CORRELATION FUNCTIONS OF CLASSICAL AND QUANTUM ARTIN SYSTEM DEFINED ON LOBACHEVSKY PLANE AND SCRAMBLING TIME

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THE CONTENTS

- 1.The motivation
- 2.The classical Artin system on Lobachevski plane-Poincare metric.

  3.The solution of the geodesic equations and fundamental region.
- 4.Chaos in Artin system and exponential instability; two point function.
- 5.The quantization of the Artin system.

THE CONTENTS

MOTIVATIONS.

- A. The experimental revolution-the enormous improvements in experimental techniques it became feasible to control quantum systems with many degrees of freedom. An entirely new arena for the study of physics of interacting quantum many-body systems.
- B. New mashines -Supercomputers and novel numerical techniques(tensor network , density matrix renormalization, Bethe Ansatz)
- C. Understanding of the quantum mechanics has improved significantly since the of von Neumann!! The new mathematical methods.

MOTIVATIONS.

- D. And finally –the problem of quantum gravity and black holes . There is the speculation that black holes should be most quantum chaotic system with maximal scrambling time $t^*=ß/2\pi$.

ARTIN DYNAMICAL SYSTEM ON LOBACHEVSKI PLANE-POINCARE METRIC.

- In 1924 Emil Artin introduced an example of Ergodic Dynamical system which is realized as geodesic flow on a compact surface Ƒ of the Lobachevsky plane. The aim of this article was to construct an example of a dynamical system in which “almost all” geodesic trajectories are quasi-ergodic , meaning that all trajectories with the exception of measure zero , during their time evolution will approach infinitely close any point and given direction on surface Ƒ . Let us consider Lobachevsky plane realized in the upper half-plane $y>0$ of the complex plane $z=x+iy$ with

ARTIN SYSTEM

- The Poincare metric which is given by the line element

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{dz\,d\bar{z}}{(\operatorname{Im} z)^2}$$

The Lobachevsky plane is a surface a constant negative curvature , because its curvature Is equal to R=-2 and it is twice the Gaussian curvature K=-1. This metric has well known properties: it is invariant with respect to all linear substitution ,which form the group G of isometries of the Lobachevsky plane

ARTIN SYSTEM

$$w = g.z = \frac{\alpha z + \beta}{\gamma z + \delta}$$

Where $\alpha, \beta, \gamma, \delta$

are real coefficients of the matrix g and the determinant of g is positive.

ARTIN SYSTEM

- The equation for the geodesic lines on curved surface the form

$$\frac{d^2x^i}{dt^2} + \Gamma^i{}_{kl} \frac{dx^k}{dt} \frac{dx^l}{dt} = 0$$

Where $\Gamma^i{}_{kl}$ are the Christoffer symbols of the Poincare metric

$$g_{ik} = \delta_{ik} \frac{1}{y^2}$$

THE SOLUTION OF THE GEODESIC EQUATIONS

- The geodesic equation takes the form

$$\frac{d^2x}{ds^2} - \frac{2}{y}\frac{dx}{ds}\frac{dy}{ds} = 0$$

$$\frac{d^2y}{ds^2} + \frac{1}{y}\left(\frac{dx}{ds}\right)^2 - \frac{1}{y}\left(\frac{dy}{ds}\right)^2 = 0$$

and has two solutions

THE SOLUTION OF GEODESIC EQUATIONS

$$x(t) - x_0 = r\tanh(t), y(t) = \frac{r}{\cosh(t)}$$

$$x(t) = x_0, y(t) = \exp(t)$$

First is the orthogonal semi-circles and second is perpendicular rays. Here

$$x_0 \in (-\infty, +\infty), t \in (-\infty, +\infty), r \in (0, \infty)$$

THE SOLUTION OF THE GEODESIC EQUATION

- Using this solution one can check the points on the geodesics curves move with unit velocity

$$\frac{ds}{dt} = 1$$

One can also observe that if we in the geodesic equation check the proper time,

$$dt = \sqrt{2}(E - U(x))dt'$$

Then the geodesic equation is coincide with equation of motion-the Newton equation

