Symmetries of deformed supersymmetric mechanics on Kähler manifolds

Hovhannes Shmavonyan

December 6, 2020

Contents

Based on
E. Ivanov, A. Nersessian, S. Sidorov, H. Shmavonyan *"Symmetries of deformed supersymmetric mechanics on Kahler manifolds"*
Phys.Rev.D 101 (2020) 2, 025003

- Kähler manifold
- SUSY mechanics on Kähler manifolds
- $SU(2|1)$ (deformed $\mathcal{N}=4$) SUSY Landau Problem
- $SU(2|1)$ Kähler superoscillator
- Examples of $SU(2|1)$ Kähler superoscillator models
- $SU(4|1)$ (deformed $\mathcal{N}=8$) SUSY Landau problem

Kähler manifold

Kähler manifold is the Hermitian manifold with the Hermitian metrics, $ds^2 = g_{a\bar{b}} dz^a d\bar{z}^b$, which also defines the symplectic structure

$$\omega = i g_{a\bar{b}} dz^a \wedge d\bar{z}^b, \qquad d\omega = 0 \quad \Rightarrow \quad g_{a\bar{b}} = \frac{\partial^2 K}{\partial z^a \partial \bar{z}^b}$$

Kähler manifold

Kähler manifold is the Hermitian manifold with the Hermitian metrics, $ds^2 = g_{a\bar{b}} dz^a d\bar{z}^b$, which also defines the symplectic structure

$$\omega = i g_{a\bar{b}} dz^a \wedge d\bar{z}^b, \qquad d\omega = 0 \quad \Rightarrow \quad g_{a\bar{b}} = \frac{\partial^2 K}{\partial z^a \partial \bar{z}^b}$$

The real function $K(z, \bar{z})$ (Kähler potential) is defined up to holomorphic and antiholomorphic functions

$$K(z, \bar{z}) \to K(z, \bar{z}) + U(z) + \bar{U}(\bar{z})$$

Kähler manifold

Kähler manifold is the Hermitian manifold with the Hermitian metrics, $ds^2 = g_{a\bar{b}} dz^a d\bar{z}^b$, which also defines the symplectic structure

$$\omega = i g_{a\bar{b}} dz^a \wedge d\bar{z}^b, \qquad d\omega = 0 \quad \Rightarrow \quad g_{a\bar{b}} = \frac{\partial^2 K}{\partial z^a \partial \bar{z}^b}$$

The real function $K(z, \bar{z})$ (Kähler potential) is defined up to holomorphic and antiholomorphic functions

$$K(z, \bar{z}) \to K(z, \bar{z}) + U(z) + \bar{U}(\bar{z})$$

Isometries of Kähler structure preserves complex and symplectic structures and are generated by the holomorphic Hamiltonian vector fields

$$\mathbf{V}_\mu = \{\mathtt{h}_\mu, \} = V_\mu^a(z) \frac{\partial}{\partial z^a} + V_\mu^{\bar{a}}(\bar{z}) \frac{\partial}{\partial \bar{z}^a}, \qquad V_\mu^a = i g^{\bar{b}a} \partial_{\bar{b}} \mathtt{h}_\mu(z, \bar{z})$$

Real function $\mathtt{h}_\mu(z, \bar{z})$ is called Killing potential satisfying

$$\frac{\partial^2 \mathtt{h}_\mu}{\partial z^a \partial z^b} - \Gamma^c_{ab} \frac{\partial \mathtt{h}_\mu}{\partial z^c} = 0$$

SUSY mechanics on Kähler manifolds

A supersymmetric particle is moving in presence of a constant magnetic field and potential field.

SUSY mechanics on Kähler manifolds

A supersymmetric particle is moving in presence of a constant magnetic field and potential field. $(2N|MN)_C$-dimensional phase superspace equipped with the symplectic structure

$$\Omega = d\pi_a \wedge dz^a + d\bar{\pi}_a \wedge d\bar{z}^a - i(Bg_{a\bar{b}} - R_{a\bar{b}c\bar{d}}\eta^{c\alpha}\bar{\eta}^d_\alpha)dz^a \wedge d\bar{z}^b + ig_{a\bar{b}}D\eta^{a\alpha} \wedge D\bar{\eta}^b_\alpha$$

where $\alpha = 1, \dots M$ are spinorial indices, $D\eta^{a\alpha} = d\eta^{a\alpha} + \Gamma^a_{bc}\eta^{b\alpha}dz^c$, and $\Gamma^a_{bc},\ R_{a\bar{b}c\bar{d}}$ are the components of connection and curvature.

SUSY mechanics on Kähler manifolds

A supersymmetric particle is moving in presence of a constant magnetic field and potential field. $(2N|MN)_C$-dimensional phase superspace equipped with the symplectic structure

$$\Omega = d\pi_a \wedge dz^a + d\bar{\pi}_a \wedge d\bar{z}^a - i(Bg_{a\bar{b}} - R_{a\bar{b}c\bar{d}}\eta^{c\alpha}\bar{\eta}^d_\alpha)dz^a \wedge d\bar{z}^b + ig_{a\bar{b}}D\eta^{a\alpha} \wedge D\bar{\eta}^b_\alpha$$

where $\alpha = 1, \dots M$ are spinorial indices, $D\eta^{a\alpha} = d\eta^{a\alpha} + \Gamma^a_{bc}\eta^{b\alpha}dz^c$, and Γ^a_{bc}, $R_{a\bar{b}c\bar{d}}$ are the components of connection and curvature.

The Poisson brackets corresponding to the symplectic structure amount to the relations

$$\{\pi_a, z^b\} = \delta^b_a, \quad \{\pi_a, \eta^{b\alpha}\} = -\Gamma^b_{ac}\eta^{c\alpha}$$

$$\{\pi_a, \bar{\pi}_b\} = i(Bg_{a\bar{b}} - R_{a\bar{b}c\bar{d}}\eta^{c\alpha}\bar{\eta}^d_\alpha) \quad \{\eta^{a\alpha}, \bar{\eta}^b_\beta\} = ig^{a\bar{b}}\delta^\alpha_\beta$$

SUSY mechanics on Kähler manifolds

A supersymmetric particle is moving in presence of a constant magnetic field and potential field. $(2N|MN)_C$-dimensional phase superspace equipped with the symplectic structure

$$\Omega = d\pi_a \wedge dz^a + d\bar{\pi}_a \wedge d\bar{z}^a - i(Bg_{a\bar{b}} - R_{a\bar{b}c\bar{d}}\eta^{c\alpha}\bar{\eta}^d_\alpha)dz^a \wedge d\bar{z}^b + ig_{a\bar{b}}D\eta^{a\alpha} \wedge D\bar{\eta}^b_\alpha$$

where $\alpha = 1, \ldots M$ are spinorial indices, $D\eta^{a\alpha} = d\eta^{a\alpha} + \Gamma^a_{bc}\eta^{b\alpha}dz^c$, and Γ^a_{bc}, $R_{a\bar{b}c\bar{d}}$ are the components of connection and curvature.

