_2\, \mathrm{d}\phi_2^2) + \lambda^2 (\mathrm{d}\psi + \cos\theta_1\, \mathrm{d}\phi_1 + \cos\theta_2\, \mathrm{d}\phi_2)^2.$$

$$B = \frac{1}{2} B_{ij}\, \mathrm{d}y^i \wedge \mathrm{d}y^j = \lambda^2 (\cos\theta_1\, \mathrm{d}\phi_1 + \mathrm{d}\psi) \wedge (\cos\theta_2\, \mathrm{d}\phi_2 + \mathrm{d}\psi).$$

$\lambda_1^2 = \lambda_2^2 = 3\lambda^2/2 \ \leftarrow$ Einstein manifold

$\lambda_1 = \lambda_2 = \lambda \quad \leftarrow$ conformal GMM model

Explicit form of the Lax connection in terms of

$$(\theta_1, \theta_2, \phi_1, \phi_2, \psi)$$

$$\widehat{\mathcal{L}}_\pm = \widehat{\mathcal{L}}^a_\pm I_a \qquad \widetilde{\mathcal{L}}_\pm = \widetilde{\mathcal{L}}^a_\pm I_a$$

$$\partial_+\hat{\mathcal{L}}_-(z) - \partial_-\hat{\mathcal{L}}_+(z) + [\hat{\mathcal{L}}_+(z), \hat{\mathcal{L}}_-(z)] = 0,$$
$$\partial_+\tilde{\mathcal{L}}_-(z) - \partial_-\tilde{\mathcal{L}}_+(z) + [\tilde{\mathcal{L}}_+(z), \tilde{\mathcal{L}}_-(z)] = 0$$

$$\widehat{\mathcal{L}}^1_+ = \frac{(\lambda^2-\lambda_1^2)z}{\lambda^2 z^2 - \lambda_1^2}\sin\theta_1\,\partial_+\phi_1, \qquad \widehat{\mathcal{L}}^2_+ = \frac{(\lambda^2-\lambda_1^2)z}{\lambda^2 z^2 - \lambda_1^2}\partial_+\theta_1,$$
$$\widehat{\mathcal{L}}^3_+ = \frac{1}{\lambda^2 z^2 - \lambda_1^2}\big((\lambda^2-\lambda_1^2)\cos\theta_1\,\partial_+\phi_1 - \lambda^2(z^2-1)(\cos\theta_2\,\partial_+\phi_2 + \partial_+\psi)\big),$$
$$\widehat{\mathcal{L}}^1_- = \frac{\sin\theta_1\,\partial_-\phi_1}{z}, \qquad \widehat{\mathcal{L}}^2_- = \frac{\partial_-\theta_1}{z}, \qquad \widehat{\mathcal{L}}^3_- = \cos\theta_1\,\partial_-\phi_1$$

$$\widetilde{\mathcal{L}}^1_+ = z\sin\theta_2\,\partial_+\phi_2, \qquad \widetilde{\mathcal{L}}^2_+ = -z\,\partial_+\theta_2, \qquad \widetilde{\mathcal{L}}^3_+ = -\cos\theta_2\,\partial_+\phi_2,$$
$$\widetilde{\mathcal{L}}^1_- = -\frac{(\lambda^2-\lambda_2^2)z}{\lambda_2^2 z^2 - \lambda^2}\sin\theta_2\,\partial_-\phi_2, \qquad \widetilde{\mathcal{L}}^2_- = \frac{(\lambda^2-\lambda_2^2)z}{\lambda_2^2 z^2 - \lambda^2}\partial_-\theta_2,$$
$$\widetilde{\mathcal{L}}^3_- = \frac{1}{\lambda_2^2 z^2 - \lambda^2}\big((\lambda^2-\lambda_2^2)z^2\cos\theta_2\,\partial_-\phi_2 + \lambda^2(z^2-1)(\cos\theta_1\,\partial_-\phi_1 + \partial_-\psi)\big)$$

Spinning strings on $T^{1,1}$

$$B = k\,(\cos\theta_1\,\mathrm{d}\phi_1 + \mathrm{d}\psi)\wedge(\cos\theta_2\,\mathrm{d}\phi_2 + \mathrm{d}\psi)$$

arbitrary

$k = 0$ embeds into $\mathrm{AdS}_5 \times \mathrm{T}^{1,1}$
near-horizon limit of the Klebanov-Witten setup