In this case the potential is

$$U(x) = \frac{1}{y^2}$$

THE FUNDAMENTAL REGION Ƒ

- In order to construct a compact surface Ƒ on Lobachevsky plane , one can identify all points in upper half of the plane which are related to each other by the substitution g with integer coefficient and a unit determinant . These transformation form a modular group .Thus we consider two points z and w to be “identical” if

$$w = \frac{mz+n}{pz+q}$$

THE FUNDAMENTAL REGION Ƒ

- With integers m,n,p,q constrained by the condition mq-pn=1 which is unit determinant condition-w=dz ,det(d)=1. These elements d form discrete group SL(2,Z) which is discrete subgroup of the isometry transformation SL(2,R).The identification creates a regular tessellation of the Lobachevsky plane by congrount hyperbolic triangles. The Lobachevsky plane is covered by the infinite-order triangular tiling.

THE FUNDAMENTAL REGION

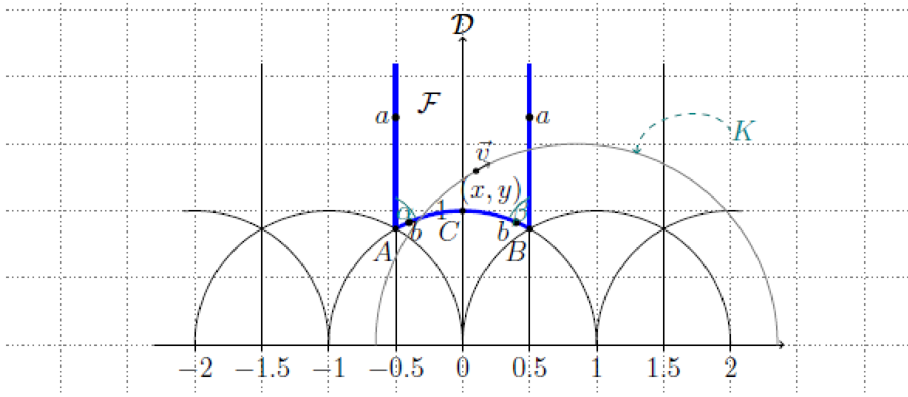


Figure 1: The non-compact fundamental region $\mathcal{F}$ of a finite area is represented by the hyperbolic triangle ABD. The vertex D is at infinity of the y axis and corresponds to a cusp. The edges of the triangle are the arc AB, the rays AD and BD. The points on the edges AD and BD and the points of the arks AC with CB should be identified by the transformations $w = z + 1$ and $w = -1/z$ in order to form a *closed non-compact surface* $\bar{\mathcal{F}}$ by "gluing" the opposite edges of the modular triangle together. The hyperbolic triangle OAB can be considered equally well as the fundamental region. The modular transformations of the fundamental region $\mathcal{F}$ create a regular tessellation of the whole Lobachevsky plane by congruent hyperbolic triangles. K is the geodesic trajectory passing through the point (x, y) of $\mathcal{F}$ in the $\vec{v}$ direction.

THE FUNDAMENTAL REGION Ƒ

- One of these triangles can be chosen as a fundamental region. That fundamental region
- Ƒ of the modular group SL(2,Z),is the well known "modular triangle" consisting of those
- Points between lines x=- 1/2, and x==1/2 which lie outside the unit circle in the Fig.The modular triangle has two equal angles
- and with the third one equal to zero,$\alpha = \beta = \frac{\pi}{3}$ $\gamma = 0$

THE FUNDAMENTAL REGION

- Thus

$$\alpha + \beta + \gamma = \frac{2\pi}{3} < \pi$$

The area of the fundamental region is finite and equal to $\frac{\pi}{3}$

Inside the modular triangle Ƒ there is exactly one representative among all equivalent points of the Lobachevsky plane with exception of the points on the triangle edges are opposite to each other . These points should be identified to form a closed compact surface $\overline{F}$

THE FUNDAMENTAL REGION

- By “gluing” the opposite edges of the mudular triangle together. The main goal of the construction is to consider now the behavior of the geodesic trajectories defined on the surface
- constant negative curvature.
- In order to describe the behaviour of the geodesic trajectories on the fundamental region one can use the knowledge of the geodesic trajectories on whole Lobachevsky plane. Arbitrary point on (x,y) Ƒ and velocity vector ∈