The Poisson brackets corresponding to the symplectic structure amount to the relations

$$\{\pi_a, z^b\} = \delta^b_a, \quad \{\pi_a, \eta^{b\alpha}\} = -\Gamma^b_{ac}\eta^{c\alpha}$$

$$\{\pi_a, \bar{\pi}_b\} = i(Bg_{a\bar{b}} - R_{a\bar{b}c\bar{d}}\eta^{c\alpha}\bar{\eta}^d_\alpha) \quad \{\eta^{a\alpha}, \bar{\eta}^b_\beta\} = ig^{a\bar{b}}\delta^\alpha_\beta$$

The Hamiltonian

$$\mathcal{H} = g^{\bar{a}b}\bar{\pi}_a\pi_b + \mathcal{U}(z, \bar{z}, \eta, \bar{\eta}),$$

$SU(2|1)$ (deformed $\mathcal{N}=4$) SUSY Landau Problem

We choose the standard chiral supercharges and the $SU(2)$ R-charges

$$Q^\alpha = \pi_a \eta^{a\alpha}\ , \quad \overline{Q}_\alpha = \bar{\pi}_a \bar{\eta}^a_\alpha, \quad \mathcal{R}^\alpha_\beta = g_{a\bar{b}} \eta^{a\alpha} \bar{\eta}^b_\beta - \frac{1}{2} \delta^\alpha_\beta g_{a\bar{b}} \eta^{a\gamma} \bar{\eta}^b_\gamma\ .$$

$SU(2|1)$ (deformed $\mathcal{N}=4$) SUSY Landau Problem

We choose the standard chiral supercharges and the $SU(2)$ R-charges

$$Q^{\alpha} = \pi_a \eta^{a\alpha}\ , \quad \overline{Q}_{\alpha} = \bar{\pi}_a \bar{\eta}^{a}_{\alpha}, \quad \mathcal{R}^{\alpha}_{\beta} = g_{a\bar{b}} \eta^{a\alpha} \bar{\eta}^{b}_{\beta} - \frac{1}{2} \delta^{\alpha}_{\beta} g_{a\bar{b}} \eta^{a\gamma} \bar{\eta}^{b}_{\gamma}\ .$$

The closure of their Poisson brackets yields the superalgebra

$$\{Q^{\alpha}, Q^{\beta}\} = 0, \quad \{\mathcal{R}^{\alpha}_{\beta}, \mathcal{R}^{\gamma}_{\delta}\} = -i\delta^{\gamma}_{\beta}\mathcal{R}^{\alpha}_{\delta} + i\delta^{\alpha}_{\delta}\mathcal{R}^{\gamma}_{\beta}, \quad \{Q^{\alpha}, \mathcal{R}^{\beta}_{\gamma}\} = i\delta^{\alpha}_{\gamma} Q^{\beta} - \tfrac{i}{2}\delta^{\beta}_{\gamma} Q^{\alpha}$$

$$\{Q^{\alpha}, \overline{Q}_{\beta}\} = i\delta^{\alpha}_{\beta}\mathcal{H}_0 + iB\mathcal{R}^{\alpha}_{\beta}, \quad \{Q^{\alpha}, \mathcal{H}_0\} = i\tfrac{B}{2} Q^{\alpha}, \quad \{\mathcal{R}^{\alpha}_{\beta}, \mathcal{H}_0\} = 0$$

$SU(2|1)$ (deformed $\mathcal{N}=4$) SUSY Landau Problem

We choose the standard chiral supercharges and the $SU(2)$ R-charges

$$Q^{\alpha}=\pi_a\eta^{a\alpha}\ ,\quad \overline{Q}_{\alpha}=\bar{\pi}_a\bar{\eta}^a_{\alpha},\quad \mathcal{R}^{\alpha}_{\beta}=g_{a\bar{b}}\eta^{a\alpha}\bar{\eta}^b_{\beta}-\frac{1}{2}\delta^{\alpha}_{\beta}g_{a\bar{b}}\eta^{a\gamma}\bar{\eta}^b_{\gamma}\ .$$

The closure of their Poisson brackets yields the superalgebra

$$\{Q^{\alpha},Q^{\beta}\}=0,\quad \{\mathcal{R}^{\alpha}_{\beta},\mathcal{R}^{\gamma}_{\delta}\}=-i\delta^{\gamma}_{\beta}\mathcal{R}^{\alpha}_{\delta}+i\delta^{\alpha}_{\delta}\mathcal{R}^{\gamma}_{\beta},\quad \{Q^{\alpha},\mathcal{R}^{\beta}_{\gamma}\}=i\delta^{\alpha}_{\gamma}Q^{\beta}-\tfrac{i}{2}\delta^{\beta}_{\gamma}Q^{\alpha}$$

$$\{Q^{\alpha},\overline{Q}_{\beta}\}=i\delta^{\alpha}_{\beta}\mathcal{H}_0+iB\mathcal{R}^{\alpha}_{\beta},\quad \{Q^{\alpha},\mathcal{H}_0\}=i\tfrac{B}{2}Q^{\alpha},\quad \{\mathcal{R}^{\alpha}_{\beta},\mathcal{H}_0\}=0$$

where $\mathcal{H}_0$ is the Hamiltonian

$$\mathcal{H}_0=g^{\bar{a}b}\bar{\pi}_a\pi_b-\frac{1}{2}R_{a\bar{b}c\bar{d}}\eta^{a\alpha}\bar{\eta}^b_{\alpha}\eta^{c\beta}\bar{\eta}^d_{\beta}+\frac{B}{2}g_{a\bar{b}}\eta^{a\alpha}\bar{\eta}^b_{\alpha}$$

