COMPUTATION ON THE CONFORMAL MODULUS OF QUADRILATERALS

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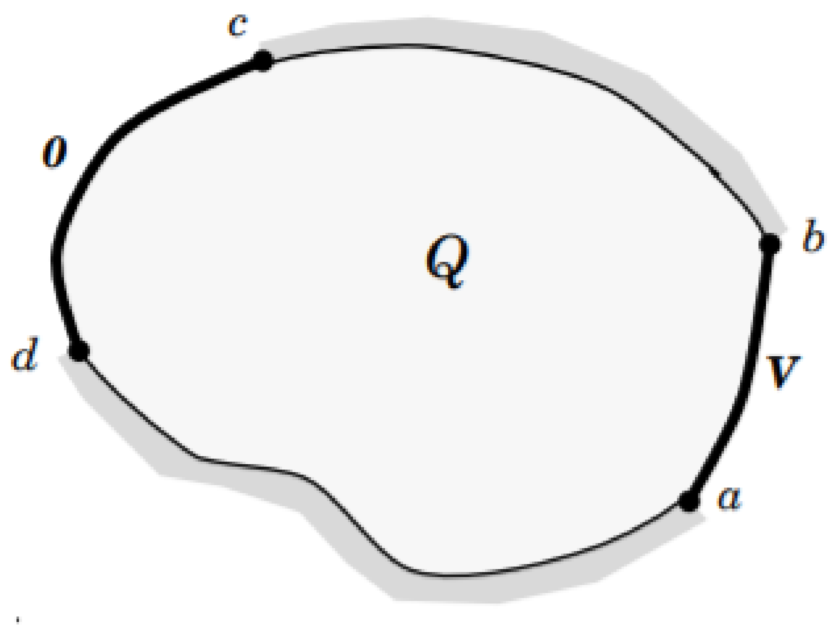
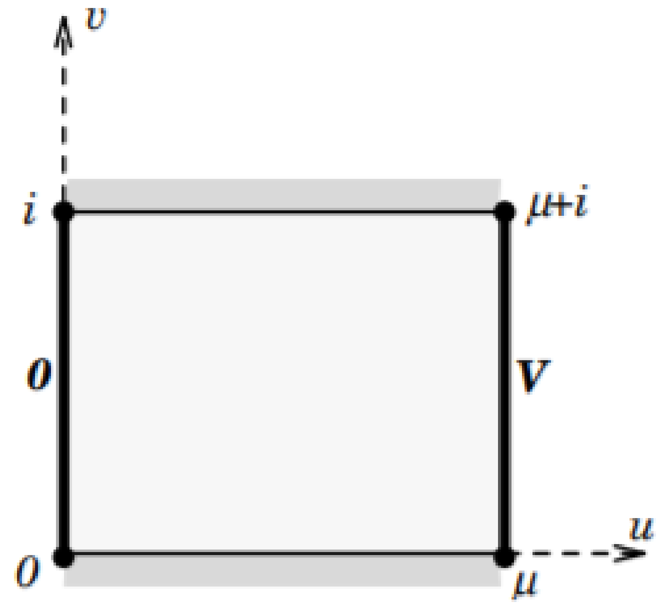
MOTIVATION (Physical interpretation)



z plane



w plane

$$V = \mu I$$

The plan of the presentation

1. Conformal equivalence relation
2. Conformal classification of Quadrilaterals
3. Conformal modulus of Quadrilaterals
4. The new method of calculation
5. The particular case
6. The relation to AMG functions
7. Summary