Spinning string ansatz

$$\theta_i = \theta_i(x), \qquad \phi_i = \omega_i t + \widetilde{\phi}_i(x), \qquad \psi = \psi(x)$$

Evolution equations for non-isometric coordinates

$$\ddot{\theta}_1 = \omega_1 \sin\theta_1 \left(\omega_1 \left(\left(\frac{\lambda^2}{\lambda_1^2} - 1\right) + \frac{k^2}{\lambda^2\lambda_1^2}\left(\left(1 - \frac{\lambda^2}{\lambda_1^2}\right) + \frac{\lambda_1^2}{\sin^4\theta_1}\right)\right)\cos\theta_1 - \omega_2 \frac{k^2-\lambda^4}{\lambda^2\lambda_1^2}\cos\theta_2\right),$$
$$\ddot{\theta}_2 = \omega_2 \sin\theta_2 \left(\omega_2 \left(\left(\frac{\lambda^2}{\lambda_2^2} - 1\right) + \frac{k^2}{\lambda^2\lambda_2^2}\left(\left(1 - \frac{\lambda^2}{\lambda_2^2}\right) + \frac{\lambda_2^2}{\sin^4\theta_2}\right)\right)\cos\theta_2 - \omega_1 \frac{k^2-\lambda^4}{\lambda^2\lambda_2^2}\cos\theta_1\right).$$

For $k = \lambda^2$ equations separate

Derivation

Canonical fields on $T^*G \simeq G \times \mathfrak{g}$

$\mathbb{D}$ is either the real line $\mathbb{R}$ or the circle S^1

Consider fields on $\mathbb{D}$ with values in $T^*G \simeq G \times \mathfrak{g}$

$$g(x), X(x), \quad \text{where} \quad x \in \mathbb{D}$$

$$\{g_{\underbrace{ij}_{\underline{1}}}(x), g_{\underbrace{kl}_{\underline{2}}}(y)\}$$

Canonical Poisson bracket

$$\begin{aligned} \{g_{\underline{\mathbf{1}}}(x), g_{\underline{\mathbf{2}}}(y)\} &= 0, \\ \{X_{\underline{\mathbf{1}}}(x), g_{\underline{\mathbf{2}}}(y)\} &= g_{\underline{\mathbf{2}}}(x) C_{\underline{\mathbf{12}}} \delta_{xy}, \\ \{X_{\underline{\mathbf{1}}}(x), X_{\underline{\mathbf{2}}}(y)\} &= [C_{\underline{\mathbf{12}}}, X_{\underline{\mathbf{1}}}(x)] \delta_{xy}, \end{aligned}$$

$$\delta_{xy} = \delta(x-y)$$

$(I_a)_{a \in \{1,\dots,n\}}$ is a basis of $\mathfrak{g}$

$(I^a)_{a \in \{1,\dots,n\}}$ is the dual of this basis with respect to κ

$$C_{\underline{\mathbf{12}}} = I_a \otimes I^a \in \mathfrak{g} \otimes \mathfrak{g} \quad \Leftarrow \text{split Casimir}$$

Canonical fields on $T^*G \simeq G \times \mathfrak{g}$

Define the following $\mathfrak{g}$-valued current:

$$j(x) = g^{-1}(x)\partial_x g(x)$$

It satisfies the Poisson brackets

$$\begin{aligned}
\{g_{\underline{\mathbf{1}}}(x), j_{\underline{\mathbf{2}}}(y)\} &= 0, \\
\{j_{\underline{\mathbf{1}}}(x), j_{\underline{\mathbf{2}}}(y)\} &= 0, \\
\{X_{\underline{\mathbf{1}}}(x), j_{\underline{\mathbf{2}}}(y)\} &= [C_{\underline{\mathbf{12}}}, j_{\underline{\mathbf{1}}}(x)]\delta_{xy} - C_{\underline{\mathbf{12}}}\delta'_{xy}
\end{aligned}$$

Momentum

$$\mathcal{P}_G = \int_{\mathbb{D}} \mathrm{d}x \; \kappa(j(x), X(x))$$

generates the spatial derivatives of the fields g and X

$$\{\mathcal{P}_G, g(x)\} = \partial_x g(x) \qquad \text{and} \qquad \{\mathcal{P}_G, X(x)\} = \partial_x X(x)$$