CHAOS IN ARTIN SYSTEM

$$\vec{v} = (\cos\vartheta, \sin\vartheta)$$

These are the coordinates of the phase space x,y, and theta
belong to phase space M. and they uniquely determine the geodesic trajectory as the orthogonal circle K in the whole Lobachevsky plane. As trajectory “hits” the edges of the fundamental region and goes outside of it, one should apply the modular transformation to that parts of the circle K which are outside of Ƒ in order to return them back to the Ƒ. That algorithm will define the whole trajectory on fundamental region for arbitrary time . One should observe that this description of the trajectory on fundamental

CHAOS IN ARTIN SYSTEM

- Under the action of the modular group on the
- initial circle K. One should join together the
- parts of the geodesic circles K’ which lie inside
- of Ƒ into a unique continues trajectory on Ƒ
- with boundaries . In this context the quasi-ergodicity of the trajectory K on compact
- surface will mean that among all circle {K’}
- there are those which are approaching arbitrary close to any given circle C!!

CHAOS IN ARTIN SYSTEM

- The geodesic trajectories are bounded to propagate on the fundamental hyperbolic triangle.

  The geodesic flow in this fundamental region represents one of the most chaotic dynamical systems with exponential instability of its trajectories , has mixing of all orders , Lebesgue spectrum and non-zero Kolmogorov entropy---Hedlund, Anosov, Hopf,Gelfand,Fomin,
- Kolmogorov…

THE TWO POINT CORRELATION FUNCTIONS

- The earlier investigation of the correlation functions such a Anosov systems was performed by- Pollicot , Moore , Dolgopyat…
- using different approaches including Fourier series for the SL(2,R) group , the methods unitary representation theory. In our analyses we shall use the time evolution equation, the properties
- of automorphic functions on Ƒ and shall estimate a
- decay exponent in terms of the phase space curvature and the transformation properties of function . The correlation function can be defined as an integral over
- a pair of functions/observables in which the first one is stationary and the second one evolves with geodesic flow:

THE TWO POINT FUNCTION

$$\mathcal{D}_t(f_1, f_2) = \int_{\mathcal{M}} f_1(g)\overline{f_2(gg_t)}d\mu$$

The f' s are our physical observables and are the function on
phase space . Here the measure and all integral are

$$d\mu = \frac{dxdy}{y^2}d\theta,$$

$$\mathcal{D}_t(f_1, f_2) == \int_0^{2\pi}\int_{\mathcal{F}} f_1[x, y, \theta]\,\overline{f_2[x'(x, y, \theta, t), y'(x, y, \theta, t), \theta'(\theta, t)]}\frac{dxdy}{y^2}d\theta.$$

THE TWO POINT FUNCTION

- The SL(2,R)(SL(2,Z)) invariants fix the functions/observables in upper half plane , these are [Gelfand ,Fomin] Poincare theta function of weight n -

$$f(\omega_2, \overline{\tau}, \tau) = f(\theta, x, y)$$

$$f(\omega_2, \tau, \overline{\tau}) = \omega_2^n (\overline{\tau} - \tau)^n \Theta(\overline{\tau}) = \frac{1}{\overline{\omega_2}^n} \Theta(\overline{\tau})$$

THE TWO POINT FUNCTION

- This theta function satisfy the condition

$$\Theta(\frac{m\overline{\tau}+n}{p\overline{\tau}+q}) = \Theta(\overline{\tau})(p\overline{\tau}+q)^n$$

Still the last point , we should define the time evolution of $\{f(x, y, \theta)\}$ the physical observables . The simplest evolution is

$$z_1(t) = g_1(t)\cdot i = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \cdot i = ie^t \ , \quad t \in (-\infty, +\infty)$$