$SU(2|1)$ (deformed $\mathcal{N}=4$) SUSY Landau Problem

We observe, however, that the supercharges do not commute with the Hamiltonian. This drawback can be remedied via the appropriate modification of the Hamiltonian:

$$\widetilde{\mathcal{H}}_0 = \mathcal{H}_0 - \frac{B}{2} g_{a\bar{b}} \eta^{a\alpha} \bar{\eta}^{b}_{\alpha} = g^{a\bar{b}} \pi_a \bar{\pi}_b - \frac{1}{2} R_{a\bar{b}c\bar{d}} \eta^{a\alpha} \bar{\eta}^{b}_{\alpha} \eta^{c\beta} \bar{\eta}^{d}_{\beta} + B g_{a\bar{b}} \eta^{a\alpha} \bar{\eta}^{b}_{\alpha}$$

$$\{Q^{\alpha}, \widetilde{\mathcal{H}}_0\} = 0$$

$SU(2|1)$ (deformed $\mathcal{N}=4$) SUSY Landau Problem

We observe, however, that the supercharges do not commute with the Hamiltonian. This drawback can be remedied via the appropriate modification of the Hamiltonian:

$$\widetilde{\mathcal{H}}_0 = \mathcal{H}_0 - \frac{B}{2} g_{a\bar{b}} \eta^{a\alpha} \bar{\eta}^{b}_{\alpha} = g^{a\bar{b}} \pi_a \bar{\pi}_b - \frac{1}{2} R_{a\bar{b}c\bar{d}} \eta^{a\alpha} \bar{\eta}^{b}_{\alpha} \eta^{c\beta} \bar{\eta}^{d}_{\beta} + B g_{a\bar{b}} \eta^{a\alpha} \bar{\eta}^{b}_{\alpha}$$

$$\{Q^{\alpha}, \widetilde{\mathcal{H}}_0\} = 0$$

The last term in the Hamiltonians is obviously Zeeman term describing interaction of spin with an external magnetic field.

$SU(2|1)$ (deformed $\mathcal{N}=4$) SUSY Landau Problem

We observe, however, that the supercharges do not commute with the Hamiltonian. This drawback can be remedied via the appropriate modification of the Hamiltonian:

$$\widetilde{\mathcal{H}}_0 = \mathcal{H}_0 - \frac{B}{2} g_{a\bar{b}} \eta^{a\alpha} \bar{\eta}^{b}_{\alpha} = g^{a\bar{b}} \pi_a \bar{\pi}_b - \frac{1}{2} R_{a\bar{b}c\bar{d}} \eta^{a\alpha} \bar{\eta}^{b}_{\alpha} \eta^{c\beta} \bar{\eta}^{d}_{\beta} + B g_{a\bar{b}} \eta^{a\alpha} \bar{\eta}^{b}_{\alpha}$$

$$\{Q^{\alpha}, \widetilde{\mathcal{H}}_0\} = 0$$

The last term in the Hamiltonians is obviously Zeeman term describing interaction of spin with an external magnetic field.
From the mathematical point of view, the shift is the new R-symmetry $U(1)$ generator $\mathcal{R} =: \frac{1}{2} g_{a\bar{b}} \eta^{a\alpha} \bar{\eta}^{b}_{\alpha}$. It extends $SU(2)$ R-symmetry generated by $\mathcal{R}^{\alpha}_{\beta}$ to $U(2)$ R-symmetry.

$SU(2|1)$ (deformed $\mathcal{N}=4$) SUSY Landau Problem

We observe, however, that the supercharges do not commute with the Hamiltonian. This drawback can be remedied via the appropriate modification of the Hamiltonian:

$$\widetilde{\mathcal{H}}_0 = \mathcal{H}_0 - \frac{B}{2} g_{a\bar{b}} \eta^{a\alpha} \bar{\eta}^b_\alpha = g^{a\bar{b}} \pi_a \bar{\pi}_b - \frac{1}{2} R_{a\bar{b}c\bar{d}} \eta^{a\alpha} \bar{\eta}^b_\alpha \eta^{c\beta} \bar{\eta}^d_\beta + B g_{a\bar{b}} \eta^{a\alpha} \bar{\eta}^b_\alpha$$

$$\{Q^\alpha, \widetilde{\mathcal{H}}_0\} = 0$$

The last term in the Hamiltonians is obviously Zeeman term describing interaction of spin with an external magnetic field.
From the mathematical point of view, the shift is the new R-symmetry $U(1)$ generator $\mathcal{R} =: \frac{1}{2} g_{a\bar{b}} \eta^{a\alpha} \bar{\eta}^b_\alpha$. It extends $SU(2)$ R-symmetry generated by $\mathcal{R}^\alpha_\beta$ to $U(2)$ R-symmetry.
Since $\tilde{H}_0$ commutes with all other generators of the extended superalgebra, it can be interpreted as the central charge generator.

Symmetries of $SU(2|1)$ SUSY Landau Problem

All the generators of $su(2|1)$ superalgebra (and of its central extension) are manifestly invariant under the action of the isometry current

$$\mathcal{J}_\mu = J_\mu + \frac{\partial^2 \mathtt{h}_\mu}{\partial z^c \partial \bar{z}^d} \eta^{c\alpha} \bar{\eta}^d_\alpha,$$

$$\{Q^\alpha, \mathcal{J}_\mu\} = \{\overline{Q}_\alpha, \mathcal{J}_\mu\} = \{\mathcal{R}^\alpha_\beta, \mathcal{J}_\mu\} = \{\mathcal{H}_0, \mathcal{J}_\mu\} = 0$$

where J_μ is the bosonic isometry

$$J_\mu = V^a_\mu \pi_a + \bar{V}^{\bar{a}}_\mu \bar{\pi}_{\bar{a}} - B\mathtt{h}_\mu(z\bar{z}), \quad V^a_\mu = ig^{\bar{b}a}\partial_{\bar{b}}\mathtt{h}_\mu(z, \bar{z})$$