CONFORMAL MAPPING on COMPLEX PLANE

CONDITION 1: HOLOMORPHISM

$$\frac{\partial f(z)}{\partial \bar{z}} = 0$$

CONDITION 2: LOCAL BIJECTION

$$\frac{\partial f(z)}{\partial z} \neq 0$$

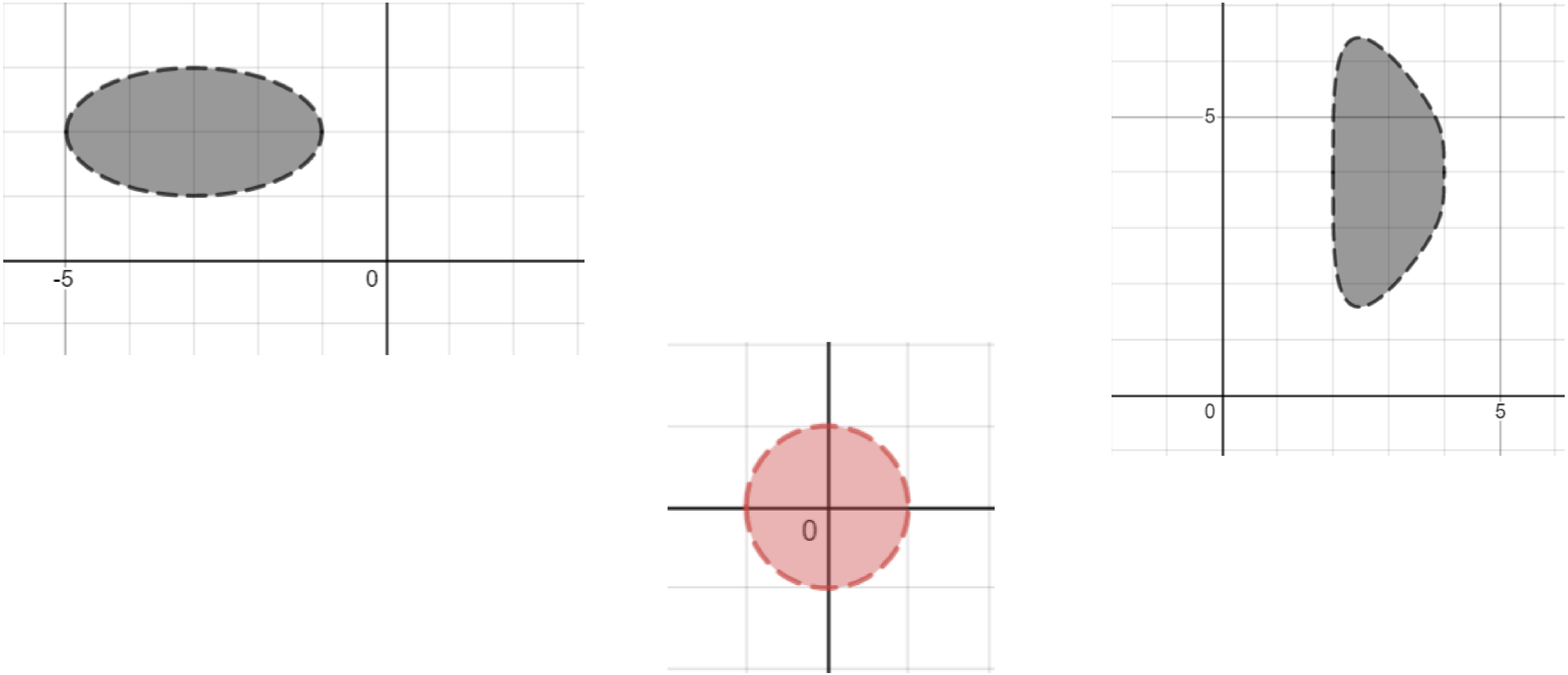
Conformal Equivalence Relation

Identity - Reflection: $f(z) = z$ is conformal

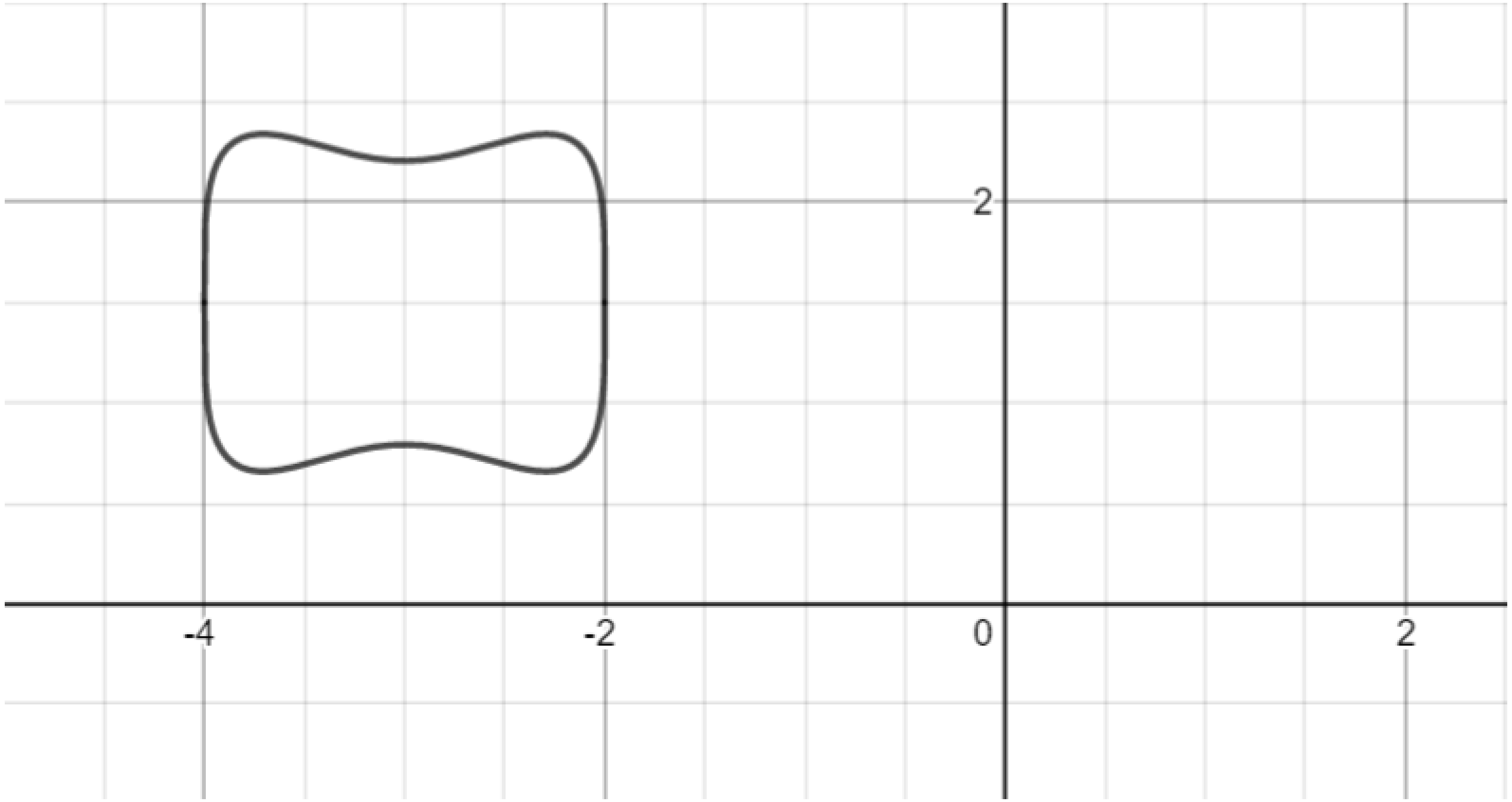
Inverse - Symmetry: If f is conf. so is f^{-1}

