RDP Online Workshop on Mathematical Physics

Solitons in normal Fermi liquid

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Presentation Outline

- Energy Spectrum for Almost Ideal Fermi Liquid
- Kinetic Equation
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- Summary

It is well known that at temperatures 1–2 K only two neutral quantum liquids exist in nature, the isotopes of helium He3 and He4

We study properties of almost ideal Fermi gas. We consider only a pair interaction between particles, since triple collisions contribute to energy only in a higher approximation.

Interaction U(r) is independent of particle spin. In the limiting case of slow collisions, the mutual scattering amplitude of particles with mass m tends to a constant limit, that is called scattering length:

$$a = \frac{mU_0}{4\pi\hbar^2}$$

$$U_0 = \int U(\vec{r})d\vec{r}$$

Using diagram technique, V.M. Galitski calculated the quasiparticle energy spectrum of almost ideal Fermi gas in the ordinary perturbation theory and derived energy expression which involves only the scattering amplitude:

$$\varepsilon(p) = \frac{p^2}{2m} + \frac{2\pi\hbar^2 an}{m}$$

Energy Spectrum for Almost Ideal Fermi Liquid

Landau created the theory of Fermi liquid. He took into account only the weakly excited energy levels lying fairly close to the ground state, assuming that any weakly excited state of a macroscopic body can be represented as an assembly of separate elementary excitation (quasiparticles).

This elementary excitation are represented as the collective motion of atoms in liquid and it can not be represented as excitation of individual atoms.

Landau's theory of Fermi liquids was generalized by incorporation the De Broglie waves diffraction (Madelung term in the kinetic equation).

$$\frac{\partial f}{\partial t} + (\vec{v}\nabla)f + \frac{\partial \vec{p}}{\partial t}\frac{\partial f}{\partial \vec{p}} = 0 \qquad \frac{\partial \vec{p}}{\partial t} = -\nabla\varepsilon + \frac{\hbar^2}{2m}\nabla\frac{1}{\sqrt{n}}\Delta\sqrt{n} \tag{1}$$

Madelung term is placed in the force expression.
With relation (1) the corresponding kinetic equation has form:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\nabla)\vec{u} + \frac{\varepsilon_F}{m}\nabla(N)^{\frac{2}{3}} + \nabla\left(\frac{2\pi\hbar^2 a n_0}{m^2}N - \frac{\hbar^2}{2m^2}\frac{1}{\sqrt{N}}\Delta\sqrt{N}\right) = 0 \tag{2}$$

Kinetic Equation

We discuss some properties of slightly non ideal Fermi gas with repulsion (a > 0) or attraction (a < 0) between the particles.

We construct hydrodynamic equations from last formula and get equation of continuity and motion for macroscopic quantities

$$\frac{\partial N}{\partial t} + \nabla(N\vec{u}) = 0 \qquad (3)$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\nabla)\vec{u} + \frac{\varepsilon_F}{m}\nabla(N)^{\frac{2}{3}} + \nabla\left(\frac{2\pi\hbar^2 a n_0}{m^2}N - \frac{\hbar^2}{2m^2}\frac{1}{\sqrt{N}}\Delta\sqrt{N}\right) = 0 \qquad (4)$$

For the pressure term we used

$$P = \frac{1}{5}(2\pi^2)^{\frac{2}{3}}\frac{\hbar^2}{m}n^{5/3}$$

N is normalized density

$$N = \frac{n}{n_0}$$

$$u = \frac{1}{n}\int \frac{2dp^3}{(2\pi\hbar)^3}\vec{v}f(\vec{r},\vec{p},t)$$

u is macroscopic velocity of Fermi liquid

Hydrodynamic Equations

Introducing perturbations u = δu and N = 1 + δN and linearization hydrodynamic equations (3) and (4) with will result dispersion relation

$$\omega^2 = \left(\frac{2\pi\hbar^2 a n_0}{m^2} + \frac{2\varepsilon_F}{3m}\right)k^2 + \frac{\hbar^2}{4m^2}k^4 \qquad (5)$$

We now consider nonlinear one dimensional traveling waves in a single direction, so-called simple waves. For simple waves it is well known that for the waves with any amplitude, velocity can be expressed as the function of the density u = u (N).

$$N = \left[1 + \frac{u}{\sqrt{3}v_F}\left(1 - \frac{3A}{2v_F^2}\right)\right]^3 \qquad (6)$$

$$\frac{v_F^2}{3A} \gg 1$$

$$A = \frac{2\pi\hbar^2 a n_0}{m^2}$$

From this expression it is clear that when we have attraction between particles: A < 0, density increases, and in the opposite case when we have repulsion A > 0, density decreases.