Symmetries of $SU(2|1)$ SUSY Landau Problem

All the generators of $su(2|1)$ superalgebra (and of its central extension) are manifestly invariant under the action of the isometry current

$$\mathcal{J}_\mu = J_\mu + \frac{\partial^2 \mathtt{h}_\mu}{\partial z^c \partial \bar{z}^d} \eta^{c\alpha} \bar{\eta}^d_\alpha,$$

$$\{Q^\alpha, \mathcal{J}_\mu\} = \{\overline{Q}_\alpha, \mathcal{J}_\mu\} = \{\mathcal{R}^\alpha_\beta, \mathcal{J}_\mu\} = \{\mathcal{H}_0, \mathcal{J}_\mu\} = 0$$

where J_μ is the bosonic isometry

$$J_\mu = V^a_\mu \pi_a + \bar{V}^{\bar{a}}_\mu \bar{\pi}_{\bar{a}} - B\mathtt{h}_\mu(z\bar{z}), \quad V^a_\mu = ig^{\bar{b}a}\partial_{\bar{b}}\mathtt{h}_\mu(z,\bar{z})$$

This means that the supersymmetric system constructed inherits all the kinematical symmetries of the initial system. In particular, in the case of $\mathbb{CP}^N$-Landau problem the extended system respects $SU(N+1)$ symmetry.

Bosonic Kähler oscillator

Bosonic Kähler oscillator in a constant magnetic field can be introduced

$$H_{osc} = g^{\bar{a}b}\left(\bar{\pi}_a \pi_b + |\omega|^2 \partial_{\bar{a}} K \partial_b K\right), \quad \{\pi_a, z^b\} = \delta_a^b, \quad \{\pi_a, \bar{\pi}_b\} = iB\, g_{a\bar{b}}$$

Bosonic Kähler oscillator

Bosonic Kähler oscillator in a constant magnetic field can be introduced

$$H_{osc} = g^{\bar{a}b}\left(\bar{\pi}_a\pi_b + |\omega|^2\partial_{\bar{a}}K\partial_b K\ \right), \quad \{\pi_a, z^b\} = \delta_a^b, \quad \{\pi_a, \bar{\pi}_b\} = iB\, g_{a\bar{b}}$$

We obtain non-equivalent Hamiltonians parameterized by an arbitrary holomorphic function $U(z)$ using the "gauge" transformation

$$K(z,\bar{z}) \to K(z,\bar{z}) + \frac{1}{\omega}U(z) + \frac{1}{\bar{\omega}}\bar{U}(\bar{z})$$

Bosonic Kähler oscillator

Bosonic Kähler oscillator in a constant magnetic field can be introduced

$$H_{osc} = g^{\bar{a}b}\left(\bar{\pi}_a\pi_b + |\omega|^2\partial_{\bar{a}}K\partial_b K\right), \quad \{\pi_a, z^b\} = \delta_a^b, \quad \{\pi_a, \bar{\pi}_b\} = iB\, g_{a\bar{b}}$$

We obtain non-equivalent Hamiltonians parameterized by an arbitrary holomorphic function $U(z)$ using the "gauge" transformation

$$K(z,\bar{z}) \to K(z,\bar{z}) + \frac{1}{\omega}U(z) + \frac{1}{\bar{\omega}}\bar{U}(\bar{z})$$

Examples of Kähler oscillators

- $\mathbb{C}^N$-oscillator
- $\mathbb{C}^N$-Smorodinsky-Winternitz
- $\mathbb{CP}^N$-oscillator
- $\mathbb{CP}^N$-Rosochatius

$SU(2|1)$ Kähler superoscillator

we can preserve the form of $SU(2)$ R-charges and adopt the following slightly modified expressions for the supercharges

$$\Theta^{\alpha} = \pi_a \eta^{a\alpha} + i\bar{\omega}\bar{\partial}_a K \varepsilon^{\alpha\beta}\bar{\eta}^a_{\beta}, \quad \overline{\Theta}_{\alpha} = \bar{\pi}_a \bar{\eta}^a_{\alpha} + i\omega\partial_a K \varepsilon_{\alpha\beta}\eta^{a\beta}$$

$SU(2|1)$ Kähler superoscillator

we can preserve the form of $SU(2)$ R-charges and adopt the following slightly modified expressions for the supercharges

$$\Theta^{\alpha} = \pi_a \eta^{a\alpha} + i\bar{\omega}\bar{\partial}_a K \varepsilon^{\alpha\beta}\bar{\eta}^a_{\beta}, \quad \overline{\Theta}_{\alpha} = \bar{\pi}_a \bar{\eta}^a_{\alpha} + i\omega\partial_a K \varepsilon_{\alpha\beta}\eta^{a\beta}$$

Calculating their Poisson brackets, we obtain

$$\{\Theta^{\alpha}, \overline{\Theta}_{\beta}\} = i\delta^{\alpha}_{\beta}\mathcal{H}_{osc} + iB\mathcal{R}^{\alpha}_{\beta}, \quad \{\Theta^{\alpha}, \Theta^{\beta}\} = 2i\bar{\omega}\mathcal{R}^{\alpha\beta},$$

$$\{\Theta^{\alpha}, \mathcal{R}^{\beta}_{\gamma}\} = -i\delta^{\alpha}_{\gamma}\Theta^{\beta} + \frac{i}{2}\delta^{\beta}_{\gamma}\Theta^{\alpha}$$

$SU(2|1)$ Kähler superoscillator

we can preserve the form of $SU(2)$ R-charges and adopt the following slightly modified expressions for the supercharges

$$\Theta^{\alpha} = \pi_a \eta^{a\alpha} + i\bar{\omega}\bar{\partial}_a K \varepsilon^{\alpha\beta} \bar{\eta}^a_{\beta}, \quad \overline{\Theta}_{\alpha} = \bar{\pi}_a \bar{\eta}^a_{\alpha} + i\omega\partial_a K \varepsilon_{\alpha\beta} \eta^{a\beta}$$