Composition - Transition: If f and g are conf. so is $g \circ f$

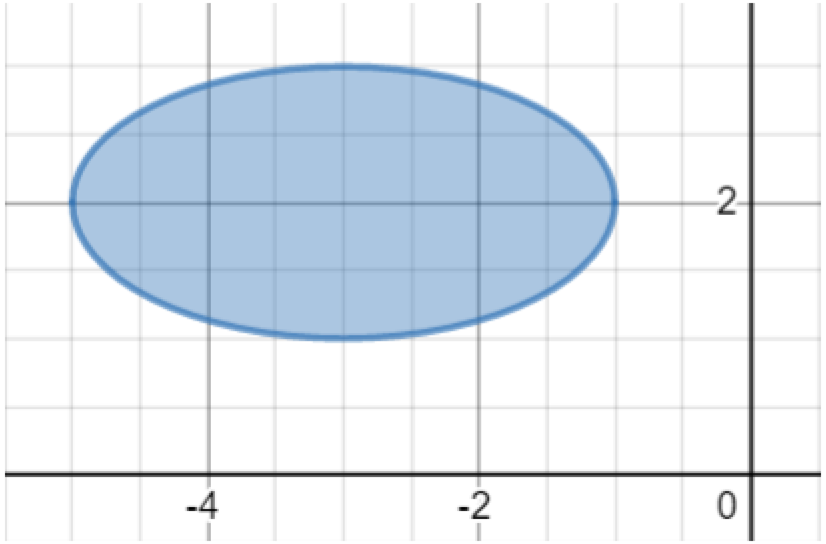
Riemann Mapping Theorem

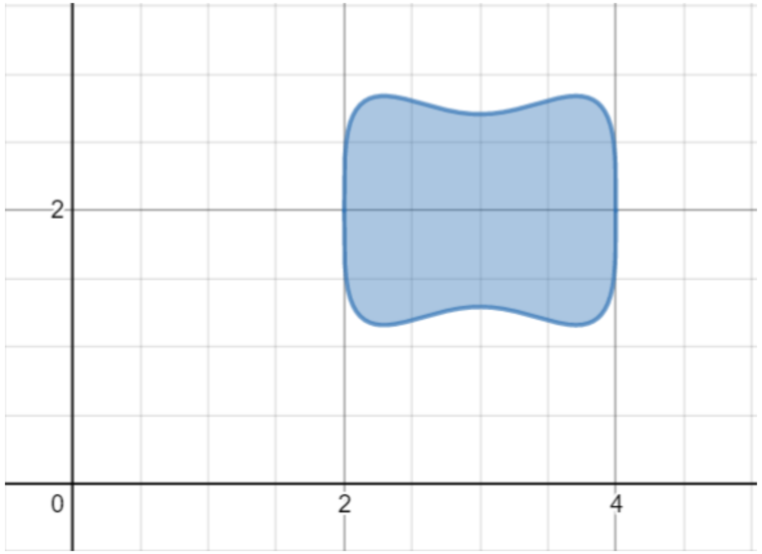


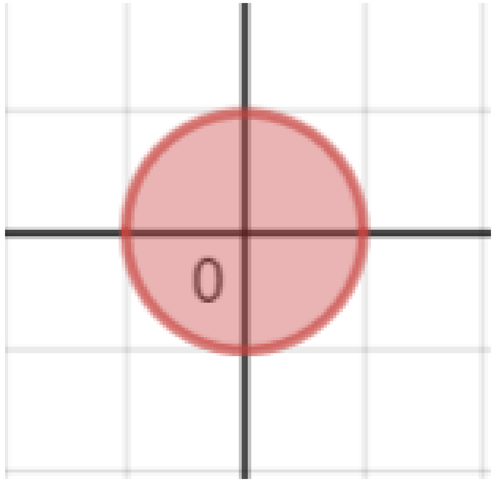
The Jordan Curve Theorem

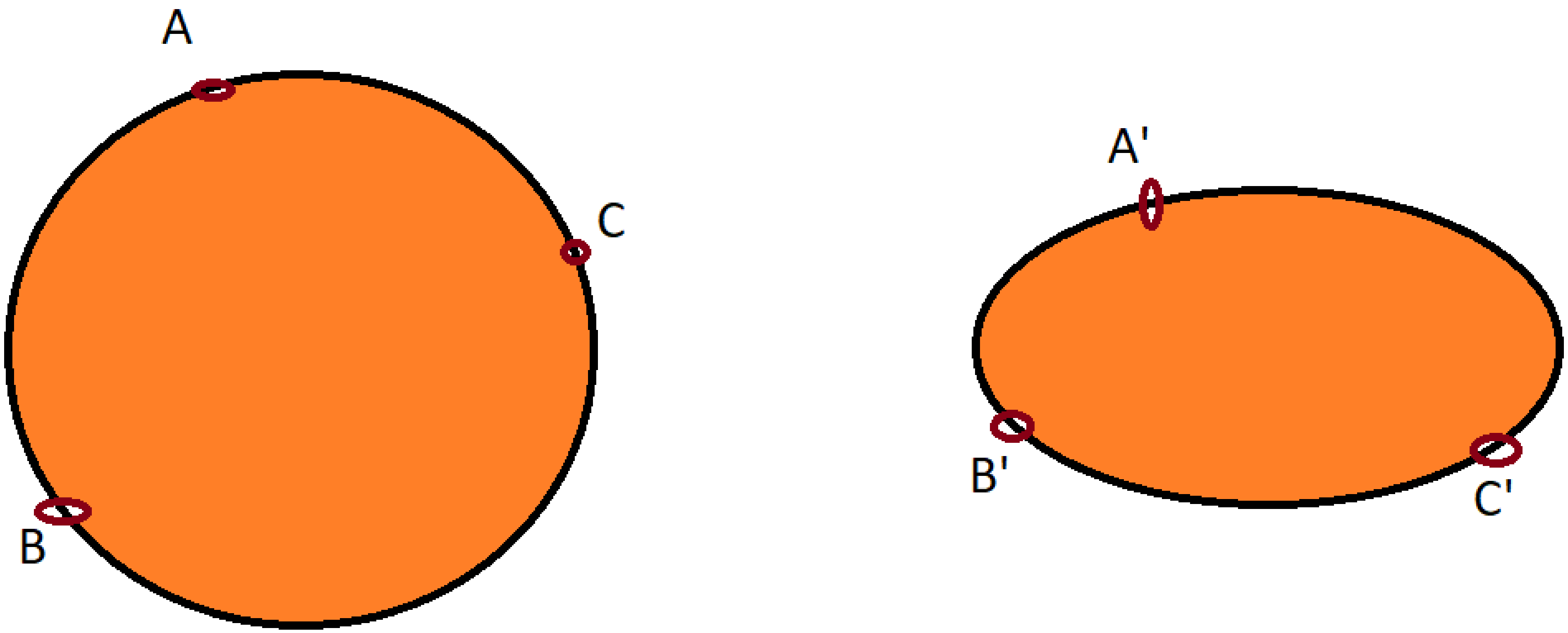


Caratheodory Extension Theorem

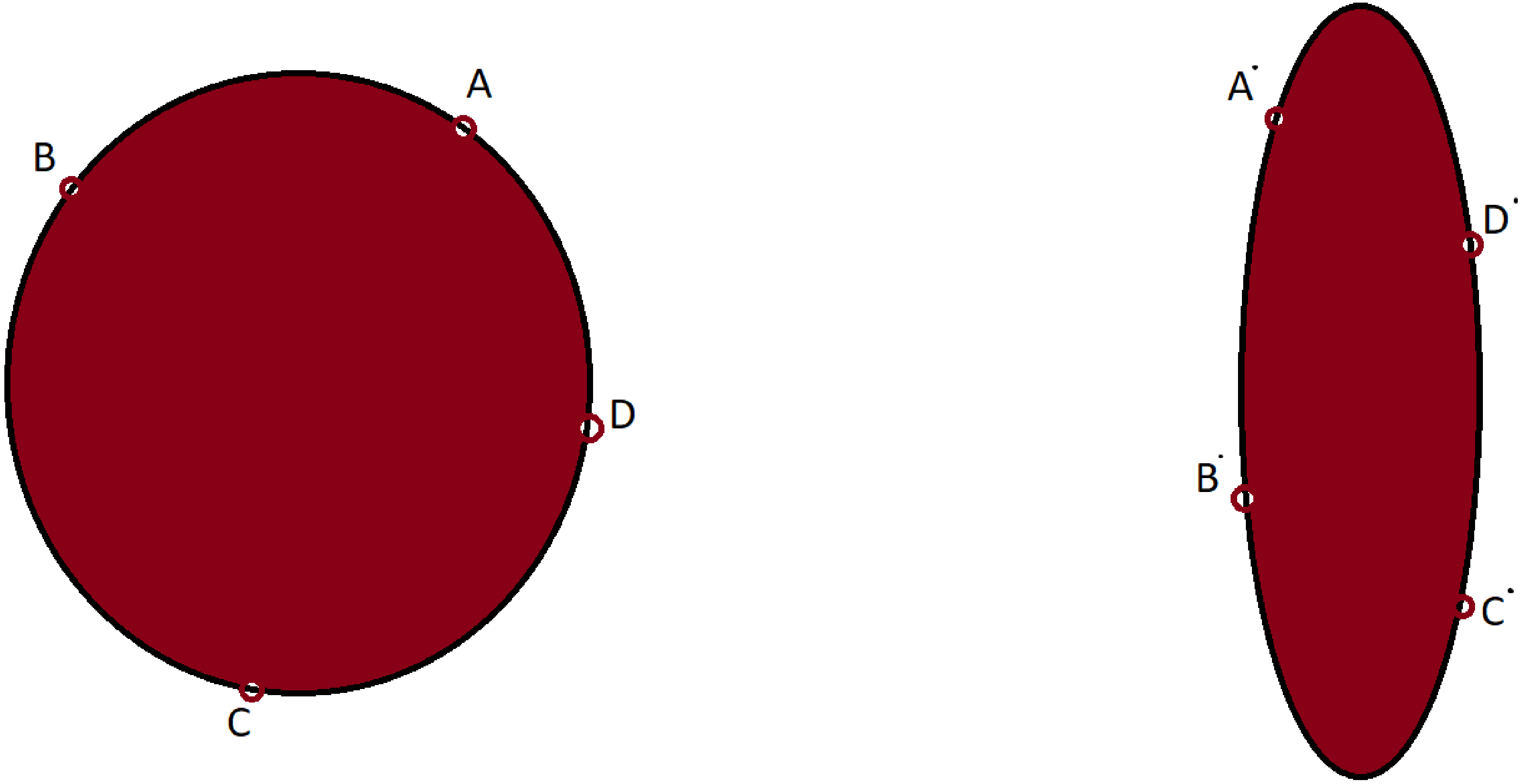








$$f:[A,B,C] \quad \rightarrow \quad [A^*,B^*,C^*]$$

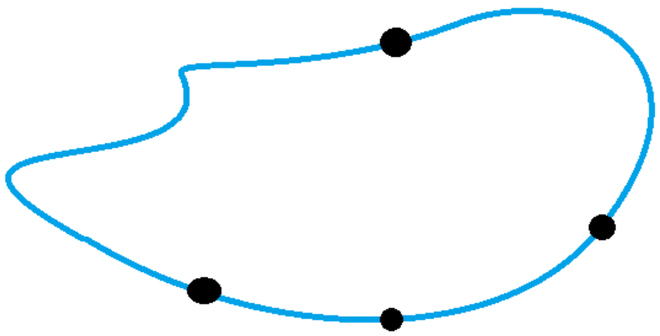


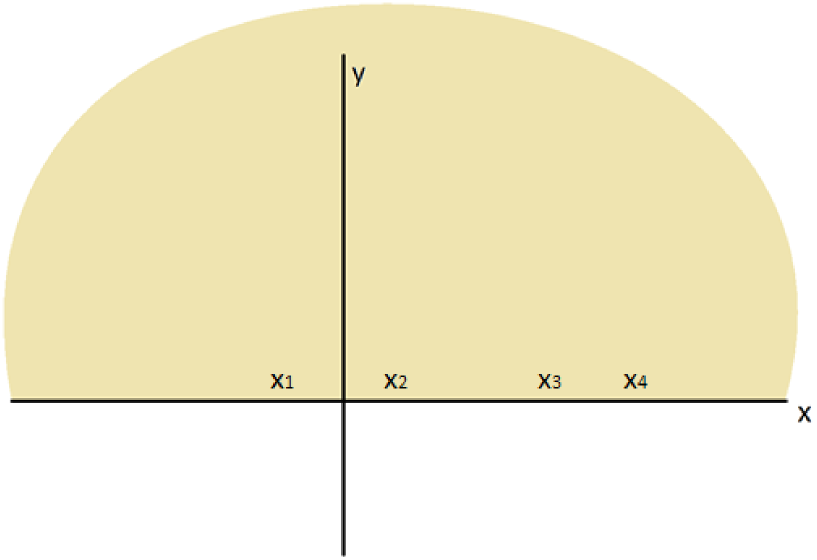
$$f: [A, B, C, D] \quad \rightarrow \quad [A^*, B^*, C^*, D^*]$$

Definition: Suppose M is some set. It is called the set of conformal invariants if there exists bijection

$$M \cong_{bij} \{Q/C\}$$

Example: ξ is a conformal invariant





$$\{x_1, x_2, x_3, x_4\} \to \{0, 1, \xi, \infty\}$$

$$\xi = \frac{x_3 - x_1}{x_3 - x_4} \frac{x_2 - x_4}{x_2 - x_1} > 1$$

Proof: ξ is the same for conformally equivalent quadrilaterals

$$Q\{q_1, q_2, q_3, q_4\} \simeq_{conf} H\{x_1, x_2, x_3, x_4\}$$

$$H\{x_1, x_2, x_3, x_4\} \simeq_{conf} H\{x_1', x_2', x_3', x_4'\}$$

$$Q'\{q_1', q_2', q_3', q_4'\} \simeq_{conf} H\{x_1', x_2', x_3', x_4'\}$$

