REGIONAL DOCTORAL PROGRAM IN THEORETICAL AND EXPERIMENTAL PARTICLE PHYSICS

RDP Online Workshop on Mathematical Physics

December 5-6, 2020

Exactly solvable models in quantum mechanics and their application in quantum computing

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Topics...

I — The exactly solvable models

1. Methods for obtaining exact and approximate solutions of the evolution of quantum-mechanical problems.

The cyclic evolution of quantum systems described by time-periodic Hamiltonians.

2 . Construction of a class of time-periodic Hamiltonians in the close analytical form.

Calculation of corresponding cyclic solution

II — Quantum computing

3. Construction of a universal set of gates for quantum computers used the exactly solvable time-dependent Hamiltonian.

A method for obtaining entanglement operators.

Construction of a Time-Dependent Hamiltonian

Let the state $\Psi(r, t)$ of a quantum system evolve according to the matrix Schrödinger equation

$$i\frac{d\Psi(r,t)}{dt} = H(r,t)\Psi(r,t), \quad (1)$$

where $H(r, t)$ is periodic in time t, $H(t + T) = H(t)$, and has the form

$$H(r,t) = \hat{p}_r^2 + V(r,t). \quad (2)$$

$$H(t+T) = H(t)$$

$$V(r, t+T) = V(r,t)$$

$$\hat{p}_r = -i\nabla_r$$

Here, the potential matrix $V(r, t)$ is a $d \times d$ time-periodic Hermitian matrix such that $V(r, t + T) = V(r, t)$ and $\hat{p}_r = -i\nabla_r$ is the angular momentum operator determining the kinetic energy.

In terms of the evolution operator $\mathcal{U}(t) \equiv \mathcal{U}(t, 0)$, $\mathcal{U}(0) = 1$, the solution of Eq. (1) is written as

$$\Psi(r,t) = \mathcal{U}(t)\Psi(r,0), \quad (3)$$

where $\Psi(r, 0)$ is the initial state and $\mathcal{U}(t)$ satisfies the equation

$$i\frac{d\mathcal{U}(t)}{dt} = H(t)\mathcal{U}(t), \quad (4)$$

which relates $H(t)$ and $\mathcal{U}(t)$.

Construction of a Time-Dependent Hamiltonian

$$i\frac{d\Psi(r,t)}{dt} = H(r,t)\Psi(r,t) \quad (1)$$

In addition to the evolution operator transforming the initial state vector $|\Psi(t=0)\rangle$ to the state vector $|\Psi(t)\rangle$ at an arbitrary instant, it is convenient to introduce the unitary transformation $[\mathcal{S}^\dagger(t) = \mathcal{S}^{-1}(t),\ \mathcal{S}^\dagger(t)\mathcal{S}(t) = 1]$ through the relation

$$\Psi(r,t) = \mathcal{S}(t)\tilde{\psi}(r,t) \qquad (5)$$

such that, in the equation

$$i\frac{d\tilde{\psi}(t)}{dt} = \tilde{H}(r)\tilde{\psi}(t) \qquad (6)$$

following from Eq. (1) with substitution (5), the Hamiltonian

$$\tilde{H} = \mathcal{S}^\dagger(t)H(t)\mathcal{S}(t) - i\mathcal{S}^\dagger(t)(d/dt)\mathcal{S}(t) \qquad (7)$$

is time independent; i.e., $d\tilde{H}/dt = 0$.

The condition (7) provides the equation for $\mathcal{S}(t)$,

$$\dot{\mathcal{S}}^\dagger H\mathcal{S} + \mathcal{S}^\dagger(\partial H/\partial t)\mathcal{S} + \mathcal{S}^\dagger H\dot{\mathcal{S}} - i\hbar\dot{\mathcal{S}}^\dagger\dot{\mathcal{S}} - i\hbar\mathcal{S}^\dagger\ddot{\mathcal{S}} = 0 \qquad (8)$$

where $\dot{\mathcal{S}} = (d/dt)\mathcal{S}$.

The solution of Eq. (6) for new functions $\tilde{\psi}(r,t) = \mathcal{S}^\dagger(t)\Psi(r,t)$ has the form

$$\tilde{\psi}(r,t) = \exp(-i\tilde{H}(r)t)\tilde{\psi}(r,0) \qquad (9)$$

and is determined by the solution of the time-independent problem

$$\tilde{H}|F(\tilde{\mathcal{E}})\rangle = \tilde{\mathcal{E}}|F(\tilde{\mathcal{E}})\rangle. \qquad (10)$$