Simple Wave Solution

Since the velocity u is the function of density, therefore it is different for the different points of the wave profile, i.e., the profile changes in the course of time. du/dn > 0, i.e., the velocity of propagation of a given point at the wave profile increases with density. This is the condition for the shock waves

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\left(u\left(\frac{4}{3}-\frac{A}{v_F^2}\right)+\frac{v_F}{\sqrt{3}}-\frac{\sqrt{3}A}{2v_F}\right)-\frac{\sqrt{3}\hbar^2}{4v_F m^2}\frac{\partial^3 u}{\partial x^3}=0 \qquad (7)$$

Introduce new variables and rewrite Eq. (7)

$$U_0=\frac{v_F}{\sqrt{3}}-\frac{\sqrt{3}A}{2v_F}\;,\; V=u\left(\frac{4}{3}-\frac{A}{v_F^2}\right),\; \xi=x-U_0 t,\; \beta=-\frac{\sqrt{3}\hbar^2}{4v_F m^2}$$

KDV equation:

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial \xi}+\beta\frac{\partial^3 V}{\partial \xi^3}=0 \qquad (8)$$

We are interested in solutions which describe waves with stationary profile. For such waves velocity $V(t, \xi)$ depends only on the parameter $(\xi - V_0{*}t)$, where V_0 is some constant and wave propagation speed is $U = U_0 - V_0$

Korteweg-de Vries (KdV) equation

In our case coefficient of higher derivative is negative ($\beta < 0$) and in order to solve KdV equation we need to change variables

$$\xi \to -\xi$$
$$V_0 \to -V_0$$
$$V \to -V$$

With such change of variables, equation remains identical except opposite sign of β. Those change does not change mathematical form of solution, but instead of "positive" (compressible), we have "negative" (rarefaction) solitary wave and the speed of such soliton is less than U_0

We replace ξ with the new variable Z into KdV equation taking into account that $Z = \xi - V_0 t$, $\partial Z/\partial t = -V_0$, $\partial Z/\partial \xi = 1$.

$$-V_0\frac{\partial V}{\partial Z} + V\frac{\partial V}{\partial Z} + \beta\frac{\partial^3 V}{\partial Z^3} = 0 \qquad (8)$$

This is invariant under the change with any constant V_C

$$V \to V + V_c$$
$$V_0 \to V_0 + V_c \qquad (9)$$

$$\beta\left(\frac{\partial V}{\partial Z}\right)^2 = -\frac{1}{3}(V - V_1)(V - V_2)(V - V_3) \qquad (10)$$

V1, V2 and V3 depend only on the initial conditions of perturbation and $V0 = (V1 + V2 + V3)/3$

Korteweg-de Vries (KdV) equation

We are interested only bounded solutions as unlimited increasing of V contradicts our assumption of slight non-linearity. This means all constants must be real.

Without loss of generality we can arrange V-s in order: V1 ≥ V2 ≥ V3. Also we can always take V3 = 0 using transformation (9). Than V can vary only in the range V1 ≥ V ≥ V2.

$$V = V_1 dn^2\left(\sqrt{\frac{V_1}{12\beta}}Z, s\right) \qquad (11)$$

$dn^2\left(\sqrt{\frac{V_1}{12\beta}}Z, s\right)$

elliptic Jacobi function with modulus s. Period is λ

$$\lambda = 4\sqrt{\frac{3|\beta|}{V_1}}K(s)$$

$$s = \sqrt{1 - \frac{V_2}{V_1}}$$

K(s) and E(s) are complete elliptic integral of the first and second kind

$$U = U_0 - V_0\left(1 - \frac{3}{2 - s^2}\frac{E(s)}{K(s)}\right) \qquad (12)$$

Wave propagation speed for negative β after changing variables back

The parameter s (0 < s < 1) measures non-linearity. When s << 1 solution is expressed in periodic functions. When s → 0 elliptic Jacobi function goes to the solution of linearized equations. For the other extreme case: s → 1, λ goes to infinity, i.e. we obtain solitary wave solution

$$s \to 1\ ,\ dn(\alpha, s) = 1/cosh(\alpha),\ \frac{E(s)}{K(s)} = 0$$

$$U_0 = \frac{v_F}{\sqrt{3}} - \frac{\sqrt{3}A}{2v_F}\ ,\ V = u\left(\frac{4}{3} - \frac{A}{v_F^2}\right)$$

$$V = 3V_0 cosh^{-2}\sqrt{\frac{V_0}{4\beta}}Z \qquad (13)$$

Constant $3V_0$ is the soliton amplitude. Width is decreasing as square root of V_0 with increasing amplitude.

Speed of ordinary sound waves Us can be found in the dispersion relation in the first approximation

$$\omega = U_s k$$

$$U_s = v_F/\sqrt(3) + \sqrt(3)A/2v_F$$

$$U = \frac{v_F}{\sqrt{3}} - \frac{\sqrt{3}A}{2v_F} - V_0$$

Rarefaction soliton speed is less than speed of sound waves. Interaction between particles changes that fact.

$$U - U_S = -\frac{\sqrt{3}A}{v_F} - V_0$$

If A > 0 his difference is always negative. When A < 0 the velocity difference can be positive for small amplitudes of solitary waves (V_0)

Solution for Solitary Waves

Based on the new quantum kinetic equation of Fermi particles a general quantum dispersion equation for almost ideal Fermi gas is derived and studied.

We consider only pair interactions between particles and we used energy expression derived by V.M. Galitski in the ordinary perturbation theory, which involves only the scattering amplitude.

New effect is that novel quantum term in the kinetic equation leads to the formation of rarefaction solitary waves in neutral Fermi liquid.

We can say when we have repulsion between particles ($A > 0$), rarefaction soliton speed is, as expected, always less than speed of sound waves. Moreover, it decreases even more because of pair interaction between particles.

In the opposite case, when we have attraction ($A < 0$) soliton speed increases and in the certain amplitudes can rich speed of ordinary sound waves

Rarefaction solitons can propagate in almost ideal, neutral Fermi liquid and it’s velocity and amplitude is dependent on the scattering length.

Summary

Used Literature

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[5] V.I. Karpman - "Non-Lnear Waves on Dispersive Media", Chapter 4, Pergamon Press, Oxford 1975

Thank you