$$Automorphism[H] =: f(z) = \frac{az + b}{cz + d};$$

but ξ is invariant under bilinear maps. Thus

$$\xi = \frac{x_3 - x_1}{x_3 - x_4}\frac{x_2 - x_4}{x_2 - x_1} = \frac{f(x_3) - f(x_1)}{f(x_3) - f(x_4)}\frac{f(x_2) - f(x_4)}{f(x_2) - f(x_1)} = \xi'$$

Proof: The same ξ corresponds to the same class

$$Q\{q_1, q_2, q_3, q_4\} \simeq_{conf} H\{0, 1, \xi, \infty\}$$

$$Q'\{q_1', q_2', q_3', q_4'\} \simeq_{conf} H\{0, 1, \xi, \infty\}$$

$$Q\{q_1, q_2, q_3, q_4\} \simeq_{conf} Q'\{q_1', q_2', q_3', q_4'\}$$

There is another conformal invariant...

$$H\{0, 1, \xi, \infty\} \simeq_{conf} H\{-\eta, -1, 1, \eta\} \qquad \eta = \frac{\sqrt{\xi}+1}{\sqrt{\xi}-1} > 1; \quad \mathrm{k} = \frac{1}{\eta};$$

$$\frac{dk}{d\xi} = \frac{1}{\sqrt{\xi}\left(\sqrt{\xi}+1\right)^2} > 0$$

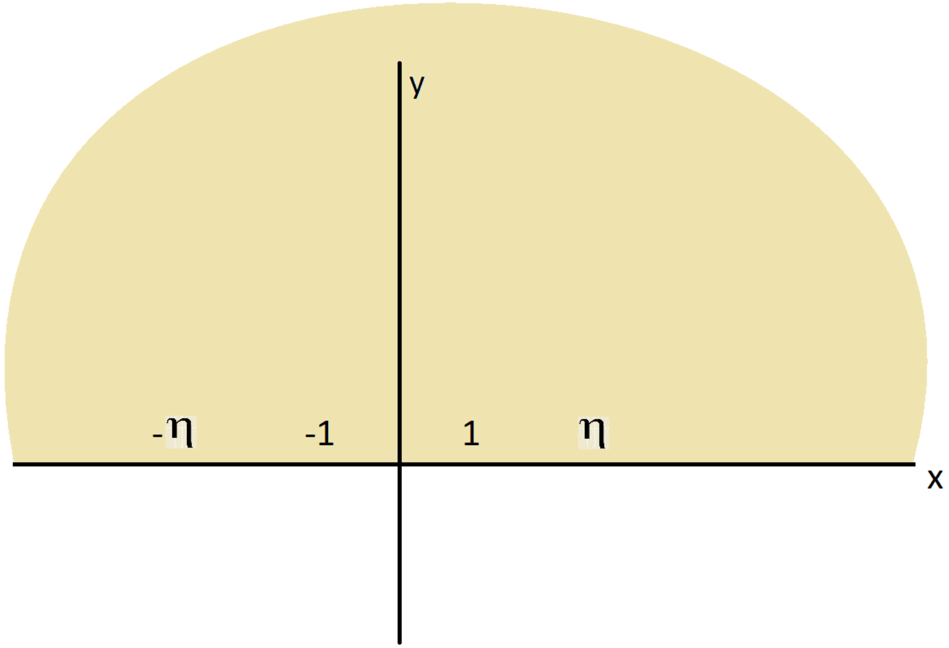
$$\xi: (1, \infty) \cong_{bij} k: (0, 1)$$

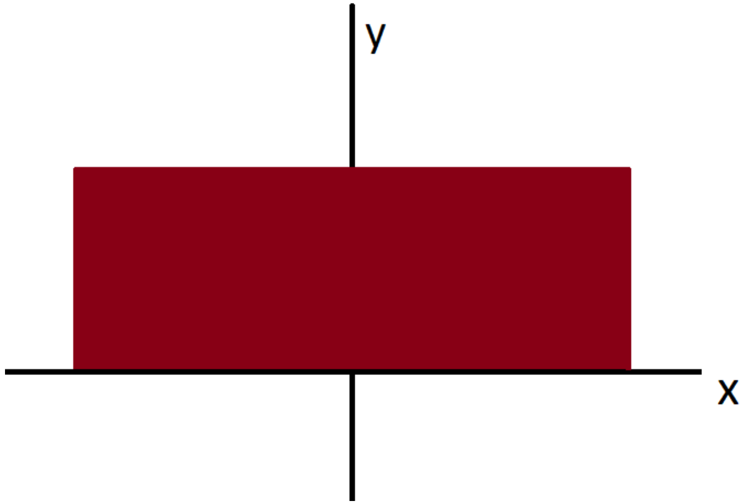
The following is true

two quadrilaterals are said to be conformally equivalent to each other iff their relevant ξ is the same!

two quadrilaterals are said to be conformally equivalent to each other iff their relevant k is the same!