Eqs. (6) and (9) ⇒

$$\Psi(r,t) = \mathcal{S}(t)\exp(-i\tilde{H}(r)t)\tilde{\psi}(r,0) \qquad (11)$$

Construction of a Time-Dependent Hamiltonian

Beginning with the time-independent equation

$$\tilde{H}|F(\tilde{\mathcal{E}})\rangle = \tilde{\mathcal{E}}|F(\tilde{\mathcal{E}})\rangle$$ ← Eq. (10)

whose solution is known numerically or analytically, specifying the transformation $\mathcal{S}(t)$ in the explicit form and using equation

$$\tilde{H} = \mathcal{S}^{\dagger}(t)H(t)\mathcal{S}(t) - i\mathcal{S}^{\dagger}(t)(d/dt)\mathcal{S}(t)$$ ← Eq. (7)

and the unitarity of $\mathcal{S}(t)$ one can determine the time-dependent Hamiltonians

$$H(t) = \mathcal{S}(t)\tilde{H}\mathcal{S}^{\dagger}(t) + i\dot{\mathcal{S}}(t)\mathcal{S}^{\dagger}(t) \qquad (12)$$

and solutions of Schrödinger equation

$$i\frac{d\Psi(r,t)}{dt} = H(r,t)\Psi(r,t)$$ ← Eq. (1)

by formula

$$\Psi(r,t) = \mathcal{S}(t)\exp(-i\tilde{H}(r)t)\tilde{\psi}(r,0)$$ ← Eq. (11)

Construction of a Time-Dependent Hamiltonian

Expression

$$H(t) = \mathcal{S}(t)\tilde{H}\mathcal{S}^{\dagger}(t) + i\dot{\mathcal{S}}(t)\mathcal{S}^{\dagger}(t)$$

immediately provides the relation between the Hamiltonian at $t = 0$ and time-independent Hamiltonian $\tilde{H}$:

$$H(0) = \mathcal{S}(0)\tilde{H}\mathcal{S}^{\dagger}(0) + i\dot{\mathcal{S}}(0)\mathcal{S}^{\dagger}(0).$$

$$\Psi(r,t) = \mathcal{U}(t)\Psi(r,0) \qquad (3)$$

$$\Psi(r,t) = \mathcal{S}(t)\tilde{\psi}(r,t) \qquad (5) \qquad \Psi(r,t) = \mathcal{S}(t)\exp(-i\tilde{H}(r)t)\tilde{\psi}(r,0) \qquad (11)$$

⇩

Comparison of Eqs. (3), (5), and (11) provides the important relation between $\mathcal{U}(t)$ and $\mathcal{S}(t)$,

$$\mathcal{U}(t) = \mathcal{S}(t)\exp(-i\tilde{H}t)\mathcal{S}^{\dagger}(0).$$

Construction of a 2 × 2 Periodic Potential Matrix

We investigate the systems of two coupled Schrödinger equations with the Hamiltonian

$$H(r, t) = h(r) + h_1(r, t). \quad (1)$$

Here, $h(r)$ is the time-independent part of the Hamiltonian, the Hamiltonian $h_1(r, t)$ is a periodic function of time, and $h_1(r, t + T) = h_1(r, t)$ and can be represented in the form

$$h_1(r, t) = \sum_i \hat{\sigma}_i B_i(r, t), \quad (2)$$

where $\hat{\sigma}_i$ are the Pauli matrices and $B_i(r, t)$ are periodic in time.

Our aim is to construct families of time-dependent Hamiltonians $H(r, t)$ of form (1) for which exact solutions of Schrodinger equation can be found.

$$i\frac{d\Psi(r,t)}{dt} = H(r,t)\Psi(r,t), \quad (3)$$

Consider the time-independent Hamiltonian

$$\tilde{H}(r) = p_r^2 + V(r),$$

where $V(r)$ is the 2×2 real symmetric potential matrix, and the time-dependent unitary transformation matrix $\mathcal{S}(t)$ that transforms known time-independent Hamiltonian to time-dependent Hamiltonian

$$H(t) = \mathcal{S}(t)\tilde{H}\mathcal{S}^\dagger(t) + i\dot{\mathcal{S}}(t)\mathcal{S}^\dagger(t) \quad (4)$$

Construction of a 2×2 Periodic Potential Matrix

Consider the transformation $\mathcal{S}(t)$ in the form of the 2×2 matrix

$$\mathcal{S}(t) = \exp(-i\mathbf{s}\cdot\mathbf{q}(t)) = \exp\left(-i\sum_{i=1}^{3} s_i q_i(t)\right), \qquad (5)$$

where $\mathbf{s} = \boldsymbol{\sigma}/2$ is the spin operator, $\boldsymbol{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$, $\hat{\sigma}_i$ are the Pauli matrices, and q_i are arbitrary real functions of time.

$$i\frac{d\Psi(r,t)}{dt} = H(r,t)\Psi(r,t)$$

$$H(t) = \mathcal{S}(t)\tilde{H}\mathcal{S}^{\dagger}(t) + i\dot{\mathcal{S}}(t)\mathcal{S}^{\dagger}(t)$$

$$\tilde{H}(r) = p_r^2 + V(r)$$

$\mathcal{S}(t)$

$$H(r,t) = p_r^2 + \exp(-i\mathbf{s}\cdot\mathbf{q}(t))V(r) \times \exp(i\mathbf{s}\cdot\mathbf{q}(t)) + \mathbf{s}\cdot\dot{\mathbf{q}}(t).$$

$$|\Psi(r,t)\rangle = \exp(-i\mathbf{s}\cdot\mathbf{q}(t)) \times \exp(-i\tilde{H}(r)t)|\tilde{\psi}(r,t=0)\rangle.$$