Calculating their Poisson brackets, we obtain

$$\{\Theta^{\alpha}, \overline{\Theta}_{\beta}\} = i\delta^{\alpha}_{\beta}\mathcal{H}_{osc} + iB\mathcal{R}^{\alpha}_{\beta}, \quad \{\Theta^{\alpha}, \Theta^{\beta}\} = 2i\bar{\omega}\mathcal{R}^{\alpha\beta},$$

$$\{\Theta^{\alpha}, \mathcal{R}^{\beta}_{\gamma}\} = -i\delta^{\alpha}_{\gamma}\Theta^{\beta} + \frac{i}{2}\delta^{\beta}_{\gamma}\Theta^{\alpha}$$

where the Hamiltonian is now given by the expression

$$\mathcal{H}_{osc} = g^{\bar{a}b}(\bar{\pi}_a \pi_b + |\omega|^2 \partial_{\bar{a}} K \partial_b K) - \frac{1}{2} R_{a\bar{b}c\bar{d}} \eta^{a\alpha} \bar{\eta}^b_{\alpha} \eta^{c\beta} \bar{\eta}^d_{\beta}$$

$$-\frac{1}{2}\omega K_{a;b} \eta^{a\alpha} \eta^b_{\alpha} - \frac{1}{2}\bar{\omega} K_{\bar{a};\bar{b}} \bar{\eta}^a_{\alpha} \bar{\eta}^{b\alpha} + \frac{B}{2} g_{a\bar{b}} \eta^{a\alpha} \bar{\eta}^b_{\alpha}$$

$SU(2|1)$ Kähler superoscillator

In order to bring this superalgebra into the conventional form we rotate the supercharges

$$Q^{\alpha} = e^{i\nu/2}\cos\lambda\Theta^{\alpha} + e^{-i\nu/2}\sin\lambda\varepsilon^{\alpha\gamma}\overline{\Theta}_{\gamma}, \quad \overline{Q}_{\alpha} = e^{-i\nu/2}\cos\lambda\overline{\Theta}_{\alpha} - e^{i\nu/2}\sin\lambda\varepsilon_{\alpha\gamma}\Theta^{\gamma}$$

where

$$\cos 2\lambda = \frac{B}{\sqrt{4|\omega|^2 + B^2}}, \quad \sin 2\lambda = -\frac{2|\omega|}{\sqrt{4|\omega|^2 + B^2}}, \qquad \omega = |\omega|e^{i\nu}\,.$$

$SU(2|1)$ Kähler superoscillator

In order to bring this superalgebra into the conventional form we rotate the supercharges

$$Q^{\alpha} = e^{i\nu/2}\cos\lambda\Theta^{\alpha} + e^{-i\nu/2}\sin\lambda\varepsilon^{\alpha\gamma}\overline{\Theta}_{\gamma}, \quad \overline{Q}_{\alpha} = e^{-i\nu/2}\cos\lambda\overline{\Theta}_{\alpha} - e^{i\nu/2}\sin\lambda\varepsilon_{\alpha\gamma}\Theta^{\gamma}$$

where

$$\cos 2\lambda = \frac{B}{\sqrt{4|\omega|^2 + B^2}}, \quad \sin 2\lambda = -\frac{2|\omega|}{\sqrt{4|\omega|^2 + B^2}}, \qquad \omega = |\omega| e^{i\nu}.$$

In terms of these quantities the $SU(2|1)$ symmetry algebra is rewritten as

$$\{Q^{\alpha}, \overline{Q}_{\beta}\} = i\delta^{\alpha}_{\beta}\mathcal{H}_{osc} + \sqrt{4|\omega|^2 + B^2}\,\mathcal{R}^{\alpha}_{\beta}, \quad \{Q^{\alpha}, \mathcal{H}_{osc}\} = \frac{i}{2}\sqrt{4|\omega|^2 + B^2}\,Q^{\alpha}$$

$$\{Q^{\alpha}, Q^{\beta}\} = \{\overline{Q}_{\alpha}, \overline{Q}_{\beta}\} = 0, \quad \{Q^{\alpha}, \mathcal{R}^{\beta}_{\gamma}\} = -i\delta^{\alpha}_{\gamma}Q^{\beta} + \frac{i}{2}\delta^{\beta}_{\gamma}Q^{\alpha}$$

$$\{\mathcal{R}^{\alpha}_{\beta}, \mathcal{R}^{\gamma}_{\delta}\} = i\delta^{\gamma}_{\beta}\mathcal{R}^{\alpha}_{\delta} - i\delta^{\alpha}_{\delta}\mathcal{R}^{\gamma}_{\beta}, \quad \{\mathcal{R}^{\alpha}_{\beta}, \mathcal{H}_{osc}\} = 0.$$

Supersymmetrization of $\mathbb{C}^N$-oscillator

We define the $\mathbb{C}^N$-harmonic oscillator defined as a Kähler oscillator with $K(z,\bar{z}) = \sum_{a=1}^{N} z^a \bar{z}^a$ and $\omega = \bar{\omega}$

$$\mathcal{H} = \sum_{a=1}^{N} \pi_a \bar{\pi}_a + \omega^2 z^a \bar{z}^a + \frac{B}{2}\eta^{a\alpha}\bar{\eta}^a_\alpha, \quad \Theta^\alpha = \sum_{a=1}^{N} \pi_a \eta^{a\alpha} + i\omega z^a \varepsilon^{\alpha\beta}\bar{\eta}^a_\beta,$$

$$\mathcal{R}^\alpha_\beta = \sum_{a=1}^{N} \eta^{a\alpha}\bar{\eta}^a_\beta - \frac{1}{2}\delta^\alpha_\beta i\eta^{a\gamma}\bar{\eta}^a_\gamma$$

Supersymmetrization of $\mathbb{C}^N$-oscillator

We define the $\mathbb{C}^N$-harmonic oscillator defined as a Kähler oscillator with $K(z,\bar{z}) = \sum_{a=1}^{N} z^a \bar{z}^a$ and $\omega = \bar{\omega}$

$$\mathcal{H} = \sum_{a=1}^{N} \pi_a \bar{\pi}_a + \omega^2 z^a \bar{z}^a + \frac{B}{2} \eta^{a\alpha} \bar{\eta}^a_\alpha, \quad \Theta^\alpha = \sum_{a=1}^{N} \pi_a \eta^{a\alpha} + i\omega z^a \varepsilon^{\alpha\beta} \bar{\eta}^a_\beta,$$

$$\mathcal{R}^\alpha_\beta = \sum_{a=1}^{N} \eta^{a\alpha} \bar{\eta}^a_\beta - \frac{1}{2} \delta^\alpha_\beta i \eta^{a\gamma} \bar{\eta}^a_\gamma$$

All constants of motion of the bosonic Hamiltonian become those of the supersymmetrized one, since all these quantities are just sums of bosonic and fermionic parts. Moreover, in the supersymmetric system there appear additional symmetry generators acting on the fermionic variables only.