FROM THE UPPER HALF PLANE INTO THE RECTANGLE





$$G(\omega_0) = A + B \int_0^{\omega_0} \frac{d\omega}{(\omega - a)^{1-\frac{\alpha}{\pi}}(\omega - b)^{1-\frac{\beta}{\pi}}\dots}$$

The rectangle: $\{D_x, D_x + iD_y, -D_x + iD_y, -D_x\}$

$$D_x = Bk\int_0^1 \frac{d\omega}{\sqrt{1-\omega^2}\sqrt{1-k^2\omega^2}} = BkK(k) > 0$$

$$D_y = Bk\int_0^1 \frac{d\omega}{\sqrt{1-\omega^2}\sqrt{1-k'^2\omega^2}} = BkK(k') > 0$$

Where $k' = \sqrt{1-k^2}$;

MODULUS AS FUNCTION OF "VERTICES"

$$[BkK(k); BkK(k) + iBkK(k'); -BkK(k) + iBkK(k'); -BkK(k)]$$

$$\mu(k) = \frac{2BkK(k)}{BkK(k')} = 2\frac{\int_0^1 \frac{d\omega}{\sqrt{1-\omega^2}\sqrt{1-k^2\omega^2}}}{\int_0^1 \frac{d\omega}{\sqrt{1-\omega^2}\sqrt{1-k'^2\omega^2}}}$$

Where $\mathrm{k} = \frac{1}{\eta} = \frac{\sqrt{\xi}-1}{\sqrt{\xi}+1}$

EQUIVALENCE OF MODULUS, "k" AND "ξ"

$$2\frac{\int_0^1 \frac{\omega^2 d\omega}{\sqrt{(1-\omega^2)}(1-k^2\omega^2)^{\frac{3}{2}}} \int_0^1 \frac{d\omega}{\sqrt{(1-\omega^2)(k^2\omega^2+1-\omega^2)}} + \int_0^1 \frac{d\omega}{\sqrt{(1-\omega^2)(1-k^2\omega^2)}} \int_0^1 \frac{\omega^2 d\omega}{\sqrt{(1-\omega^2)}(k^2\omega^2+1-\omega^2)^{\frac{3}{2}}}}{\left[\int_0^1 \frac{d\omega}{\sqrt{(1-\omega^2)(k^2\omega^2+1-\omega^2)}}\right]^2} k$$

This is always positive. So, the modulus increases when k varies from "0" to "1"

$$\xi:(1,\infty) \cong_{bij} k:(0,1) \cong_{bij} \mu:(0,\infty);$$

Thus there is one to one map between ξ, k and modulus, where

$$\xi = \frac{x_3 - x_1}{x_3 - x_4}\frac{x_2 - x_4}{x_2 - x_1}$$

An important question:

$$\mu(\xi)\ ?$$

ξ-transformation

$$\mu(q_2, q_3, q_4, q_1) = \frac{1}{\mu(q_1, q_2, q_3, q_4)}$$

$$[\mu, k, \xi] \ \leftrightarrow_{\xi} \ \left[\frac{1}{\mu}, \frac{\sqrt{\xi} - \sqrt{\xi - 1}}{\sqrt{\xi} + \sqrt{\xi - 1}}, \frac{\xi}{\xi - 1}\right]$$

k-transformation

$$\mu\left(\sqrt{1-k^2}\right) = \frac{4}{\mu(k)}$$

$$[\mu, k, \xi] \leftrightarrow_k \left[\frac{4}{\mu}, \sqrt{1-k^2}, \left(\frac{1+\sqrt{1-k^2}}{1-\sqrt{1-k^2}}\right)^2\right]$$

Non-symmetric combinations and the infinity chain

$$\dots \leftrightarrow_k \frac{1}{4\mu} \leftrightarrow_\xi 4\mu \leftrightarrow_k \frac{1}{\mu} \leftrightarrow_\xi \mu \leftrightarrow_k \frac{4}{\mu} \leftrightarrow_\xi \frac{\mu}{4} \leftrightarrow_k \frac{16}{\mu} \leftrightarrow_\xi \dots$$

$2^{2s}\mu$ or $2^{2s}\mu^{-1}$

Suppose $\mu = 1$. Using ξ-transformation

$$\xi = \frac{\xi}{\xi - 1} = 2, \qquad k = \frac{\sqrt{2}-1}{\sqrt{2}+1};$$

$$[\mu, k, \xi] \leftrightarrow_{\xi} \left[\frac{1}{\mu}, \frac{\sqrt{\xi} - \sqrt{\xi - 1}}{\sqrt{\xi} + \sqrt{\xi - 1}}, \frac{\xi}{\xi - 1}\right]$$

Suppose $\mu = 2$. Using k-transformation

$$k = \sqrt{1-k^2} = \frac{1}{\sqrt{2}}, \qquad \xi = \left(\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right)^2;$$

$$[\mu, k, \xi] \leftrightarrow_k \left[\frac{4}{\mu}, \sqrt{1-k^2}, \left(\frac{1+\sqrt{1-k^2}}{1-\sqrt{1-k^2}}\right)^2\right]$$

Two independent chains

$$1 \leftrightarrow_k 4 \leftrightarrow_\xi \frac{1}{4} \leftrightarrow_k 16 \leftrightarrow_\xi \frac{1}{16} \leftrightarrow_k 64 \leftrightarrow_\xi \frac{1}{64} \ldots$$

$$2 \leftrightarrow_\xi \frac{1}{2} \leftrightarrow_k 8 \leftrightarrow_\xi \frac{1}{8} \leftrightarrow_k 32 \leftrightarrow_\xi \frac{1}{32} \leftrightarrow_k 128 \ldots$$