Construction of a 2×2 Periodic Potential Matrix

It is convenient to represent 2×2 Hamiltonian

$$\tilde{H}(r) = p_r^2 + V(r)$$

as the sum of the diagonal and traceless matrices:

$$\tilde{H}(r) = p_r^2 \hat{I} + \begin{pmatrix} v_{11}(r) & v_{12}(r) \\ v_{21}(r) & v_{22}(r) \end{pmatrix}$$

$$= \left(p_r^2 + \frac{v_{11}(r) + v_{22}(r)}{2}\right)\hat{I} + \begin{pmatrix} \frac{v_{11}(r) - v_{22}(r)}{2} & v_{12}(r) \\ v_{21}(r) & -\frac{v_{11}(r) - v_{22}(r)}{2} \end{pmatrix}$$

$$= (p_r^2 + u(r))\hat{I} + \boldsymbol{\sigma} \cdot \mathbf{B}(r),$$

where v_{ij} are real, $v_{12} = v_{21}$, $u = (v_{11} + v_{22})/2$, $B_1 = v_{12}$, $B_2 = 0$, $B_3 = (v_{11} - v_{22})/2$, and $\hat{I}$ is the identity matrix.

If $\Psi(r,0) = \tilde{\psi}(r,0)$ ⇒ then $\mathcal{S}(0) = 1$.

$$\mathcal{U}(t) = \mathcal{S}(t)\exp(-i\tilde{H}t)\mathcal{S}^{\dagger}(0)$$

$$\mathcal{S}(t) = \exp(-i\mathbf{s}\cdot\mathbf{q}(t)) = \exp\left(-i\sum_{i=1}^{3} s_i q_i(t)\right)$$

$$\mathcal{U}(t) = \mathcal{S}(t)\exp(-i\tilde{H}t) = \exp(i\mathbf{s}\cdot\mathbf{q}(t))\exp(-i\tilde{H}t).$$

The evolution operator after the period is written as

$$\mathcal{U}(T) = \exp(-i\mathbf{s}\cdot\mathbf{q}(T))\exp(-i\tilde{H}T).$$

Particular cases

SU(2) transformations

$\mathbf{q}(t)$ are linear functions of time: $q_i(t) = \omega_i t$.

1

Particular cases

$$\mathcal{S}_3(t) = \exp(-i\hat{\sigma}_3\omega t/2) = \begin{pmatrix} \exp(-i\omega t/2) & 0 \\ 0 & \exp(i\omega t/2) \end{pmatrix},$$

where ω is the angular frequency.

$$H(r,t) = (p_r^2 + u(r))\hat{I}$$

$$+ \begin{pmatrix} \dfrac{v_{11}(r) - v_{22}(r) + \omega}{2} & v_{12}(r)\exp(-i\omega t) \\ v_{21}(r)\exp(i\omega t) & -\dfrac{v_{11}(r) - v_{22}(r) + \omega}{2} \end{pmatrix} \leftrightarrow \tilde{H}(r) = (p_r^2 + u(r))\hat{I} + \tilde{\Omega}(r)\begin{pmatrix} \cos\tilde{\theta}(r) & \sin\tilde{\theta}(r) \\ \sin\tilde{\theta}(r) & -\cos\tilde{\theta}(r) \end{pmatrix}.$$

Here, $u(r) = (v_{11}(r) + v_{22}(r))/2$ can be treated as an analogue of the constant electric field, and the analogue of the magnetic field $\tilde{\mathbf{B}}(r)$ can be expressed in terms of the potential matrix as

$$\tilde{\mathbf{B}}(r) = \tilde{\Omega}(r)(\sin\tilde{\theta}(r), 0, \cos\tilde{\theta}(r)),$$

$$\tilde{\Omega}(r) = (1/2)\sqrt{(v_{11}(r) - v_{22}(r))^2 + 4v_{12}(r)^2}$$

$$= \sqrt{B_3^2(r) + B_1^2(r)},$$

where

$$\sin\tilde{\theta}(r) = \frac{v_{12}(r)}{\tilde{\Omega}(r)},$$

$$\cos\tilde{\theta}(r) = \frac{v_{11}(r) - v_{22}(r)}{2\tilde{\Omega}(r)}.$$

Construction of a 2×2 Periodic Potential Matrix

$$H(t,r) = (p_r^2 + u(r))\hat{I} + \Omega(r)\begin{pmatrix} \cos\theta(r) & \sin\theta(r)\exp(-i\omega t) \\ \sin\theta(r)\exp(i\omega t) & -\cos\theta(r) \end{pmatrix}$$

Particular cases

1

$$H(t) = \boldsymbol{\sigma}\cdot\mathbf{B}(t)$$

$$= \Omega\begin{pmatrix} \cos\theta & \sin\theta\exp(-i\omega t) \\ \sin\theta\exp(i\omega t) & -\cos\theta \end{pmatrix},$$

where

$$\mathbf{B}(t) = \Omega(\sin\theta\cos(\omega t), \sin\theta\sin(\omega t), \cos\theta)$$

$$\tilde{H} = \boldsymbol{\sigma}\cdot\tilde{\mathbf{B}} = \tilde{\Omega}\begin{pmatrix} \cos\tilde{\theta} & \sin\tilde{\theta} \\ \sin\tilde{\theta} & -\cos\tilde{\theta} \end{pmatrix}$$