Canonical fields on $T^*G \simeq G \times \mathfrak{g}$

Wess-Zumino term and current $W(x)$

closed

$$I_{\mathrm{WZ}}[g] = \iiint_{\mathbb{B}} \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}\xi\;\kappa\Big(\big[g^{-1}\partial_x g, g^{-1}\partial_t g\big], g^{-1}\partial_\xi g\Big)$$

Extend the space-time $\mathbb{D} \times \mathbb{R}$ to a 3-dimensional manifold $\mathbb{B}$ with boundary $\partial\mathbb{B} = \mathbb{D} \times \mathbb{R}$

$$I_{\mathrm{WZ}}[g] = \iint_{\mathbb{D}\times\mathbb{R}} \mathrm{d}x\,\mathrm{d}t\;\kappa(W, g^{-1}\partial_t g)$$

W is a $\mathfrak{g}$-valued current depending g and its spatial derivatives

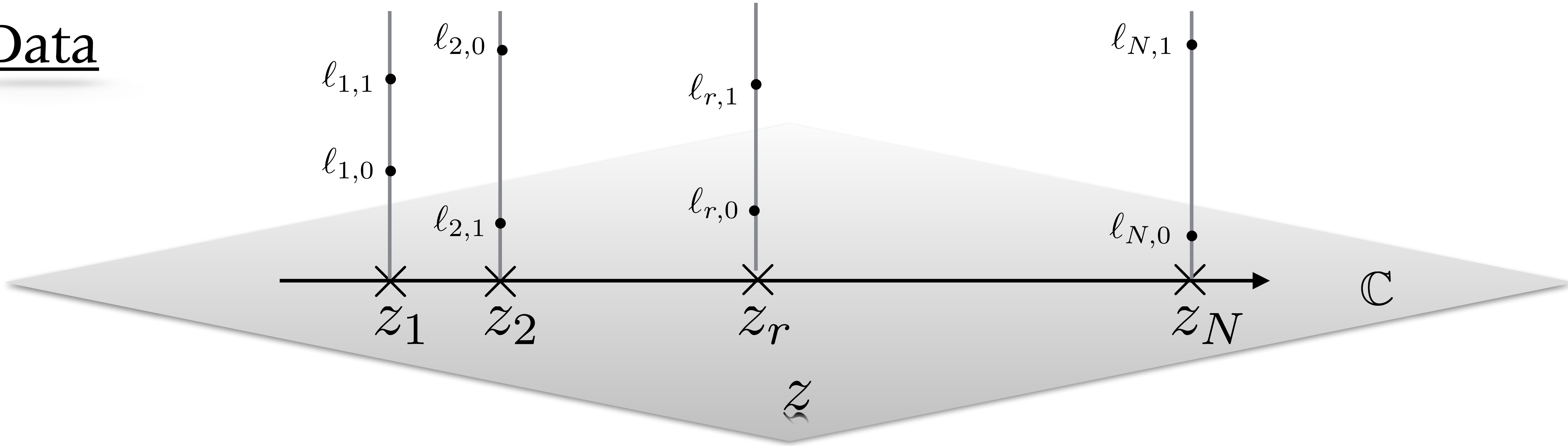
$$\{g_{\underline{\mathbf{1}}}(x), W_{\underline{\mathbf{2}}}(y)\} = 0, \qquad \{j_{\underline{\mathbf{1}}}(x), W_{\underline{\mathbf{2}}}(y)\} = 0$$

$$\{X_{\underline{\mathbf{1}}}(x), W_{\underline{\mathbf{2}}}(y)\} + \{W_{\underline{\mathbf{1}}}(x), X_{\underline{\mathbf{2}}}(y)\} = [C_{\underline{\mathbf{12}}}, W_{\underline{\mathbf{1}}}(x) - j_{\underline{\mathbf{1}}}(x)]\delta_{xy}$$

$$\kappa\big(j(x), W(x)\big) = 0$$

Affine Gaudin models $T = 2$

Data



$i)$ N sites at $\{z_1, \ldots, z_r, \ldots z_N\}$, $\quad z_r \in \mathbb{R}^*$

$ii)$ Each site has multiplicity two :

to each site two levels (real numbers) are associated $\ell_{r,0} \in \mathbb{R}$ and $\ell_{r,1} \in \mathbb{R}^*$