THE TWO POINT FUNCTION

- The general evolution which map any geodesic to other and is the element of the isometry group SL(2,R) is

$$z(t) = gg_1(t) \cdot i = \begin{pmatrix} \alpha e^{t/2} & \beta e^{-t/2} \\ \gamma e^{t/2} & \delta e^{-t/2} \end{pmatrix} \cdot i, \quad z(t) = \frac{i\alpha e^t + \beta}{i\gamma e^t + \delta}$$

Now we can combine all these things together and after some calculation we will arrive the final formula for the two point function when the time is $t \to \pm\infty$

THE TWO POINT FUNCTION

$$|\mathcal{D}_t(f_1, f_2)| \leq \int_0^{2\pi} d\theta \int_{\mathcal{F}} |\Theta_1(\overline{\tau})\overline{\Theta_2(\overline{\tau'})}| \; \frac{y^{n-2} dx dy}{2^n[(\alpha\cos\theta + \gamma\sin\theta)^2]^{\frac{n}{2}}} \; e^{-\frac{n}{2}t}$$

$$|\mathcal{D}_t(f_1, f_2)| \leq \; M_{\Theta_1\Theta_2}(\epsilon) \; e^{-\frac{n}{2}|t|}$$

In order to exclude the apparent singularity at the angle which solves the

$$\xi\cos\theta_0 + \sin\theta_0 = 0$$

If the surface has a negative curvature K

THE POINT FUNCTION

$$ds^2 = \frac{dx^2 + dy^2}{Ky^2}$$

Then in last formula the exponential factor will take the form

$$|\mathcal{D}_t(f_1, f_2)| \leq \; M_{\Theta_1 \Theta_2}(\epsilon) \; e^{-\frac{n}{2} K|t|}$$

Which shoes that when surface has larger negative curvature, then divergency of the trajectories is stronger.

THE QUANTUM ARTIN SYSTEM

$$S = \int L dt = \int \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{y} dt$$

$$\frac{d}{dt} \frac{\dot{x}}{y\sqrt{\dot{x}^2 + \dot{y}^2}} = 0,$$

$$\frac{d}{dt} \frac{\dot{y}}{y\sqrt{\dot{x}^2 + \dot{y}^2}} + \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{y^2} = 0$$

This the action of the Artin system with equation of motion . We should notice the
invariance of the action and equation of motions under time reparametrizations

$$t \to t(\tau)$$

THE QUANTUM ARTIN SYSTEM

- Presence of a local "gauge" symmetry indicates that we have a constrained dynamical system . One particularly convenient choice of gauge fixing specifying the time parameter t proportional to the proper time , is archived by imposing the condition

$$2H = \frac{\dot{x}^2+\dot{y}^2}{y^2}$$

THE QUANTUM ARTIN SYSTEM

- Where H is a constant . Defining the canonical
- momenta conjugate to the coordinates x , y as

$$p_x = \frac{\dot{x}}{y^2}, p_y = \frac{\dot{y}}{y^2}$$

We shall get the geodesic equations in the Hamiltionian form

$$\dot{p}_x = 0, \dot{p}_y = -\frac{2H}{y}$$

THE QUANTUM ARTIN SYSTEM

- Indeed , after defining the Hamiltonian as

$$H = \frac{y^2}{2}(p_x{}^2 + p_y{}^2)$$

The corresponding equation of motions will take the form as above , and advantage
Of the gauge which we have,is that the Hamiltonian coincides with the constraint.
Now it is fairly standard to quantize this Hamiltonian system we simply replace

$$p_x = -i\frac{\partial}{\partial x}, p_y = -i\frac{\partial}{\partial y}$$

THE QUANTUM ARTIN SYSTEM

- And consider the Schrodinger equation

$$H\Psi = E\Psi$$

$$-y^2(\partial_x{}^2 + \partial_y{}^2)\Psi = E\Psi$$

In this equation one easily recognises the Laplace operator with an extra minus sign,in Poincare metric. It is convenient to introduce parametrization E=s(1-s), as far E is real and Semi-positive and parametrization is symmetric with respect to s to 1-s and opposite .So , th the parameter s should be chosen within the range

$$s \in [\tfrac{1}{2}, 1]$$

$$s = \tfrac{1}{2} + iu$$

$$u \in [0, \infty]$$

THE QUANTUM ARTIN SYSTEM

- One should impose the “periodic” boundary condition on the wave function with respect to the modular group