with the spatially uniform magnetic field

$$\tilde{\mathbf{B}} = \tilde{\Omega}(\sin\tilde{\theta}, 0, \cos\tilde{\theta}).$$

$$|\Psi_1(t)\rangle = \begin{pmatrix} \exp[-(i\omega t/2) - i\tilde{\mathcal{E}}_1 t]\cos(\tilde{\theta}/2) \\ \exp[(i\omega t/2) - i\tilde{\mathcal{E}}_1 t]\sin(\tilde{\theta}/2) \end{pmatrix},$$

$$|\Psi_2(t)\rangle = \begin{pmatrix} -\exp[-(i\omega t/2) - i\tilde{\mathcal{E}}_2 t]\sin(\tilde{\theta}/2) \\ \exp[(i\omega t/2) - i\tilde{\mathcal{E}}_2 t]\cos(\tilde{\theta}/2) \end{pmatrix}.$$

$$|\Phi_1\rangle = \begin{pmatrix} \cos(\tilde{\theta}/2) \\ \sin(\tilde{\theta}/2) \end{pmatrix}, \quad |\Phi_2\rangle = \begin{pmatrix} -\sin(\tilde{\theta}/2) \\ \cos(\tilde{\theta}/2) \end{pmatrix}$$

$$\tilde{\mathcal{E}}_1 = +\tilde{\Omega}, \quad \tilde{\mathcal{E}}_2 = -\tilde{\Omega}.$$

2

Particular cases

$$\mathcal{S}_2(t) = \exp(-i\hat{\sigma}_2\omega_2 t/2) = \begin{pmatrix} \cos(\omega_2 t/2) & -\sin(\omega_2 t/2) \\ \sin(\omega_2 t/2) & \cos(\omega_2 t/2) \end{pmatrix}$$

$$H(r,t) = (p_r^2 + u(r))\hat{I}$$

$$+\begin{pmatrix} \frac{v_{11}-v_{22}}{2}\cos\omega_2 t - v_{12}\sin\omega_2 t & v_{12}\cos\omega_2 t + \frac{v_{11}-v_{22}}{2}\sin\omega_2 t - i\omega_2/2 \\ v_{12}\cos\omega_2 t + \frac{v_{11}-v_{22}}{2}\sin\omega_2 t + i\omega_2/2 & -\left(\frac{v_{11}-v_{22}}{2}\cos\omega_2 t - v_{12}\sin\omega_2 t\right) \end{pmatrix}.$$

$$|\Psi_1(t)\rangle = \begin{pmatrix} \cos(\omega_2 t/2)\exp(-i\tilde{\mathcal{E}}_1 t)\cos\tilde{\theta}/2 - \sin(\omega_2 t/2)\exp(-i\tilde{\mathcal{E}}_2 t)\sin\tilde{\theta}/2 \\ \sin(\omega_2 t/2)\exp(-i\tilde{\mathcal{E}}_1 t)\cos\tilde{\theta}/2 + \cos(\omega_2 t/2)\exp(-i\tilde{\mathcal{E}}_2 t)\sin\tilde{\theta}/2 \end{pmatrix},$$

$$|\Psi_2(t)\rangle = \begin{pmatrix} -\cos(\omega_2 t/2)\exp(-i\tilde{\mathcal{E}}_1 t)\sin\tilde{\theta}/2 - \sin(\omega_2 t/2)\exp(-i\tilde{\mathcal{E}}_2 t)\cos\tilde{\theta}/2 \\ -\sin(\omega_2 t/2)\exp(-i\tilde{\mathcal{E}}_1 t)\sin\tilde{\theta}/2 + \cos(\omega_2 t/2)\exp(-i\tilde{\mathcal{E}}_2 t)\cos\tilde{\theta}/2 \end{pmatrix}.$$

Particular cases

3

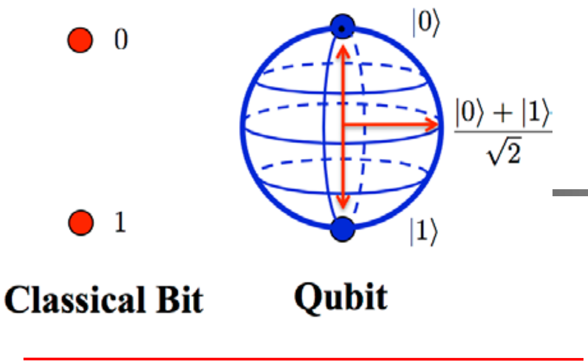
$$\mathcal{S}_1(t) = \exp(-i\hat{\sigma}_1\omega_1 t/2) = \begin{pmatrix} \cos(\omega_1 t/2) & -i\sin(\omega_1 t/2) \\ -i\sin(\omega_1 t/2) & \cos(\omega_1 t/2) \end{pmatrix}.$$

$$H(r,t) = (p_r^2 + u(r))\hat{I} + \begin{pmatrix} \frac{v_{11}-v_{22}}{2}\cos(\omega_1 t) & v_{12} - \frac{\omega_1}{2} + i\frac{v_{11}-v_{22}}{2}\sin(\omega_1 t) \\ v_{12} - \frac{\omega_1}{2} - i\frac{v_{11}-v_{22}}{2}\sin(\omega_1 t) & -\frac{v_{11}-v_{22}}{2}\cos(\omega_1 t) \end{pmatrix}.$$