Data are encoded in the twist function

$$\varphi(z) = \frac{1}{2} \sum_{r=1}^{N} \sum_{p=0}^{1} \sum_{k=0}^{1} \frac{(-1)^k \ell_{r,p}}{((-1)^k z - z_r)^{p+1}}$$

The sum over $k \in \{0, 1\}$ and the factors $(-1)^k$ encodes the $T = 2$ dihedrality of the model

Takiff currents and phase space

Two each site one associates two $\mathfrak{g}$-valued fields $\mathcal{J}_{r,[0]}(x)$ and $\mathcal{J}_{r,[1]}(x)$, called Takiff currents

$$\begin{aligned}
\{\mathcal{J}_{r,[0]\underline{\mathbf{1}}}(x), \mathcal{J}_{s,[0]\underline{\mathbf{2}}}(y)\} &= \delta_{rs}\left([C_{\underline{\mathbf{12}}}, \mathcal{J}_{r,[0]\underline{\mathbf{1}}}(x)]\delta_{xy} - \ell_{r,0} C_{\underline{\mathbf{12}}}\delta'_{xy}\right)\\
\{\mathcal{J}_{r,[0]\underline{\mathbf{1}}}(x), \mathcal{J}_{s,[1]\underline{\mathbf{2}}}(y)\} &= \delta_{rs}\left([C_{\underline{\mathbf{12}}}, \mathcal{J}_{r,[1]\underline{\mathbf{1}}}(x)]\delta_{xy} - \ell_{r,1} C_{\underline{\mathbf{12}}}\delta'_{xy}\right)\\
\{\mathcal{J}_{r,[1]\underline{\mathbf{1}}}(x), \mathcal{J}_{s,[1]\underline{\mathbf{2}}}(y)\} &= 0.
\end{aligned}$$

Phase space of AGM

Impose the first-class condition

$$\sum_{r=1}^{N} \ell_{r,0} = 0$$

Gaudin Lax matrix

$$\Gamma(z,x) = \frac{1}{2}\sum_{r=1}^{N}\sum_{p=0}^{1}\sum_{k=0}^{1}\frac{(-1)^k\sigma^k\mathcal{J}_{r,[p]}(x)}{((-1)^k z - z_r)^{p+1}}$$

$$\{\Gamma_{\underline{\mathbf{1}}}(z,x),\Gamma_{\underline{\mathbf{2}}}(w,y)\} = [\mathcal{R}^0_{\underline{\mathbf{12}}}(z,w),\Gamma_{\underline{\mathbf{1}}}(z,x)]\delta_{xy} - [\mathcal{R}^0_{\underline{\mathbf{21}}}(w,z),\Gamma_{\underline{\mathbf{2}}}(w,x)]\delta_{xy}$$
$$- \Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(z,w)\varphi(z) + \mathcal{R}^0_{\underline{\mathbf{21}}}(w,z)\varphi(w)\Big)\,\delta'_{xy}$$

$$\mathcal{R}^0_{\underline{\mathbf{12}}}(z,w) = \frac{1}{2}\sum_{k=0}^{1}\frac{\sigma^k_{\underline{\mathbf{1}}}C_{\underline{\mathbf{12}}}}{w-(-1)^k z}$$

$\mathcal{R}^0_{\underline{\mathbf{12}}}$ satisfies the classical Yang-Baxter equation

$$[\mathcal{R}^0_{\underline{\mathbf{12}}}(z_1,z_2),\mathcal{R}^0_{\underline{\mathbf{13}}}(z_1,z_3)] + [\mathcal{R}^0_{\underline{\mathbf{12}}}(z_1,z_2),\mathcal{R}^0_{\underline{\mathbf{23}}}(z_2,z_3)] + [\mathcal{R}^0_{\underline{\mathbf{32}}}(z_3,z_2),\mathcal{R}^0_{\underline{\mathbf{13}}}(z_1,z_3)] = 0$$