Some discrete values of conformal invariants

$$\xi = 33.9705627485; k = 0.707106781187; \mu = 2;$$

$$\xi = 1.03033008589; k = 0.0074696667295; \mu = \frac{1}{2};$$

$$\xi = 5.1391447246 \times 10^9; k = 0.99997210165; \mu = 8;$$

$$\xi = 1.00000000019; k = 4.8646258927 \times 10^{-11}; \mu = \frac{1}{8};$$

$$\ldots$$

$$\xi = 2; k = 0.171572875254; \mu = 1;$$

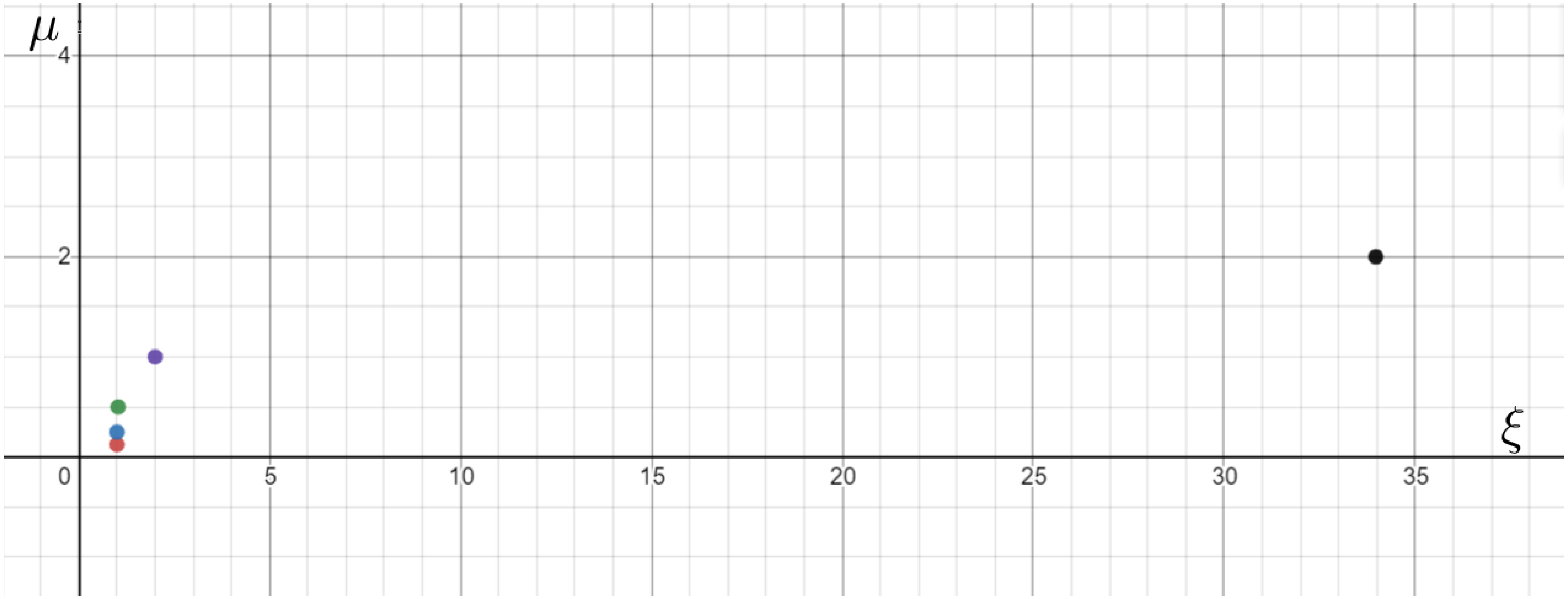
$$\xi = 17922.4570753; k = 0.985171431009; \mu = 4;$$

$$\xi = 1.00005579903; k = 0.000013949369424\,2; \mu = \frac{1}{4};$$

$$\xi = 4.2257246934 \times 10^{20}; k = 0.999999999903; \mu = 16;$$

$$\ldots$$

Discrete graph of “$\mu(\xi)$”



It is well known that

$$\int_0^1 \frac{d\omega}{\sqrt{(1-\omega^2)(1-k^2\omega^2)}} = \frac{\pi/2}{M\left(1,\sqrt{1-k^2}\right)}; \quad \rightarrow \quad \mu(k) = 2\frac{M(1,k)}{M\left(1,\sqrt{1-k^2}\right)};$$

where

$$M(a_0,b_0) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n\,;$$

$$a_0 \geq b_0 > 0; \qquad a_n = \frac{a_{n-1}+b_{n-1}}{2}; b_n = \sqrt{a_{n-1}b_{n-1}};$$

Properties of AGM

1) Symmetry: $M(a,b) = M(b,a)$
2) Bilinearity: $\mathrm{q}M(a,b) = M(qa,qb)$
3) Sequence invariance:

$$\ldots = M(a_{-1},b_{-1}) = M(a_{-1},b_{-1}) = M(a_0,b_0) = M(a_1,b_1) = M(a_2,b_2) = \cdots$$

Let $a_0 = 1$ and $b_0 = k$, then

$$\ldots = \left(1+\sqrt{1-k^2}\right) M_{-1}\left(1, \frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}\right) = M_0(1,k) = \frac{1+k}{2} M_1\left(1, \frac{2\sqrt{k}}{1+k}\right) = \cdots$$

General form $\ldots v_{-1}(k) M_{-1}\big(1, \tau_{-1}(k)\big) = v_0(k) M_0\big(1, \tau_0(k)\big) = v_1(k) M_1\big(1, \tau_1(k)\big) \ldots$

Recurrent formulas:

Let $\tau_r(k) = \begin{cases} \frac{2\sqrt{\tau_{r-1}(k)}}{1+\tau_{r-1}(k)}; \\ k; \\ \frac{1-\sqrt{1-\tau_{r+1}^2(k)}}{1+\sqrt{1-\tau_{r+1}^2(k)}}; \end{cases}$ $\quad v_r(k) = \begin{cases} \prod_{p=0}^{r-1} \frac{1+\tau_p(k)}{2}; & r > 0 \\ 1; & r = 0 \\ \prod_{q=r+1}^{0} \left(1 + \sqrt{1 - \tau_q^2(k)}\right); & r < 0 \end{cases}$