$$|\Psi_1(t)\rangle = \begin{pmatrix} \cos(\omega_1 t/2)\exp(-i\tilde{\mathcal{E}}_1 t)\cos\tilde{\theta}/2 - i\sin(\omega_1 t/2)\exp(-i\tilde{\mathcal{E}}_2 t)\sin\tilde{\theta}/2 \\ -i\sin(\omega_1 t/2)\exp(-i\tilde{\mathcal{E}}_1 t)\cos\tilde{\theta}/2 + \cos(\omega_1 t/2)\exp(-i\tilde{\mathcal{E}}_2 t)\sin\tilde{\theta}/2 \end{pmatrix},$$

$$|\Psi_2(t)\rangle = \begin{pmatrix} -\cos(\omega_1 t/2)\exp(-i\tilde{\mathcal{E}}_1 t)\sin\tilde{\theta}/2 - i\sin(\omega_1 t/2)\exp(-i\tilde{\mathcal{E}}_2 t)\cos\tilde{\theta}/2 \\ i\sin(\omega_1 t/2)\exp(-i\tilde{\mathcal{E}}_1 t)\sin\tilde{\theta}/2 + \cos(\omega_1 t/2)\exp(-i\tilde{\mathcal{E}}_2 t)\cos\tilde{\theta}/2 \end{pmatrix}.$$

Quantum Computation



The classical bit is a system having two values, 0 and 1. Any quantum system having at least two states can be taken as the qubit. One state is denoted as $|0\rangle$, and the other state, orthogonal to the first, is denoted as $|1\rangle$. An arbitrary state of the qubit can be represented as

$$|\phi\rangle = a|0\rangle + b|1\rangle, \qquad |a|^2 + |b|^2 = 1$$

where a and b are complex numbers. The components of two-dimensional vectors $|0\rangle$, $|1\rangle$, and $|\phi\rangle$ are written in the form of the columns

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |\phi\rangle = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

A convenient example of the qubit is the nuclear or electron spin $\mathbf{s} = \frac{1}{2}\boldsymbol{\sigma}$.

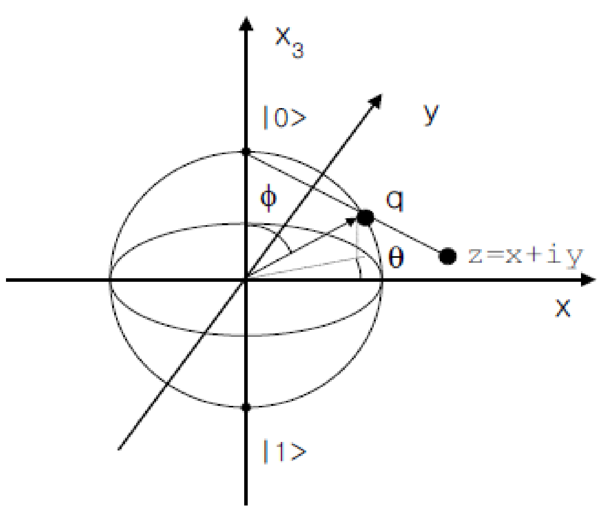
This spin in the constant external magnetic field has two energy levels corresponding to the spin directions along and against the field.

In this case, the Hamiltonian has the form $\tilde{H} = \boldsymbol{\sigma} \cdot \tilde{\mathbf{B}} = \tilde{\Omega}\begin{pmatrix} \cos\tilde{\theta} & \sin\tilde{\theta} \\ \sin\tilde{\theta} & -\cos\tilde{\theta} \end{pmatrix}$

$$|\phi_1\rangle = \begin{pmatrix} \cos(\tilde{\theta}/2) \\ \sin(\tilde{\theta}/2) \end{pmatrix} = \cos(\tilde{\theta}/2)|0\rangle + \sin(\tilde{\theta}/2)|1\rangle, \quad |\phi_2\rangle = \begin{pmatrix} -\sin(\tilde{\theta}/2) \\ \cos(\tilde{\theta}/2) \end{pmatrix} = -\sin(\tilde{\theta}/2)|0\rangle + \cos(\tilde{\theta}/2)|1\rangle.$$

Here, $a = \cos(\tilde{\theta}/2)$ and $b = \sin(\tilde{\theta}/2)$ ⟵ $|\phi_1\rangle$

$|\phi_2\rangle$ ⟶ $a = -\sin(\tilde{\theta}/2)$ and $b = \cos(\tilde{\theta}/2)$



Quantum Computation

$|\phi\rangle \xrightarrow{\mathcal{U}} |\phi'\rangle$

The vector $|\phi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ is transformed to the vector $|\phi'\rangle = \begin{pmatrix} a' \\ b' \end{pmatrix}$ by means of the second-order unitary matrix $\mathcal{U}$

$$|\phi'\rangle = \mathcal{U}(2 \times 2)|\phi\rangle.$$

One - qubit quantum gate $\mathbb{C}^2 \to \mathbb{C}^2$

Two-qubit quantum gate $\mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$

Main conditions necessary for constructing the quantum computer is the existence of a universal set of one- and two-qubit quantum gates that can provide an arbitrary unitary transformation $\mathcal{U}(t)$ of an n-qubit quantum system (register) in the $N = 2^n$ - dim Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$, where $\mathcal{H}_j \cong \mathbb{C}^2$.