$j_r = g_r^{-1}\partial_x g_r$

Realization of AGM

$$T^*G^N: \begin{cases} N\ G\text{-valued fields } g_1(x), \cdots, g_N(x) \\ N\ \mathfrak{g}\text{-valued fields } X_1(x), \cdots, X_N(x) \end{cases}$$

$$\mathcal{J}_{r,[0]}(x) = X_r(x) + \frac{\ell_{r,0}}{2} j_r(x) + \frac{\ell_{r,0}}{2} W_r(x)$$

$$\mathcal{J}_{r,[1]}(x) = \ell_{r,1}\, j_r(x)$$

WZ current

Takiff brackets are satisfied

$$\varphi(z) = \frac{1}{2}\sum_{r=1}^{N}\sum_{p=0}^{1}\sum_{k=0}^{1}\frac{(-1)^k \ell_{r,p}}{((-1)^k z - z_r)^{p+1}}$$

$\varphi(z)\mathrm{d}z$ has zeroes at zero and infinity and it has $4N$ poles

By Riemann-Hurwitz $\#\text{zeros} - \#\text{poles} = 2g - 2 = -2 \longrightarrow \#\text{zeros} = 4N - 2 = \underbrace{4(N-1)}_{\{\zeta_i, -\zeta_i\}, \quad i = 1, \ldots, 2N-2} + 2$

$$\boxed{\varphi(z) = 2K \frac{z\prod_{i=1}^{2N-2}(z^2 - \zeta_i^2)}{\prod_{r=1}^{N}(z^2 - z_r^2)^2}} \qquad K = \frac{1}{2}\sum_{r=1}^{N} z_r\,(z_r\,\ell_{r,0} + 2\,\ell_{r,1})$$

$$\mathcal{Q}(z) = -\frac{1}{2\varphi(z)}\int_{\mathbb{D}} \mathrm{d}x\ \kappa(\Gamma(z,x), \Gamma(z,x)) \qquad \leftarrow \text{ quadratic in currents}$$

Local charges in involution

$$\begin{cases} \mathcal{Q}_{\pm i} = \underset{z=\pm\zeta_i}{\mathrm{res}}\ \mathcal{Q}(z)\mathrm{d}z, \qquad i = 1, \ldots, 2N-2, \\ \mathcal{Q}_0 = \underset{z=0}{\mathrm{res}}\ \mathcal{Q}(z)\mathrm{d}z \\ \mathcal{Q}_\infty = \underset{z=\infty}{\mathrm{res}}\ \mathcal{Q}(z)\mathrm{d}z. \end{cases} \qquad \mathcal{Q}_i = \mathcal{Q}_{-i}$$

Naive Hamiltonian

$$\mathcal{H} = \epsilon_0 \mathcal{Q}_0 + 2\sum_{i=1}^{2N-2}\epsilon_i \mathcal{Q}_i + \epsilon_\infty \mathcal{Q}_\infty \qquad \leftarrow \text{real}$$

$\epsilon_i, \epsilon_\infty = \{\pm 1\} \quad \longrightarrow \quad$ relativistic invariance

Constraint

$$\mathcal{C}(x) = -\operatorname*{res}_{z=\infty} \Gamma(z,x)\mathrm{d}z = \lim_{u\to 0} \frac{1}{u}\Gamma\left(\frac{1}{u}, x\right)$$

$$\mathcal{C}(x) = \sum_{r=1}^{N} \mathcal{J}^{(0)}_{r,[0]}(x) \;\in \mathfrak{g}^{(0)}$$

The models we are interested in are defined on a reduced phase space, obtained from canonical fields on T^*G^N by imposing

$$\mathcal{C}(x) \approx 0$$

We have

$$\{\mathcal{Q}_i, \mathcal{C}(x)\} = 0, \qquad \forall\, i \in \{0, \cdots, 2N-2\}. \qquad\qquad \{\mathcal{Q}_\infty, \mathcal{C}(x)\} = \partial_x \mathcal{C}(x)$$

$$\{\mathcal{H}, \mathcal{C}(x)\} = \epsilon_\infty \partial_x \mathcal{C}(x) \qquad \Longrightarrow \qquad \{\mathcal{H}, \mathcal{C}(x)\} \approx 0$$