$$\Psi\left(\frac{az+b}{cz+d}\right)=\Psi(z)$$

In order to have wave function which is propoerly defined on the fundamental region. Taking into account that transformation T:z to z+1 belongs to SL(2,Z),one has to impose the periodicity condition

$$\Psi(z)=\Psi(z+1)$$

Thus we have a Fourier expansion

THE QUANTUM ARTIN SYSTEM

$$\Psi(x,y)=\sum_{n=-\infty}^{\infty} f_n(y)\exp(2\pi i n x)$$

Inserting this to Schrodinger equation one get the solution for Fourier component

$$\frac{d^2 f_n(y)}{dy^2}+s(1-s)-4\pi^2 n^2) f_n(y)=0$$

$$f_n(y)=\sqrt{y}K_{s-\frac{1}{2}}(2\pi n\,|\,y\,|)$$

THE QUANTUM ARTIN SYSTEM

- For the case n=0 one simply gets and combining all together we get

$$f_0(y) = c_0 y^s + c_0' y^{1-s}$$

$$\Psi(x,y) = c_0 y^s + c_0' y^{1-s} +$$

$$\sqrt{y} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n K_{s-\frac{1}{2}}(2\pi n\,|\,y\,|) \times \exp(2\pi i n x)$$

THE QUANTUM ARTIN SYSTEM

- Where all c's should be defined such that the wave function will fulfill the boundary conditions

$$\Psi\left(\frac{az+b}{cz+d}\right)=\Psi(z)$$

How we should solve this problem???
H.Maas(1949)

THE QUANTUM ARTIN SYSTEM AND THE MAAS FORMULA

$$\Psi_s(z) = \tfrac{1}{2}\sum \gamma . y^s$$

$$\Psi_s(z) = y^s + \tfrac{\vartheta(1-s)}{\vartheta(s)}\, y^{1-s}$$

$$+ \tfrac{4\sqrt{y}}{\vartheta(s)} \sum_{l=1}^{\infty} \tau_{s-\frac{1}{2}}(l) K_{s-\frac{1}{2}}(2\pi l y) \cos(2\pi l x)$$

THE MAAS FORMULA

- Where we should define the several functions and element of SL(2,Z)

$$\gamma.z = \frac{az+b}{cz+d}, ad - bc = 1$$

$$\vartheta(s) = \pi^{-s}\varsigma(2s)\Gamma(s)$$

$$\tau_\nu(n) = \sum_{a.b=n} (\tfrac{a}{b})^\nu$$

THE CONTINUOUS AND DISCRETE SPECTRUM

- This wave function is well defined in the complex s plane and has a simple pole at s=1. The physical continuous spectrum is defined by

$$s \in [\tfrac{1}{2}, 1],$$

$$s = \tfrac{1}{2} + iu,$$

$$u \in [0, \infty]$$

$$E = s(1-s) = \tfrac{1}{4} + u^2$$

THE DISCRETE SPECTRUM

- The wave function of the discrete spectrum has a form

$$\Psi_n(z)=\sum_{l=1}^{\infty}c_l(n)\sqrt{y}K_{iu_n}(2\pi ly)\{\cos(2\pi lx),\sin(2\pi lx)\}$$

And the coefficients c's are not known analytically but where computed numerically for many values of n [D.A.Hejhal]. For the calculation they use the condition

$$\Psi_n(z)=\Psi_n(-\tfrac{1}{z})$$

$$E_n=\tfrac{1}{4}+u_n{}^2$$

CORRELATION FUNCTIONS

- The Two –point Functions.
- Having explicit expressions of the wave functions one can analyze the quantum-mechanical behavior of the correlation functions .

$$D_2(\beta,t) = \langle A(t)B(0)\exp(-\beta H)\rangle =$$

$$\sum_n \langle n|\exp(iHt)A(0)\exp(-iHt)B(0)\exp(-\beta H)|n\rangle =$$

$$\sum_{n,m} (\exp(i(E_n - E_m)t - \beta E_n)\langle n|A(0)|m\rangle\langle m|B(0)|n\rangle$$