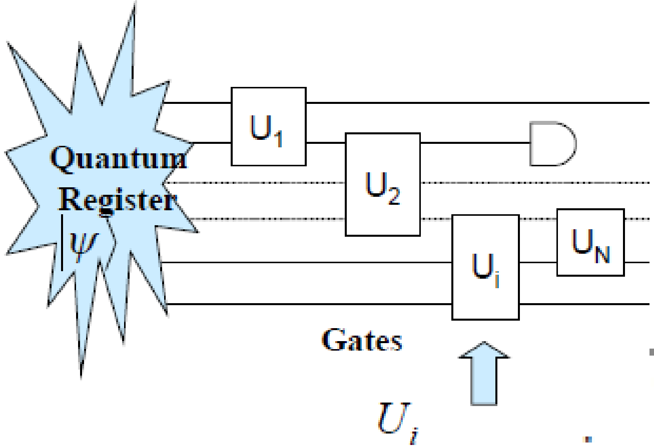
The basis consisting of 2^n vectors is taken in the form

$$|j_0\rangle = |0\rangle \otimes |0\rangle \ldots \otimes |0\rangle$$
$$|j_1\rangle = |0\rangle \otimes |0\rangle \ldots \otimes |1\rangle$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$|j_{2^n-1}\rangle = |1\rangle \otimes |1\rangle \ldots \otimes |1\rangle.$$

$|i_1, i_2, \ldots, i_n\rangle = |i_1\rangle \otimes \ldots |i_n\rangle \equiv |j\rangle,\ i_1, i_2, \ldots, i_n = \{0, 1\}$ $\Rightarrow$ $|\Psi\rangle = \sum_{k=0}^{2^n-1} a_k |j_k\rangle.$

a_i are the projections of the $|\Psi\rangle$ vector on the unit vectors $|j_0\rangle, |j_1\rangle, \ldots, |j_{2^n-1}\rangle$, and $\sum_k a_k^2 = 1.$

Quantum Computation



$U_i = \exp - i\int H dt$ ⇐ **Dynamics**

U_i ⇐ **Generation of Gates**

The computation process on the quantum computer is considered as the transformation of the initial state vector $|\Psi_0\rangle$ to the final vector $|\Psi_f\rangle$ by means of the $2^n \times 2^n$ unitary evolution matrix $\mathcal{U}(t)$:

$$|\Psi_f\rangle = \mathcal{U}(2^n \times 2^n)|\Psi_0\rangle.$$

It is convenient to take an initial state such that all its qubits are in the $|0\rangle$ state: $|\Psi_0\rangle = |0_1\rangle \otimes |0_2\rangle \ldots \otimes |0_n\rangle = |0_1, 0_2, \ldots, 0_n\rangle \equiv |j_0\rangle$. This operation is called preparation or *initialization* of the quantum register. The matrices $\mathcal{U}(2^n \times 2^n)$ determine the dynamic evolution of the quantum system. The algorithm of solving the problem is specified by the $\mathcal{U}(2^n \times 2^n)$ transformation matrix, which ensures the quantum computation process at each given instant.

The possibility of the decomposition of the $\mathcal{U}(2^n \times 2^n)$ matrix in the ordered product of the second- and fourth-order matrices is usually considered:

$$\mathcal{U}(2^n \times 2^n) = \prod_{i,j} \mathcal{U}_i(2 \times 2) \otimes \mathcal{U}_j(2^2 \times 2^2).$$

The second-order matrices $\mathcal{U}_i(2 \times 2)$ transform the state vectors of one qubit:

$$\begin{pmatrix} a' \\ b' \end{pmatrix}_i = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}_i \begin{pmatrix} a \\ b \end{pmatrix}_i.$$

The fourth-order matrices $\mathcal{U}_j(2^2 \times 2^2)$ transform the state vectors of pairs of qubits.

Consider the quantum system that evolves according to Schrödinger equation

$$i\frac{d\Psi(r,t)}{dt} = H(r,t)\Psi(r,t)$$

with Hamiltonian

$$H(t) = \lambda\begin{pmatrix} \cos\tilde{\theta}\cos(\omega_1 t) & \sin\tilde{\theta} - \omega_1/2\lambda + i\cos\tilde{\theta}\sin(\omega_1 t) \\ \sin\tilde{\theta} - \omega_1/2\lambda - i\cos\tilde{\theta}\sin(\omega_1 t) & -\cos\tilde{\theta}\cos(\omega_1 t) \end{pmatrix}$$