1st class

$$\{\mathcal{C}_{\underline{\mathbf{1}}}(x), \mathcal{C}_{\underline{\mathbf{2}}}(y)\} = \left[C^{(00)}_{\underline{\mathbf{12}}}, \mathcal{C}_{\underline{\mathbf{1}}}(x)\right]\delta_{xy}. \qquad\qquad C^{(00)}_{\underline{\mathbf{12}}} \in \mathfrak{g}^{(0)} \otimes \mathfrak{g}^{(0)}$$

$$-\left(\textstyle\sum_{r=1}^{N} \ell_{r,0}\right) C^{(00)}_{\underline{\mathbf{12}}} \delta'_{xy} \quad \to 0$$

Hamiltonian

$$\mathcal{H}_T = \mathcal{H} + \int_{\mathbb{D}} dx\, \kappa\big(\mu(x), \mathcal{C}(x)\big)$$

Lagrangian multiplier

Gauge symmetry

$$\delta_\epsilon \mathcal{O} = \left\{ \int_{\mathbb{D}} dx\, \kappa\big(\epsilon(x,t), \mathcal{C}(x)\big), \mathcal{O} \right\} = \int_{\mathbb{D}} dx\, \kappa\big(\epsilon(x,t), \{\mathcal{C}(x), \mathcal{O}\}\big) \qquad \epsilon(x,t) \in \mathfrak{g}^{(0)}$$

Infinitesimal :

$$\begin{cases} \delta_\epsilon g_r(x) = g_r(x)\epsilon(x,t) \\ \delta_\epsilon Y_r(x) = [Y_r(x), \epsilon(x,t)] + \dfrac{\ell_{r,0}}{2}\partial_x \epsilon(x,t) \end{cases} \qquad Y_r = X_r + \ell_{r,0} W_r/2$$

Global :

$$g_r \longmapsto g_r h \quad \text{and} \quad Y_r \longmapsto h^{-1} Y_r h + \frac{\ell_{r,0}}{2} h^{-1}\partial_x h \qquad h(x,t) \in G^{(0)}$$

The gauge symmetry acts on $(g_1, \cdots, g_N) \in G^N$ by right translation of the diagonal subgroup

$$G^{(0)}_{\text{diag}} = \big\{(h, \cdots, h),\ h \in G^{(0)}\big\}$$

Hamiltonian reduction

- Initial phase space $\mathcal{P} = T^* G^N$ with symmetry $G^{(0)}_{\text{diag, gauge}}$
- Moment map $\mathcal{C}(x)$
- Reduced phase space

$$\mathcal{P}_r = \left\{ T^* G^N ,\ \mathcal{C}(x) = 0 \right\} \Big/ G^{(0)}_{\text{diag, gauge}}$$

The "physical" coordinate fields of the model are fields on the quotient $G^N / G^{(0)}_{\text{diag}}$

The constraint $\mathcal{C}(x) = 0$ eliminates the corresponding superfluous conjugate momentum fields

$$\mathcal{P}_r = T^* (G^N / G^{(0)}_{\text{diag}})$$

Lagrangian model is defined on $G^N / G^{(0)}_{\text{diag}}$

Lax connection and Maillet bracket

Define $$\mathcal{L}(z,x) = \frac{\Gamma(z,x)}{\varphi(z)}$$

$$\partial_t \mathcal{L}(z,x) - \partial_x \mathcal{M}(z,x) + [\mathcal{M}(z,x), \mathcal{L}(z,x)] = 0.$$

$$\mathcal{M}(z,x) \approx \frac{\epsilon_0}{\varphi'(0)} \frac{\Gamma(0,x)}{z} + \sum_{i=1}^{2N-2} \sum_{k=0}^{1} \frac{\epsilon_i}{\varphi'(\zeta_i)} \frac{(-1)^k \sigma^k \big(\Gamma(\zeta_i,x)\big)}{z-(-1)^k \zeta_i} - \epsilon_\infty \frac{\mathcal{B}_1(x)}{2K} - \epsilon_\infty \frac{\mathcal{B}(x)}{2K} z + \mu(x)$$

$$\mathcal{B}(x) = -\sum_{r=1}^{N} \left(z_r \mathcal{J}^{(1)}_{r,[0]} + \mathcal{J}^{(1)}_{r,[1]} \right)$$

$$\mathcal{B}_1(x) = -\sum_{r=1}^{N} z_r \left(z_r \mathcal{J}^{(0)}_{r,[0]} + 2\mathcal{J}^{(0)}_{r,[1]} \right)$$