$$J_{a\bar{b}} = i\pi_a z^b - i\bar{\pi}_b \bar{z}^a - B z^b \bar{z}^a, \quad I_{a\bar{b}} = \pi_a \bar{\pi}_b + \omega^2 \bar{z}^a z^b, \quad \mathcal{R}_{a\bar{b}} = \sum_\alpha \eta^{b\alpha} \bar{\eta}^a_\alpha$$

Supersymmetrization of $\mathbb{C}^N$-Smorodinsky-Winternitz

This system can be identified as a Kähler oscillator with the following Kähler potential

$$K = z\bar{z} + \frac{g_a}{\omega}\log z^a + \frac{\bar{g}_a}{\bar{\omega}}\log \bar{z}^a, \qquad \arg\omega = \arg\sum_{a=1}^{N} g_a + \pi/2.$$

$$\mathcal{H}_{SW} = \sum_{a=1}^{N} \pi_a\bar{\pi}_a + |\omega|^2 z^a\bar{z}^a + \frac{|g_a|^2}{z^a\bar{z}^a} + \frac{g_a}{2}\frac{\eta^{a\alpha}\eta^a_\alpha}{z^a z^a} + \frac{\bar{g}_a}{2}\frac{\bar{\eta}^a_\alpha\bar{\eta}^{a\alpha}}{\bar{z}^a\bar{z}^a} + \frac{B}{2}\eta^{a\alpha}\bar{\eta}^a_\alpha$$

Supersymmetrization of $\mathbb{C}^N$-Smorodinsky-Winternitz

This system can be identified as a Kähler oscillator with the following Kähler potential

$$K = z\bar{z} + \frac{g_a}{\omega}\log z^a + \frac{\bar{g}_a}{\bar{\omega}}\log \bar{z}^a, \qquad \arg\omega = \arg\sum_{a=1}^{N} g_a + \pi/2.$$

$$\mathcal{H}_{SW} = \sum_{a=1}^{N}\pi_a\bar{\pi}_a + |\omega|^2 z^a\bar{z}^a + \frac{|g_a|^2}{z^a\bar{z}^a} + \frac{g_a}{2}\frac{\eta^{a\alpha}\eta^a_\alpha}{z^a z^a} + \frac{\bar{g}_a}{2}\frac{\bar{\eta}^a_\alpha\bar{\eta}^{a\alpha}}{\bar{z}^a\bar{z}^a} + \frac{B}{2}\eta^{a\alpha}\bar{\eta}^a_\alpha$$

This supersymmetric system possesses N manifest $U(1)$ symmetries

$$\mathcal{J}_{a\bar{a}} = J_{a\bar{a}} + \eta^{a\alpha}\bar{\eta}^a_\alpha$$

Supersymmetrization of $\mathbb{C}^N$-Smorodinsky-Winternitz

This system can be identified as a Kähler oscillator with the following Kähler potential

$$K = z\bar{z} + \frac{g_a}{\omega}\log z^a + \frac{\bar{g}_a}{\bar{\omega}}\log \bar{z}^a, \qquad \arg\omega = \arg\sum_{a=1}^{N} g_a + \pi/2.$$

$$\mathcal{H}_{SW} = \sum_{a=1}^{N}\pi_a\bar{\pi}_a + |\omega|^2 z^a\bar{z}^a + \frac{|g_a|^2}{z^a\bar{z}^a} + \frac{g_a}{2}\frac{\eta^{a\alpha}\eta^a_\alpha}{z^a z^a} + \frac{\bar{g}_a}{2}\frac{\bar{\eta}^a_\alpha\bar{\eta}^{a\alpha}}{\bar{z}^a\bar{z}^a} + \frac{B}{2}\eta^{a\alpha}\bar{\eta}^a_\alpha$$

This supersymmetric system possesses N manifest $U(1)$ symmetries

$$\mathcal{J}_{a\bar{a}} = J_{a\bar{a}} + \eta^{a\alpha}\bar{\eta}^a_\alpha$$

Extension of hidden symmetries

$$\mathcal{I}_{ab} = I_{ab} + \frac{g_a}{2}\frac{z^b\bar{z}^b}{z^a z^a}\eta^{a\alpha}\eta^a_\alpha + \frac{\bar{g}_a}{2}\frac{z^b\bar{z}^b}{\bar{z}^a\bar{z}^a}\bar{\eta}^a_\alpha\bar{\eta}^{a\alpha} + \frac{g_b}{2}\frac{z^a\bar{z}^a}{z^b z^b}\eta^{b\alpha}\eta^b_\alpha + \frac{\bar{g}_b}{2}\frac{z^a\bar{z}^a}{\bar{z}^b\bar{z}^b}\bar{\eta}^b_\alpha\bar{\eta}^{b\alpha}$$

Supersymmetrization of $\mathbb{CP}^N$-oscillator

We can choose the Kähler potential to be $K = \log(1 + z\bar{z})$.
The relevant Hamiltonian reads

$$\mathcal{H}_{osc} = g^{\bar{a}b}\bar{\pi}_a\pi_b + |\omega|^2 z\bar{z} - \frac{1}{2}(g_{a\bar{b}}g_{c\bar{d}} + g_{c\bar{b}}g_{a\bar{d}})\eta^{a\alpha}\bar{\eta}^b_\alpha\eta^{c\beta}\bar{\eta}^d_\beta$$

$$-\frac{\omega}{2}\frac{\bar{z}^a\bar{z}^b\eta^{a\alpha}\eta^b_\alpha}{(1+z\bar{z})^2} - \frac{\bar{\omega}}{2}\frac{z^a z^b\bar{\eta}^a_\alpha\bar{\eta}^{b\alpha}}{(1+z\bar{z})^2} + \frac{B}{2}g_{a\bar{b}}\eta^{a\alpha}\bar{\eta}^b_\alpha$$