CORRELATION FUNCTIONS

- The energy eigenvalues are parametrized

$$n = \tfrac{1}{2} + iu$$

$$m = \tfrac{1}{2} + iv$$

Thus we get for two point function

$$D_2(\beta, t) = \int_0^\infty du \int_0^\infty dv \exp(i(u^2 - v^2)t - \beta(\tfrac{1}{4} + u^2)) \times$$

$$\int_{\overline{F}} \Psi_{\frac{1}{2}-iu}(z) A \Psi_{\frac{1}{2}+iv}(z) d\mu(z) \int_{\overline{F}} \Psi_{\frac{1}{2}-iv}(w) B \Psi_{\frac{1}{2}+iu}(w) d\mu(w)$$

CORRELATION FUNCTIONS

- Defining the basic matrix element as

$$A_{uv} = \int_{F} \Psi_{\frac{1}{2}-iu}(z) A \Psi_{\frac{1}{2}+iv}(z) d\mu(z) =$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} \Psi_{\frac{1}{2}-iu}(z) A \Psi_{\frac{1}{2}+iv}(z)$$

$$D_2(\beta,t) = \int_{-\infty}^{\infty} \exp(i(u^2 - v^2)t - \beta(\tfrac{1}{4} + u^2)) A_{uv} B_{vu} du dv$$

CORRELATION FUNCTIONS

This expression is very convenient for the numerical calculation .One should choose also appropriate observables A and B . The operator

$$y^{-2}$$

seems very appropriate because ,the convergence of the integrals over fundamental region F will be much stronger.

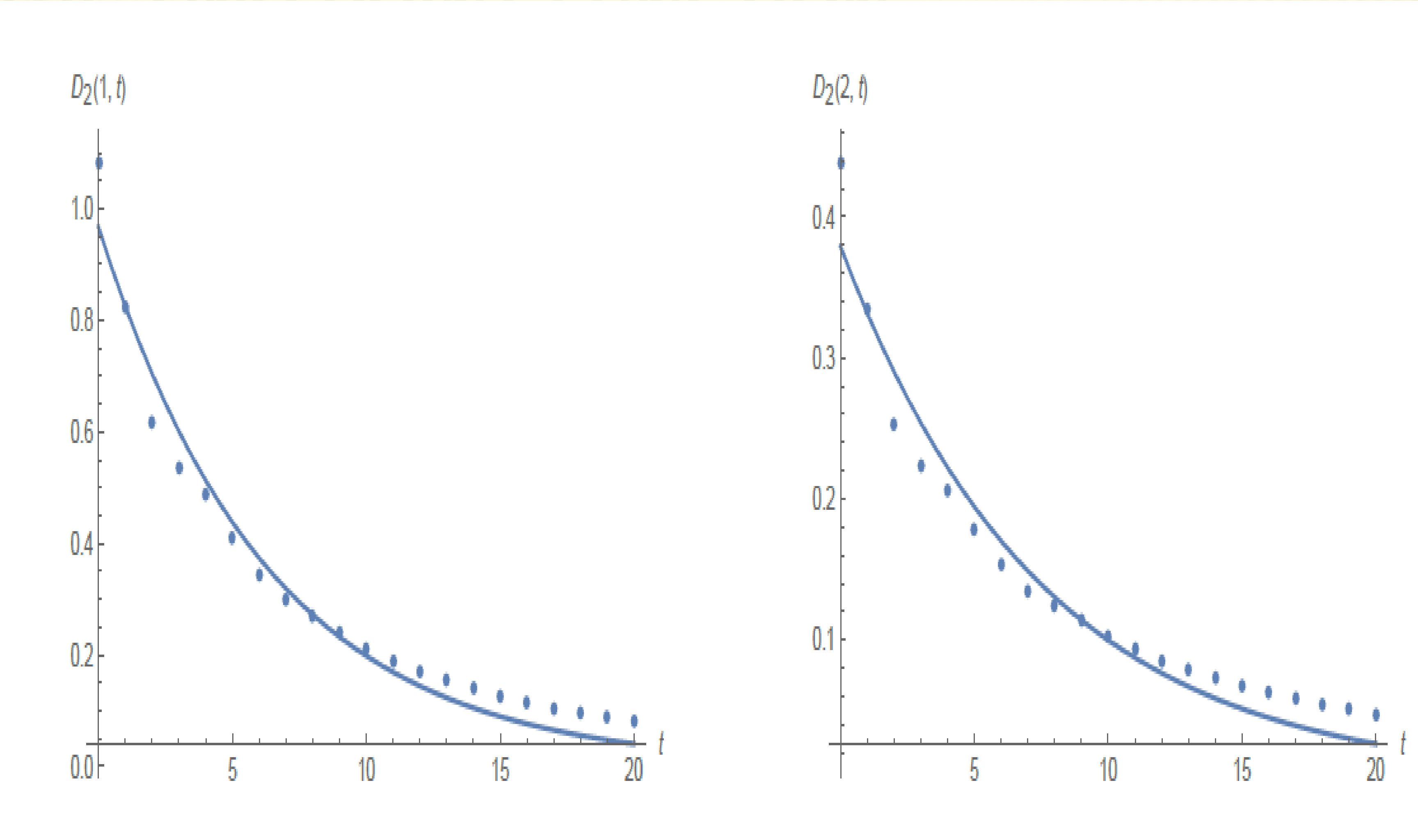
- It is expected that the two point correlation function should decay as