$$\begin{aligned}\{\mathcal{L}_{\underline{1}}(z,x), \mathcal{L}_{\underline{2}}(w,y)\} = {} & [\mathcal{R}_{\underline{12}}(z,w), \mathcal{L}_{\underline{1}}(z,x)]\delta_{xy} - [\mathcal{R}_{\underline{21}}(w,z), \mathcal{L}_{\underline{2}}(w,x)]\delta_{xy} \\ & - (\mathcal{R}_{\underline{12}}(z,w) + \mathcal{R}_{\underline{21}}(w,z))\delta'_{xy},\end{aligned}$$

$$\mathcal{R}_{\underline{12}}(z,w) = \mathcal{R}^0_{\underline{12}}(z,w)\varphi(w)^{-1}$$

$$[\mathcal{R}_{\underline{12}}(z_1,z_2), \mathcal{R}_{\underline{13}}(z_1,z_3)] + [\mathcal{R}_{\underline{12}}(z_1,z_2), \mathcal{R}_{\underline{23}}(z_2,z_3)] + [\mathcal{R}_{\underline{32}}(z_3,z_2), \mathcal{R}_{\underline{13}}(z_1,z_3)] = 0$$

Lagrangian formulation

$Y_r = X_r + \frac{\ell_{r,0}}{2} W_r$

N=2 : $\quad G \times G \big/ G^{(0)}_{\text{diag}}$ $\qquad \epsilon_0 = -1, \epsilon_1 = -1, \epsilon_2 = +1, \epsilon_\infty = +1$

The passage to the Lagrangian formulation is done by means of the inverse Legendre transform

$$S[g_1, g_2] = \sum_{r=1}^{2} \iint \mathrm{d}x\, \mathrm{d}t\; \kappa\left(X_r, j_{0,r}\right) - \int \mathrm{d}t\; \mathcal{H}$$

$j_{0,r} = g_r^{-1} \partial_t g_r$

$$S[g_1, g_2] = \sum_{r=1}^{2} \iint \mathrm{d}x\, \mathrm{d}t\; \kappa\left(Y_r, j_{0,r}\right) - \int \mathrm{d}t\; \mathcal{H} - \sum_{r=1}^{2} \frac{\ell_{r,0}}{2}\, I_{\mathrm{WZ}}[g_r]$$

$$j_{0,r} = g_r^{-1}\{\mathcal{H}_T, g_r\} = \sum_{s=1}^{2} \sum_{k=0}^{1} b^{(k)}_{rs} j^{(k)}_s + 2c^{(k)}_{rs} Y^{(k)}_s + \mu$$

$$Y^{(0)}_1 + Y^{(0)}_2 = -\frac{\ell_{1,0}}{2} j^{(0)}_1 - \frac{\ell_{2,0}}{2} j^{(0)}_2 \qquad \leftarrow \text{constraint}$$

$\left.\begin{array}{l} Y_1 = Y^{(0)}_1 + Y^{(1)}_1 \\ Y_2 = Y^{(0)}_2 + Y^{(1)}_2 \end{array}\right\}$ are solved in terms of $j_{0,r} = g^{-1}\partial_t g$ and $j_r = g^{-1}\partial_x g$

Lagrangian formulation

$j_{\pm,r} = g_r^{-1}\partial_\pm g_r = j_{0,r} \pm j_r$

$$\boxed{S[g_1,g_2] = \sum_{r,s=1}^{2} \iint \mathrm{d}x\,\mathrm{d}t \left(\rho^{(0)}_{rs}\,\kappa\left(j^{(0)}_{+,r}, j^{(0)}_{-,s}\right) + \rho^{(1)}_{rs}\,\kappa\left(j^{(1)}_{+,r}, j^{(1)}_{-,s}\right)\right) + \Bbbk\, I_{\mathrm{WZ}}[g_1] - \Bbbk\, I_{\mathrm{WZ}}[g_2]}$$

$$\rho^{(0)}_{11} = \rho^{(0)}_{22} = \frac{K}{2}\frac{\zeta_-^2 - \zeta_+^2}{(1-x^2)^2}, \qquad \rho^{(0)}_{12} = K\frac{\left(1-\zeta_+^2\right)\left(x^2-\zeta_-^2\right)}{\left(1-x^2\right)^3}, \qquad \rho^{(0)}_{21} = -K\frac{\left(1-\zeta_-^2\right)\left(x^2-\zeta_+^2\right)}{\left(1-x^2\right)^3}$$