Then $\mu_r = 2\frac{M_0(1,k)}{M_r(1,\sqrt{1-k^2})} = 2v_r(k)$

where k is the solution of the following equation: $\sqrt{1-k^2} = \tau_r(k)$

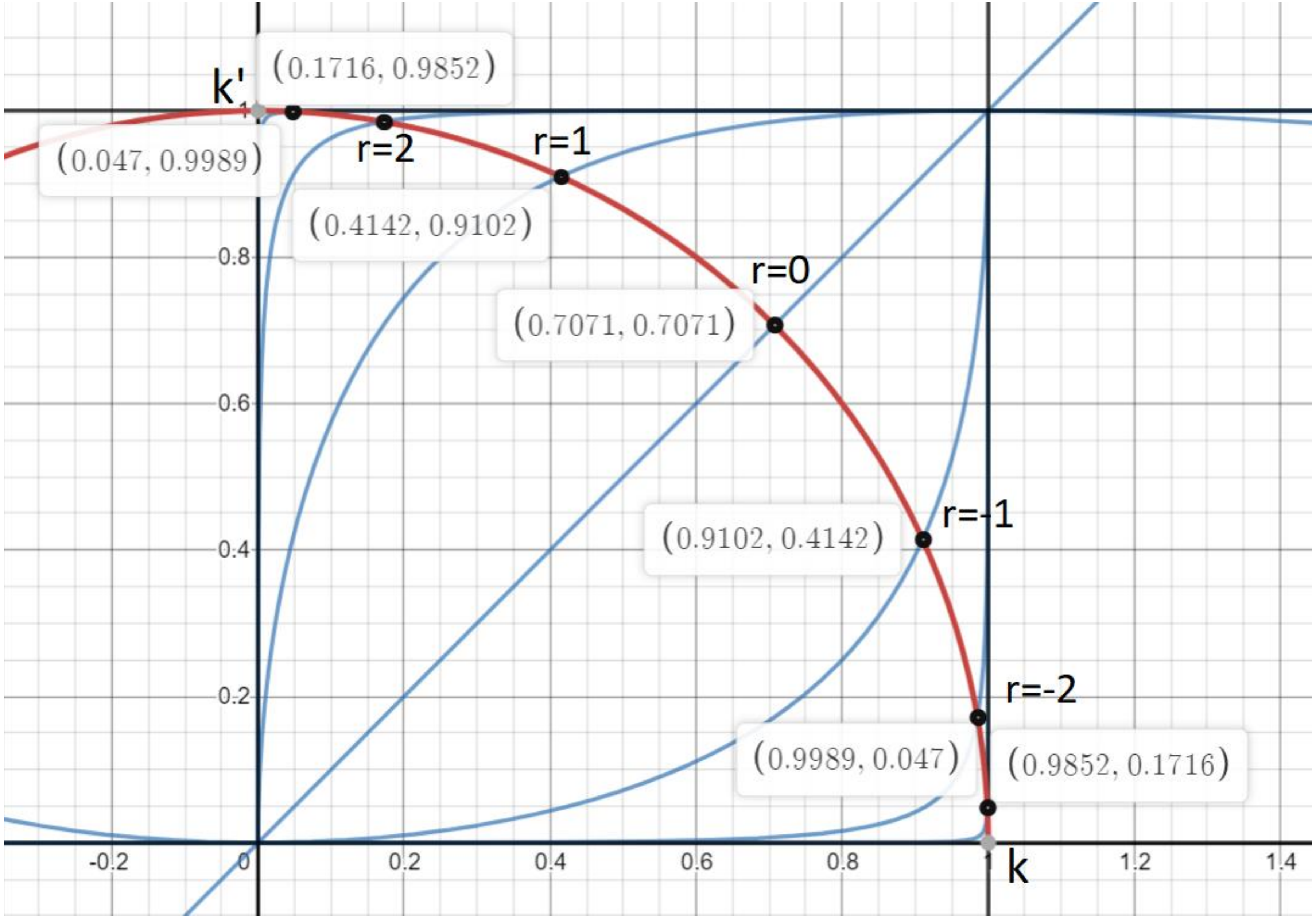
The DISCRETE SPECTRUM of k parameter

For each r there exist only one solution

One r

One k_r

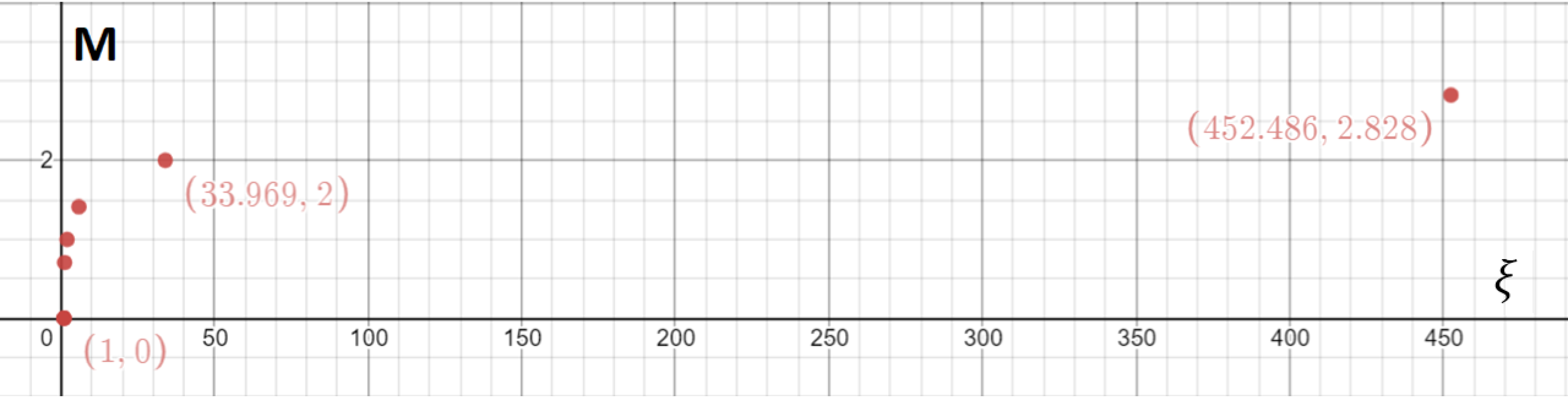
One μ_r



The DISCRETE SPECTRUM of the Modulus ("AMG" chain)

$r = 3;$	$\xi = 1.2070;$	$k = 0.047;$	$\mu = 0.7070;$
$r = 2;$	$\xi = 2.0002;$	$k = 0.1716;$	$\mu = 1.0000;$
$r = 1;$	$\xi = 5.8280;$	$k = 0.4142;$	$\mu = 1.4142;$
$r = 0;$	$\xi = 33.9687;$	$k = 0.7071;$	$\mu = 2;$
$r = -1;$	$\xi = 452.4858;$	$k = 0.9102;$	$\mu = 2.8284;$
$r = -2;$	$\xi = 17992.2344;$	$k = 0.9852;$	$\mu = 3.9998;$
$r = -3;$	$\xi = 3302149.76033;$	$k = 0.9989;$	$\mu = 5.6574;$