$$D_2(\beta,t) \propto K \exp(-\frac{t}{t_d})$$

- . In figure one can see the exponential decay of the two-point correlation function with time at different temperatures with the

$$t_d = \beta$$

THE TWO POINT FUNCTION



THE FOUR- POINT FUNCTION

- The out-of-time four-point correlation function is[Maldacena , Shenker , Stenford]

$$D_4(\beta,t) = \langle A(t)B(0)A(t)B(0)\exp(-\beta H\rangle =$$

$$\int_{-\infty}^{\infty} \exp(i(u^2 - v^2 + l^2 - r^2)t - \beta(\tfrac{1}{4} + u^2))A_{uv}B_{vl}A_{lr}B_{ru}dudvdldr$$

As well as the behavior of the commutator and square of commutator.

$$\langle [A(t), B(0)]\exp(-\beta H)\rangle$$

THE FOUR POINT FUNCTION

- It was speculated in [Maldacena , Shenker , Stenford] , that the most important correlation functions including the races of the classical chaotic dynamics in quantum regime is

$$C(t) = -\langle [A(t), B(0)]^2 \exp(-\beta H) \rangle =$$

$$-D_4(\beta,t) + D_4'(\beta,t) + D_4''(\beta,t) - D_4'''(\beta,t)$$

THE SCRAMLING TIME

- These formulas and the explicit form of the wave functions allow to calculate the above correlation functions at least using numerical integration. Where we use so called plane wave approximation(we use only first two terms in wave function).Thus by performing numerical integration we observed that a two point correlation functions decays exponentially and that a four-point function demonstrate tendency to decay with a lower pace. With numerical data available to us it is impossible to estimate the scrambling time $t_* = \frac{\beta}{2\pi}$

THE SCRAMBLING TIME

- Where the scrambling time is define as

$$D_4(\beta,t) \propto 1 - f_0 \exp(\tfrac{t}{t_*})$$

CONCLUSION

- 1.We recall the classical Artin system which is defined on the Lobachevsky plane .This system is one of the most chaotic dynamical system with exponential instability, has mixing of all order.
- 2.Using the Gelfand-Fomin differential geometry methods we calculate two point correlation function . This two point function decay exponentially.
- 3. We consider the quantization of the artin system defined on the fundamental region of the Lobachevsky plane.

CONCLUSION

- 4.By performing a numerical integration we observed that the two point correlation function of the quantum Artin system decays exponentially and that a four-point function demonstrates tendency to decay with a lower pace. With a the numerical data available to us it is impossible to estimate scrambling time or to confirm its existence in the hyperbolic system , but qualitatively we observe a short time exponential decay of the out-of-time correlation function to almost zero value and then an essential increase with the subsequent large fluctuations.