This Hamiltonian and corresponding cyclic solutions

$$|\Psi_1(t)\rangle = \begin{pmatrix} \cos(\omega_1 t/2)\exp(-i\tilde{\mathcal{E}}_1 t)\cos\tilde{\theta}/2 - i\sin(\omega_1 t/2)\exp(-i\tilde{\mathcal{E}}_2 t)\sin\tilde{\theta}/2 \\ -i\sin(\omega_1 t/2)\exp(-i\tilde{\mathcal{E}}_1 t)\cos\tilde{\theta}/2 + \cos(\omega_1 t/2)\exp(-i\tilde{\mathcal{E}}_2 t)\sin\tilde{\theta}/2 \end{pmatrix},$$

$$|\Psi_2(t)\rangle = \begin{pmatrix} -\cos(\omega_1 t/2)\exp(-i\tilde{\mathcal{E}}_1 t)\sin\tilde{\theta}/2 - i\sin(\omega_1 t/2)\exp(-i\tilde{\mathcal{E}}_2 t)\cos\tilde{\theta}/2 \\ i\sin(\omega_1 t/2)\exp(-i\tilde{\mathcal{E}}_1 t)\sin\tilde{\theta}/2 + \cos(\omega_1 t/2)\exp(-i\tilde{\mathcal{E}}_2 t)\cos\tilde{\theta}/2 \end{pmatrix}$$

are obtained by means of the transformation

$$\mathcal{S}(t) = \exp(-i\sigma_1\omega_1 t/2)$$

of time-independent Hamiltonian

$$\tilde{H} = \boldsymbol{\sigma}\cdot\tilde{\mathbf{B}} = \tilde{\Omega}\begin{pmatrix} \cos\tilde{\theta} & \sin\tilde{\theta} \\ \sin\tilde{\theta} & -\cos\tilde{\theta} \end{pmatrix}$$

The initial qubit state is taken as one of the States

$$\phi_1 = \cos\tilde{\theta}/2|0\rangle + \sin\theta/2|1\rangle \quad \text{or} \quad \phi_2 = -\sin\theta/2|0\rangle + \cos\tilde{\theta}/2|1\rangle$$

of Hamiltonian

$$\tilde{H} = \boldsymbol{\sigma}\cdot\tilde{\mathbf{B}} = \tilde{\Omega}\begin{pmatrix} \cos\tilde{\theta} & \sin\tilde{\theta} \\ \sin\tilde{\theta} & -\cos\tilde{\theta} \end{pmatrix}$$

The evolution matrix corresponding to Hamiltonian

$$H(t) = \lambda\begin{pmatrix} \cos\tilde{\theta}\cos(\omega_1 t) & \sin\tilde{\theta} - \omega_1/2\lambda + i\cos\tilde{\theta}\sin(\omega_1 t) \\ \sin\tilde{\theta} - \omega_1/2\lambda - i\cos\tilde{\theta}\sin(\omega_1 t) & -\cos\tilde{\theta}\cos(\omega_1 t) \end{pmatrix}$$

has the form

$$\mathcal{U}_1(t) = \begin{pmatrix} \cos(\omega_1 t/2) & -i\sin(\omega_1 t/2) \\ -i\sin(\omega_1 t/2) & \cos(\omega_1 t/2) \end{pmatrix}\begin{pmatrix} \exp(-i\lambda t) & 0 \\ 0 & \exp(i\lambda t) \end{pmatrix}.$$

This evolution matrix provides a continuous set of single-qubit gates, which is specified by the parameters ω_1 and λ.

An important single-qubit transformation is the NOT or inversion operation

$$\mathrm{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x.$$

The NOT gate can be obtained from $\mathcal{U}_1(t)$ multiplied by i at

$\omega_1 t = \pi$ and $\lambda t = 2n\pi$:

$$\mathrm{NOT} = iU_1(\omega_1 t = \pi, \lambda t = 2n\pi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Another special single-qubit gate can be obtained from $\mathcal{U}_1(t)$ by setting $\omega_1 t = \pi$ and $\lambda t = \frac{\pi}{2}$ and multiplying the result by i:

$$\mathrm{Y} = iU_1(\omega_1 t = \pi, \lambda t = \pi/2) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y.$$

The Z gate is obtained from $\mathcal{U}_1(t)$ for $\omega_1 t = 4\pi$ and $\lambda t = \frac{\pi}{2}$ after the multiplication by i:

$$\mathrm{Z} = iU_1(\omega_1 t = 4\pi, \lambda t = \pi/2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z.$$

$$\mathcal{U}_1(t) = \begin{pmatrix} \cos(\omega_1 t/2) & -i\sin(\omega_1 t/2) \\ -i\sin(\omega_1 t/2) & \cos(\omega_1 t/2) \end{pmatrix} \begin{pmatrix} \exp(-i\lambda t) & 0 \\ 0 & \exp(i\lambda t) \end{pmatrix}.$$

In quantum informatics, the Hadamard transform matrix

$$\mathrm{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$$

is often used.

$$\mathrm{H}|0\rangle = \mathrm{H}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

$$\mathrm{H}|1\rangle = \mathrm{H}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

$$\sqrt{1/2}(|0\rangle + |1\rangle) = \sqrt{1/2}\begin{pmatrix}1\\1\end{pmatrix}.$$

In this case, the evolution matrix $\mathcal{U}(t)$ corresponding to Hamiltonian $H(t)$ is

written as $\mathcal{U}(t) = \exp(-i\sigma_x\omega_1 t/2)\exp(-i\sigma_z\lambda t)\exp(-i\sigma_y\tilde{\theta}/2)$

$$= \begin{pmatrix} \cos(\omega_1 t/2) & -i\sin(\omega_1 t/2) \\ -i\sin(\omega_1 t/2) & \cos(\omega_1 t/2) \end{pmatrix}\begin{pmatrix} \exp(-i\lambda t) & 0 \\ 0 & \exp(i\lambda t) \end{pmatrix}\begin{pmatrix} \cos(\tilde{\theta}/2) & -\sin(\tilde{\theta}/2) \\ \sin(\tilde{\theta}/2) & \cos(\tilde{\theta}/2) \end{pmatrix}.$$