$$\rho^{(1)}_{11} = \frac{K}{2}\frac{\left(1-2\zeta_+^2+\zeta_-^2\zeta_+^2\right)}{\left(1-x^2\right)^2}, \qquad \rho^{(1)}_{12} = K\frac{x\left(1-\zeta_+^2\right)\left(x^2-\zeta_-^2\right)}{\left(1-x^2\right)^3},$$

$$\rho^{(1)}_{21} = -K\frac{\left(1-\zeta_-^2\right)\left(x^2-\zeta_+^2\right)}{x\left(1-x^2\right)^3}, \qquad \rho^{(1)}_{22} = \frac{K}{2}\frac{\left(x^4-2\zeta_+^2x^2+\zeta_-^2\zeta_+^2\right)}{x^2\left(1-x^2\right)^2}$$

$$\Bbbk = K\frac{2x^2+2\zeta_-^2\zeta_+^2-(1+x^2)(\zeta_-^2+\zeta_+^2)}{(1-x^2)^3} = -\ell_{1,0}/2 = \ell_{2,0}/2$$

$3N-2 = 3\times 2 - 2 = 4$ parameters $z_2 \equiv x$, $\zeta_+ \equiv \zeta_1$, $\zeta_- \equiv \zeta_2$ and K

The action has the gauge symmetry $\quad g_r(x,t) \mapsto g_r(x,t)h(x,t)$ with $h(x,t) \in G^{(0)}$

Polyakov & Wiegmann

$$I_{\mathrm{WZ}}[g_r h] = I_{\mathrm{WZ}}[g_r] + I_{\mathrm{WZ}}[h] - \frac{1}{2}\iint \mathrm{d}x\,\mathrm{d}t\left[\kappa\left(j^{(0)}_{+,r}, (\partial_- h)h^{-1}\right) - \kappa\left(j^{(0)}_{-,r}, (\partial_+ h)h^{-1}\right)\right]$$

Reformulation

$$S = \sum_{r=1}^{2} S_{\text{WZW},\,\Bbbk_r}[g_r] - 4K \iint \mathrm{d}x\,\mathrm{d}t \sum_{r,s=1}^{2} \operatorname*{res}_{w=z_s} \operatorname*{res}_{z=z_r} \kappa_{\underline{\mathbf{12}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(w,z)\varphi_+(z)\varphi_-(w), j_{+,r\underline{\mathbf{1}}}\, j_{-,s\underline{\mathbf{2}}}\Big)$$

where

$$S_{\text{WZW},\,\Bbbk}[g] = \frac{\Bbbk}{2} \iint \mathrm{d}x\,\mathrm{d}t\ \kappa\left(g^{-1}\partial_+ g, g^{-1}\partial_- g\right) + \Bbbk\, I_{\text{WZ}}[g]$$

and $\varphi_\pm(z)$ are functions defined as

$$\varphi_+(z) = \frac{z^2 - \zeta_+^2}{(z^2 - z_1^2)(z^2 - z_2^2)} \qquad \text{and} \qquad \varphi_-(z) = \frac{z(z^2 - \zeta_-^2)}{(z^2 - z_1^2)(z^2 - z_2^2)}.$$

Future Directions

New integrable models on $G^N/G^{(0)}_{\text{diag}}$ from affine Gaudin models

$$\boxed{S = \sum_{r=1}^{N} S_{\text{WZW},\,\cancel{k}_r}[g_r] - \frac{KT^3}{2} \iint \mathrm{d}x\, \mathrm{d}t \sum_{r,s=1}^{N} \operatorname*{res}_{w=z_s} \operatorname*{res}_{z=z_r} \kappa_{\underline{\mathbf{12}}}\Big(\mathcal{R}^0_{\underline{\mathbf{12}}}(w,z)\varphi_+(z)\varphi_-(w), j_{+,r\underline{\mathbf{1}}}\, j_{-,s\underline{\mathbf{2}}}\Big)}$$

★ Study RG flow. Is integrable $T^{1,1}$ flows to the GMM fixed point?

★ Integrable sigma model on Lorentzian spaces $W_{4,2} = SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/U(1)$?

★ Integrable coset sigma models based on supergroups
Interesting case $G = PSU(1,1|2)$