Supersymmetrization of $\mathbb{CP}^N$-oscillator

We can choose the Kähler potential to be $K = \log(1 + z\bar{z})$.
The relevant Hamiltonian reads

$$\mathcal{H}_{osc} = g^{\bar{a}b}\bar{\pi}_a\pi_b + |\omega|^2 z\bar{z} - \frac{1}{2}(g_{a\bar{b}}g_{c\bar{d}} + g_{c\bar{b}}g_{a\bar{d}})\eta^{a\alpha}\bar{\eta}^b_\alpha\eta^{c\beta}\bar{\eta}^d_\beta$$

$$-\frac{\omega}{2}\frac{\bar{z}^a\bar{z}^b\eta^{a\alpha}\eta^b_\alpha}{(1+z\bar{z})^2} - \frac{\bar{\omega}}{2}\frac{z^a z^b\bar{\eta}^a_\alpha\bar{\eta}^{b\alpha}}{(1+z\bar{z})^2} + \frac{B}{2}g_{a\bar{b}}\eta^{a\alpha}\bar{\eta}^b_\alpha$$

This system has the manifest $u(N)$ symmetry defined by the generators

$$\mathcal{J}_{a\bar{b}} = J_{a\bar{b}} + \frac{\partial^2 h_{a\bar{b}}}{\partial z^c \partial \bar{z}^d}\eta^{c\alpha}\bar{\eta}^d_\alpha, \qquad \{\mathcal{J}_{a\bar{b}}, \mathcal{H}_{osc}\} = 0$$

Supersymmetrization of $\mathbb{CP}^N$-oscillator

We can choose the Kähler potential to be $K = \log(1 + z\bar{z})$.
The relevant Hamiltonian reads

$$\mathcal{H}_{osc} = g^{\bar{a}b}\bar{\pi}_a\pi_b + |\omega|^2 z\bar{z} - \frac{1}{2}(g_{a\bar{b}}g_{c\bar{d}} + g_{c\bar{b}}g_{a\bar{d}})\eta^{a\alpha}\bar{\eta}^b_\alpha\eta^{c\beta}\bar{\eta}^d_\beta$$

$$-\frac{\omega}{2}\frac{\bar{z}^a\bar{z}^b\eta^{a\alpha}\eta^b_\alpha}{(1+z\bar{z})^2} - \frac{\bar{\omega}}{2}\frac{z^a z^b\bar{\eta}^a_\alpha\bar{\eta}^{b\alpha}}{(1+z\bar{z})^2} + \frac{B}{2}g_{a\bar{b}}\eta^{a\alpha}\bar{\eta}^b_\alpha$$

This system has the manifest $u(N)$ symmetry defined by the generators

$$\mathcal{J}_{a\bar{b}} = J_{a\bar{b}} + \frac{\partial^2 h_{a\bar{b}}}{\partial z^c\partial\bar{z}^d}\eta^{c\alpha}\bar{\eta}^d_\alpha, \qquad \{\mathcal{J}_{a\bar{b}}, \mathcal{H}_{osc}\} = 0$$

Straightforward generalization of the Fradkin tensor, with J_a replaced by $\mathcal{J}_a$, is not a constant of motion. So, for the time being, it is an open question whether a supersymmetric counterpart of the Fradkin tensor exists.

Supersymmetrization of $\mathbb{CP}^N$-Rosochatius

$\mathbb{CP}^N$-Rosochatius model can be considered as a spherical counterpart of the $\mathbb{C}^N$-Smorodinsky-Winternitz model. Kähler potential can be chosen

$$K = \log(1 + z\bar{z}) - \sum_{a=1}^{N}\left(\frac{\omega_a}{\omega}\log z^a + \frac{\bar{\omega}_a}{\bar{\omega}}\log \bar{z}^a\right)$$

Supersymmetrization of $\mathbb{CP}^N$-Rosochatius

$\mathbb{CP}^N$-Rosochatius model can be considered as a spherical counterpart of the $\mathbb{C}^N$-Smorodinsky-Winternitz model. Kähler potential can be chosen

$$K = \log(1 + z\bar{z}) - \sum_{a=1}^{N}\left(\frac{\omega_a}{\omega}\log z^a + \frac{\bar{\omega}_a}{\bar{\omega}}\log \bar{z}^a\right)$$

The bosonic Hamiltonian (H_{Ros}) is

$$H_{Ros} = (1 + z\bar{z})\left(\pi\bar{\pi} + (z\pi)(\bar{z}\bar{\pi}) + |\omega_0|^2 + \sum_{a=1}^{N}\frac{|\omega_a|^2}{z^a\bar{z}^a}\right) - \sum_{i=0}^{N}|\omega_i|^2.$$

Supersymmetrization of $\mathbb{CP}^N$-Rosochatius

$\mathbb{CP}^N$-Rosochatius model can be considered as a spherical counterpart of the $\mathbb{C}^N$-Smorodinsky-Winternitz model. Kähler potential can be chosen

$$K = \log(1+z\bar{z}) - \sum_{a=1}^{N}\left(\frac{\omega_a}{\omega}\log z^a + \frac{\bar{\omega}_a}{\bar{\omega}}\log\bar{z}^a\right)$$

The bosonic Hamiltonian (H_{Ros}) is

$$H_{Ros} = (1+z\bar{z})\left(\pi\bar{\pi} + (z\pi)(\bar{z}\bar{\pi}) + |\omega_0|^2 + \sum_{a=1}^{N}\frac{|\omega_a|^2}{z^a\bar{z}^a}\right) - \sum_{i=0}^{N}|\omega_i|^2.$$

The Supersymmetric Hamiltonian ($\mathcal{H}_{Ros}$) is presented

$$\mathcal{H}_{Ros} = H_{Ros} - \frac{1}{2}(g_{a\bar{b}}g_{c\bar{d}} + g_{c\bar{b}}g_{a\bar{d}})\eta^{a\alpha}\bar{\eta}^b_\alpha\eta^{c\beta}\bar{\eta}^d_\beta + \frac{B}{2}g_{a\bar{b}}\eta^{a\alpha}\bar{\eta}^b_\alpha$$

$$-\left(\frac{\omega\bar{z}^a\bar{z}^b}{1+z\bar{z}} - \frac{\omega_a\bar{z}^b}{z^a} - \frac{\omega_b\bar{z}^a}{z^b}\right)\frac{\eta^{a\alpha}\eta^b_\alpha}{2(1+z\bar{z})} - \left(\frac{\bar{\omega}z^az^b}{1+z\bar{z}} - \frac{\bar{\omega}_az^b}{\bar{z}^a} - \frac{\bar{\omega}_bz^a}{\bar{z}^b}\right)\frac{\bar{\eta}^a_\alpha\bar{\eta}^{b\alpha}}{2(1+z\bar{z})}$$