$\mathcal{U}(t)$ for $t = 0$, $\tilde{\theta} = \frac{\pi}{2}$ and any ω_1 and λ values provides the gate

$$\mathcal{U}(\omega_1, \lambda; t = 0, \tilde{\theta} = \pi/2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \Downarrow$$

$$\mathrm{H} = iU(\pi, 2\pi n, \tilde{\theta} = 0)U(\omega_1, \lambda, \tilde{\theta} = \pi/2; t = 0).$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$$

Hadamard gate H is obtained as the sequence of two transformations. The H gate applied to n qubits $|0\rangle$ generates the superposition of 2^n possible states and thus provides the binary representation of numbers from 0 to $2^n - 1$:

$$(H \otimes H \otimes \ldots \otimes H)|00\ldots0\rangle$$

$$= \frac{1}{\sqrt{2^n}}((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes \ldots \otimes (|0\rangle + |1\rangle)) = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} |j_k\rangle.$$

$\mathcal{U}(t)$ generates all $SU(2)$ matrices.

$$\mathcal{U}(t) = \exp(-i\sigma_x\omega_1 t/2)\exp(-i\sigma_z\lambda t)\exp(-i\sigma_y\tilde{\theta}/2)$$

$$= \begin{pmatrix} \cos(\omega_1 t/2) & -i\sin(\omega_1 t/2) \\ -i\sin(\omega_1 t/2) & \cos(\omega_1 t/2) \end{pmatrix} \begin{pmatrix} \exp(-i\lambda t) & 0 \\ 0 & \exp(i\lambda t) \end{pmatrix} \begin{pmatrix} \cos(\tilde{\theta}/2) & -\sin(\tilde{\theta}/2) \\ \sin(\tilde{\theta}/2) & \cos(\tilde{\theta}/2) \end{pmatrix}.$$

Quantum Computation

$$\mathcal{U}(2^n \times 2^n) = \prod_{i,j} \mathcal{U}_i(2 \times 2) \otimes \mathcal{U}_j(2^2 \times 2^2).$$

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$\{|00\rangle = |0\rangle \otimes |0\rangle, |01\rangle = |0\rangle \otimes |1\rangle, \qquad |10\rangle = |1\rangle \otimes |0\rangle, |11\rangle = |1\rangle \otimes |1\rangle\}:$$

$$|\Psi\rangle = c_{00}|00\rangle + c_{10}|10\rangle + c_{01}|01\rangle + c_{11}|11\rangle = \begin{pmatrix} c_{00} \\ c_{01} \\ c_{10} \\ c_{11} \end{pmatrix},$$

where $|c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1$.

$C = 2|c_{00}c_{11} - c_{10}c_{01}|$ Entanglement degree $\quad C = 0 \leftrightarrow |\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle.$

The reversible invertor or *controlled* NOT operator (CNOT) is one of the main two-qubit gates. This gate is described by the 4×4 matrix, which can be represented in the form

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{1} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \sigma_x = |0\rangle\langle 0| \otimes \mathbf{1} + |1\rangle\langle 1| \otimes \sigma_x,$$

where, $\mathbf{1}$ is the 2×2 identity matrix.

Consider Hamiltonian of the form

$$H = h \otimes 1 + 1 \otimes h + \epsilon A,$$

where $\epsilon \in \{0, 1\}$ and the Hamiltonian h is time independent and has the form

$$h = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

In this case, if the operator A commutes with $h \otimes 1 + 1 \otimes h$, the evolution operator is represented as

$$\mathcal{U}(t) = (e^{-iht} \otimes e^{-iht})e^{-iAt}.$$

Consider the operator $\exp(-iAt) \equiv R(t)$ in the form

$$R(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(t) & -i\sin(t) & 0 \\ 0 & -i\sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The matrix $A(t)$ is given by the expression

$$A = i\frac{dR(t)}{dt}R^{-1}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix

$$h = \sigma_3/2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

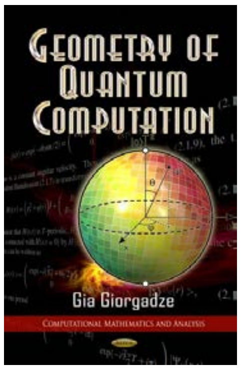
satisfies the commutation condition $[A, (h \otimes 1 + 1 \otimes h)] = 0.$

Entanglement operator

$$\mathcal{U}(t) = \begin{pmatrix} e^{it} & 0 & 0 & 0 \\ 0 & \cos(t) & -i\sin(t) & 0 \\ 0 & -i\sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & e^{-it} \end{pmatrix}$$

$$\mathcal{U}(t) = (e^{-iht} \otimes e^{-iht})e^{-iAt}.$$

References



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Tbilisi Mathematical Journal, vol.13, N2,pp.187-206, 2020

THE END

Thanks for Attention !