$\mu(\xi)$ is monotonic and slowly increasing function



$r = 0;$ $\xi = 33.9687;$ $k = 0.7071;$ $\mu = 2;$

$r = \pm 1$

$\xi = 5.82804546569; k = 0.4142; \mu = 1.4142;$

$\xi = 452.337043898; k = 0.910185893101; \mu = 2.82845424975;$

$\xi = 1.20712315307; k = 0.0470252827077; \mu = 0.707113562438;$

$\xi = 1.0022156391; k = 0.000553296990388; \mu = 0.35355;$

$\xi = 3264623.75025; k = 0.998893699443; \mu = 5.6568;$

$\xi = 1.7072087103 \times 10^{14}; k = 0.999999846931; \mu = 11.313816999;$

$\xi = 1.00000030631; k = 7.6578514845 \times 10^{-8}; \mu = 0.17677839061;$

$r = \pm 2$

$\xi = 2.00022359311; k = 0.1716; \mu = 1.00004630354;$

$\xi = 17910.9572796; k = 0.985166706705; \mu = 3.99981479442;$

$\xi = 1.99977645687; k = 0.171545753279; \mu = 0.999953698604;$

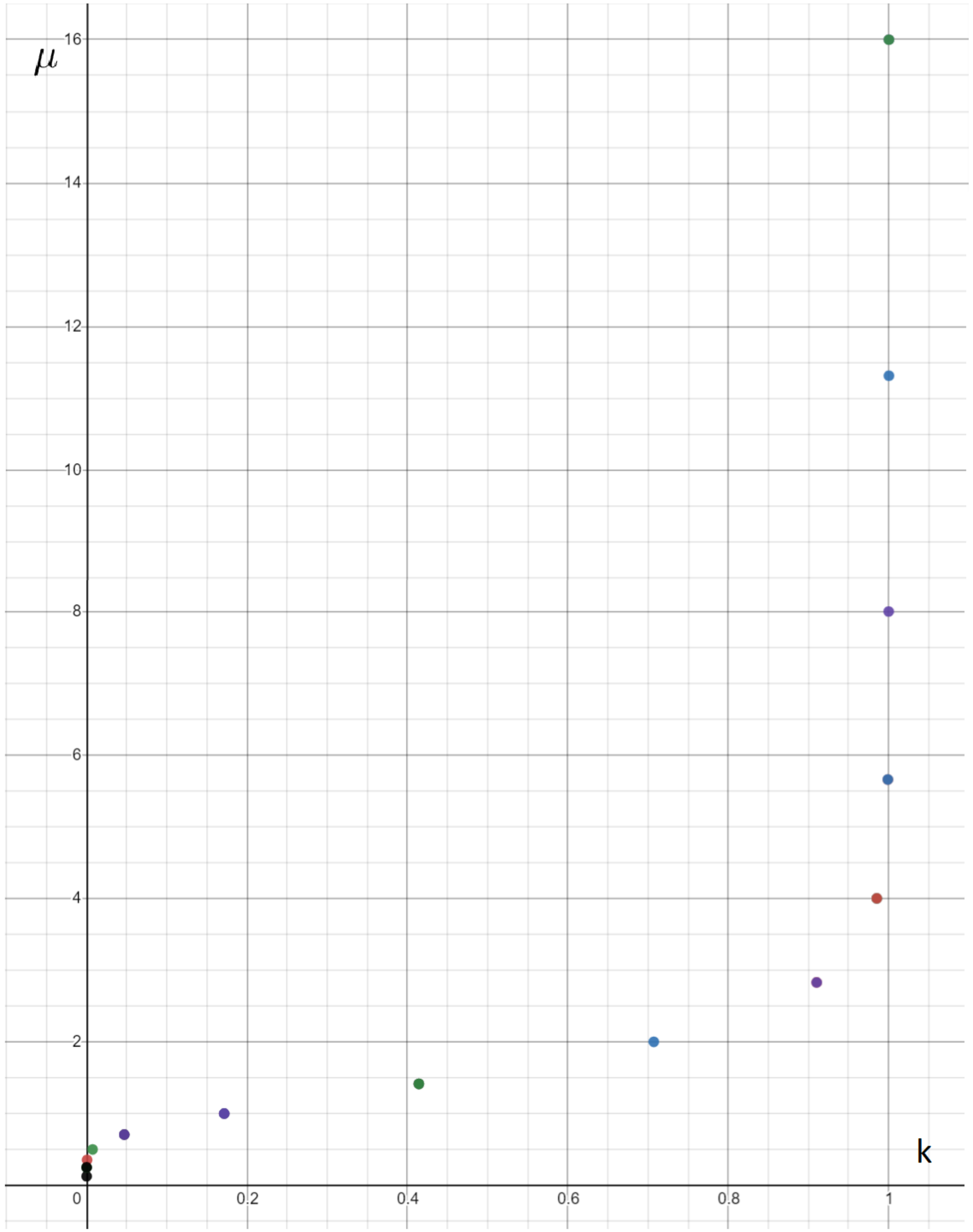
$\xi = 1.00005583486; k = 0.000013958325918; \mu = 0.250011575885;$

$\xi = 17933.9648431; k = 0.985176154062; \mu = 4.00018521416;$

$\xi = 4.2148959295 \times 10^{20}; k = 0.999999999903; \mu = 15.9992591777;$

$\xi = 1.00005576323; k = 0.0000139404182201; \mu = 0.249988424651;$

The discrete graph of $\mu(k)$



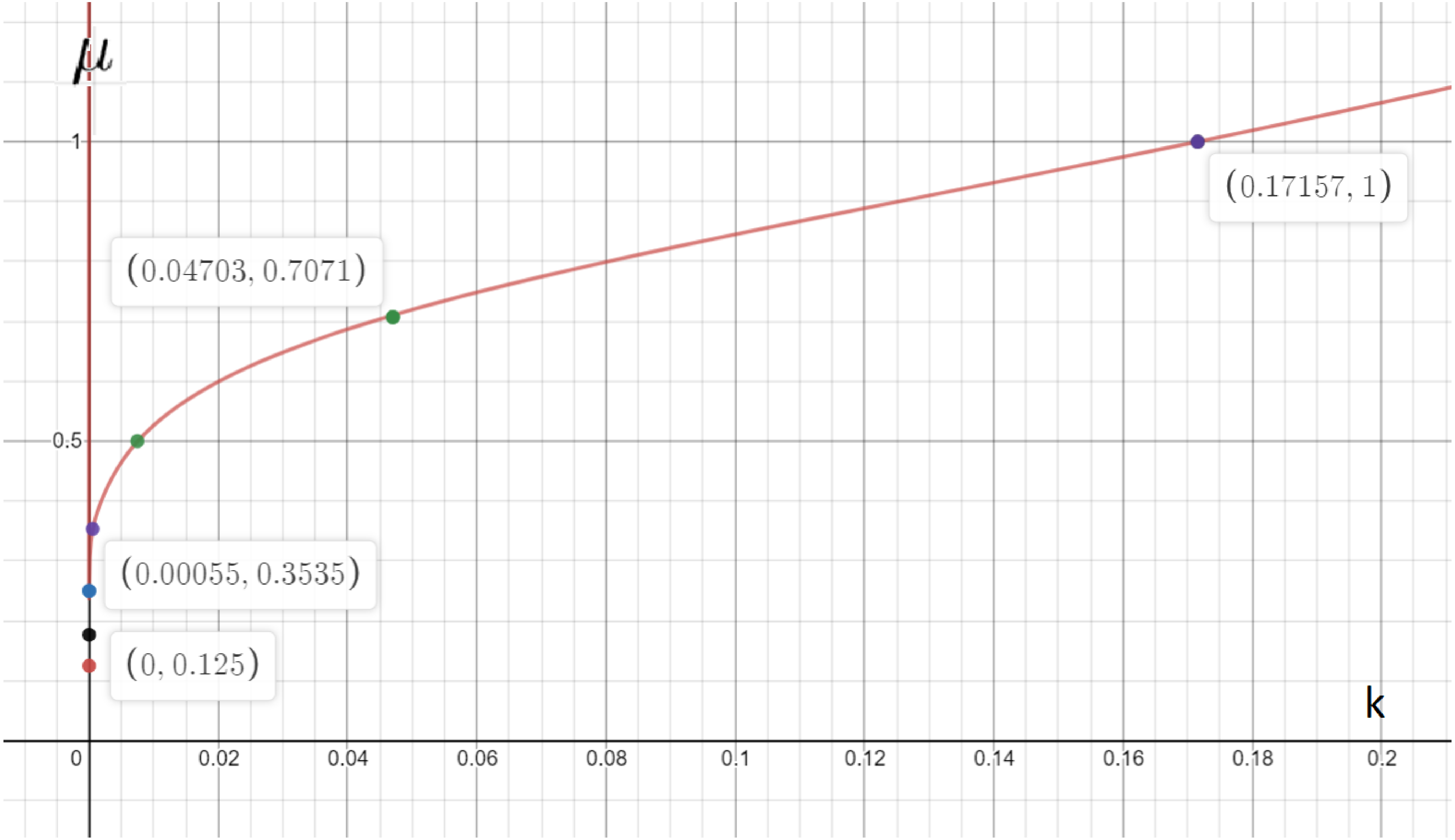
Approximate formula of modulus

$$\mu(k) = \left(\left(ln\left(\frac{1}{1-k}\right)\right)^{\frac{1}{6}} e^{k^{\frac{1}{6}}}\right) -$$

$$-\frac{1}{19}\left(\frac{ln\left(sin(\pi k)\right)}{\pi} - tg\left(\frac{4\pi}{10}\right)k + \left(\frac{cos(\pi k)+\pi k}{2\pi}\right) + 2.36\right) -$$

$$-0.7\sqrt{1-4\left(k-\frac{1}{2}\right)^2} + (1.8k)^3 + 0.05e^{-1000k}$$

For the interval $k \in (0, 0.171572875254)$ and $\mu \in (0, 1)$



μ
1
0.5
(0.17157, 1)
(0.04703, 0.7071)
(0.00055, 0.3535)
(0, 0.125)
k
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2

Summary:

1) We have proven that the modulus is conformal invariant of quadrilaterals.

2) Using the properties of the modulus we have established the new method of finding some discrete values by the known ones.

3) Using the relation to AMG functions we have established another method of calculating and joined these two methods together.

4) After all we did not give up and found the approximate formula for the further calculations on some local interval.

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THANK YOU FOR ATTENTION!