$SU(4|1)$ (deformed $\mathcal{N} = 8$) SUSY Landau problem

We perform a similar construction for $\mathcal{N} = 8$ supersymmetric mechanics on the special Kähler manifolds of the rigid type (equipped with a symmetric tensor f_{abc} satisfying compatibility conditions)

$SU(4|1)$ (deformed $\mathcal{N}=8$) SUSY Landau problem

We perform a similar construction for $\mathcal{N}=8$ supersymmetric mechanics on the special Kähler manifolds of the rigid type (equipped with a symmetric tensor f_{abc} satisfying compatibility conditions)
We define the supercharges and R-charges ($su(4)$) as

$$Q^I = \pi_a \eta^{aI} + \frac{i}{3}\,\bar{f}_{abc}\,\bar{T}^{abcI}, \quad R^I_J = \eta^{aI} g_{a\bar{b}}\,\bar{\eta}^b_J - \frac{\delta^I_J}{4}\,\eta^{aK} g_{a\bar{b}}\,\bar{\eta}^b_K$$

$SU(4|1)$ (deformed $\mathcal{N}=8$) SUSY Landau problem

We perform a similar construction for $\mathcal{N}=8$ supersymmetric mechanics on the special Kähler manifolds of the rigid type (equipped with a symmetric tensor f_{abc} satisfying compatibility conditions)
We define the supercharges and R-charges ($su(4)$) as

$$Q^I = \pi_a \eta^{aI} + \frac{i}{3}\,\bar{f}_{abc}\,\bar{T}^{abcI}, \quad R^I_J = \eta^{aI} g_{a\bar{b}}\,\bar{\eta}^b_J - \frac{\delta^I_J}{4}\,\eta^{aK} g_{a\bar{b}}\,\bar{\eta}^b_K$$

We find the $su(4|1)$ superalgebra

$$\{Q^I, \overline{Q}_J\} = i\delta^I_J \mathcal{H}_0 + iBR^I_J, \quad \{R^I_J, Q^K\} = i\delta^K_J Q^I - \frac{i}{4}\,\delta^I_J Q^K, \quad \{\mathcal{H}_0, Q^K\} = -\frac{3iB}{4}\,Q^K$$

$$\mathcal{H}_0 = g^{\bar{a}b}\bar{\pi}_a\pi_b + R_{a\bar{b}c\bar{d}}\Lambda_0^{ac\bar{b}\bar{d}} + \frac{B}{4}\,\eta^{aK} g_{a\bar{b}}\bar{\eta}^b_K - \frac{1}{3} f_{abc;d}\Lambda^{abcd} - \frac{1}{3}\,\bar{f}_{abc;d}\bar{\Lambda}^{abcd}$$

$SU(4|1)$ (deformed $\mathcal{N}=8$) SUSY Landau problem

We perform a similar construction for $\mathcal{N}=8$ supersymmetric mechanics on the special Kähler manifolds of the rigid type (equipped with a symmetric tensor f_{abc} satisfying compatibility conditions)
We define the supercharges and R-charges ($su(4)$) as

$$Q^I = \pi_a \eta^{aI} + \frac{i}{3}\,\bar{f}_{abc}\,\bar{T}^{abcI}, \quad R^I_J = \eta^{aI} g_{a\bar{b}}\,\bar{\eta}^b_J - \frac{\delta^I_J}{4}\,\eta^{aK} g_{a\bar{b}}\,\bar{\eta}^b_K$$

We find the $su(4|1)$ superalgebra

$$\{Q^I, \overline{Q}_J\} = i\delta^I_J \mathcal{H}_0 + iBR^I_J, \quad \{R^I_J, Q^K\} = i\delta^K_J Q^I - \frac{i}{4}\,\delta^I_J Q^K, \quad \{\mathcal{H}_0, Q^K\} = -\frac{3iB}{4}\,Q^K$$

$$\mathcal{H}_0 = g^{\bar{a}b}\bar{\pi}_a \pi_b + R_{a\bar{b}c\bar{d}}\Lambda_0^{ac\bar{b}\bar{d}} + \frac{B}{4}\,\eta^{aK} g_{a\bar{b}}\bar{\eta}^b_K - \frac{1}{3} f_{abc;d}\Lambda^{abcd} - \frac{1}{3}\bar{f}_{abc;d}\bar{\Lambda}^{abcd}$$

Using these additional relations that $\mathbf{V}_\mu$ preserves as well the third-order tensor $f_{abc}dz^a dz^b dz^c$, the isometry generator commutes with all elements of superalgebra

$$\{\mathcal{J}_\mu, Q_I\} = \{\mathcal{J}_\mu, \overline{Q}_I\} = \{\mathcal{J}_\mu, R^I_J\} = \{\mathcal{J}_\mu, \mathcal{H}_0\} = 0$$

Outlook

- Quantum mechanics of described models

Outlook

- Quantum mechanics of described models
- $\mathcal{N} = 8$ supersymmetric mechanics with non-zero potential on special Kähler manifolds

Outlook

- Quantum mechanics of described models
- $\mathcal{N} = 8$ supersymmetric mechanics with non-zero potential on special Kähler manifolds
- Construction of the deformed supersymmetric extensions of the Landau problem on quaternionic manifolds (i.e. $\mathbb{HP}^N$)

Outlook

- Quantum mechanics of described models
- $\mathcal{N} = 8$ supersymmetric mechanics with non-zero potential on special Kähler manifolds
- Construction of the deformed supersymmetric extensions of the Landau problem on quaternionic manifolds (i.e. $\mathbb{HP}^N$)
- The construction of the $\mathbb{H}^N$-Smorodinsky-Winternitz and $\mathbb{HP}^N$-Rosochatius

Outlook

- Quantum mechanics of described models
- $\mathcal{N} = 8$ supersymmetric mechanics with non-zero potential on special Kähler manifolds
- Construction of the deformed supersymmetric extensions of the Landau problem on quaternionic manifolds (i.e. $\mathbb{HP}^N$)
- The construction of the $\mathbb{H}^N$-Smorodinsky-Winternitz and $\mathbb{HP}^N$-Rosochatius
- Introducing the notion of quaternionic oscillator, by analogy with the Kähler one and superextensions

Thank You!