Universal formulae for simple Lie algebras and configurations of points and lines

Maneh Avetisyan,

Yerevan Physics Institute

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Introduction

The present research is rooted in the notion of *universal Lie algebra,* introduced by Vogel [*P. Vogel The universal Lie algebra. Preprint (1999)*]

Mathematically the latter is a certain tensor category having Vogel plane as a moduli space with special points corresponding to all simple Lie algebras.

Vogel plane is the quotient space P^2/S_3 of projective plane with projective coordinates α, β and γ (Vogel's parameters) by the symmetric group S_3 acting by permutations of the parameters.

Vogel's table

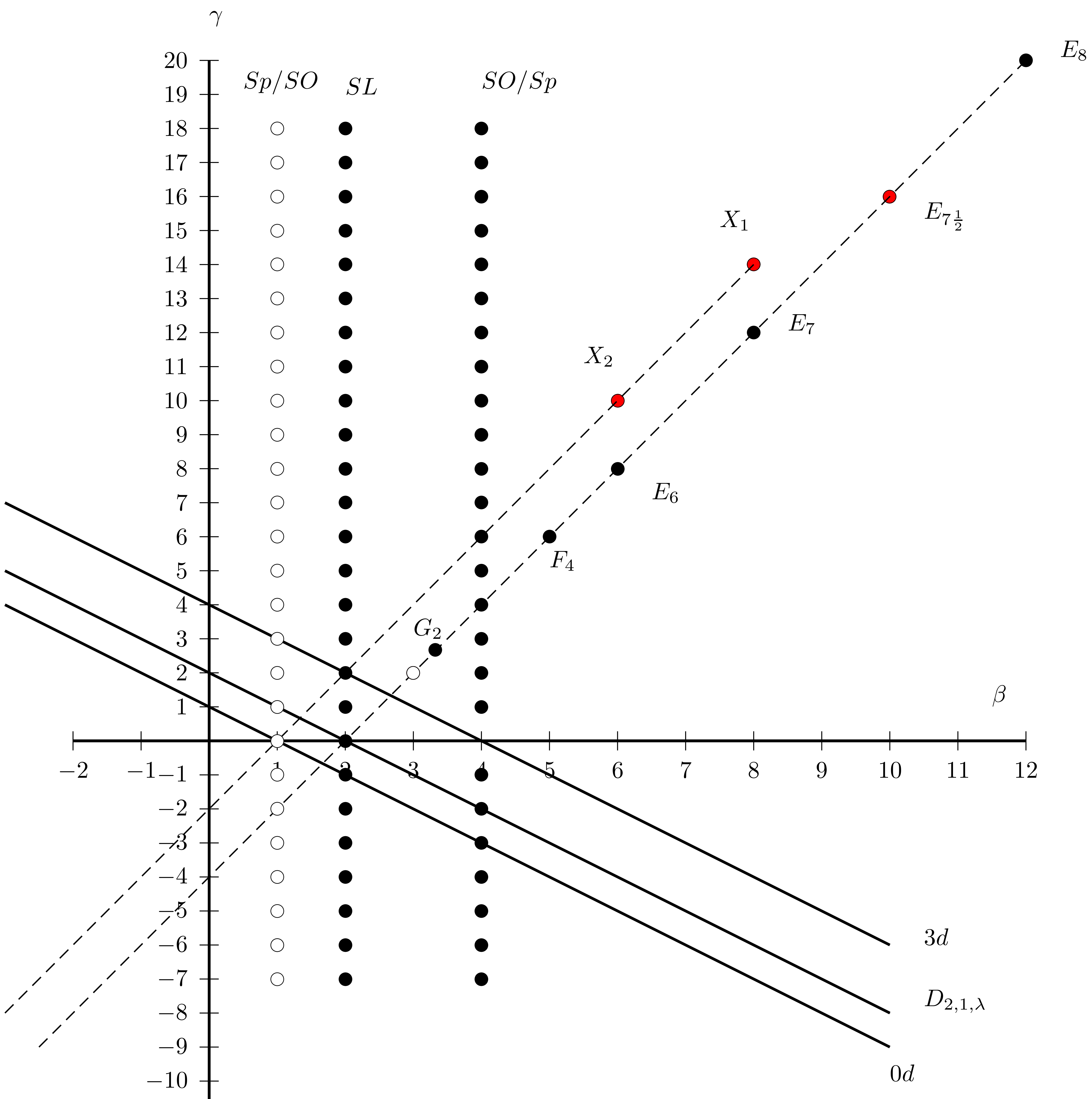
The sets of the parameters, corresponding to simple Lie algebras are given in this table, where the normalization corresponds to $\alpha = -2$.

The projective nature of the parameters corresponds to the choice of the invariant bilinear form on simple Lie algebra, which is known to be unique up to a multiple.

Type	Lie algebra	α	β	γ
A_n	$\mathfrak{sl}_{n+1}$	-2	2	$(n+1)$
B_n	$\mathfrak{so}_{2n+1}$	-2	4	$2n-3$
C_n	$\mathfrak{sp}_{2n}$	-2	1	$n+2$
D_n	$\mathfrak{so}_{2n}$	-2	4	$2n-4$
G_2	$\mathfrak{g}_2$	-2	$10/3$	$8/3$
F_4	$\mathfrak{f}_4$	-2	5	6
E_6	$\mathfrak{e}_6$	-2	6	8
E_7	$\mathfrak{e}_7$	-2	8	12
E_8	$\mathfrak{e}_8$	-2	12	20

Vogel's parameters

Distinguished lines in the Vogel plane



$$sp : \alpha + 2\beta = 0$$

$$sl : \alpha + \beta = 0$$

$$so : 2\alpha + \beta = 0$$

$$exc : \gamma - 2(\alpha + \beta) = 0$$

Universal formulae

Some numerical characteristics of simple Lie algebras can be expressed in terms of only 3 Vogel's parameters by some formulae:
these are called ***universal formulae*.**

An example is the ***Vogel's dimension formula***:

$$dim\mathfrak{g} = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma}, t = \alpha + \beta + \gamma$$

At the associated points of the Vogel plane, this function yields values for dimensionality of the corresponding Lie algebra.

Several universal formulae in the scope of the representation theory have been derived recently: [*M.Avetisyan and R.Mkrtchyan, 2018, arXiv:1812.07914, M.Avetisyan, R.Mkrtchyan, 2019, arXiv:1909.02076*]

Particularly, for the Cartan powers of arbitrary powers of the X_2 and $\mathfrak{g}$ representations, appearing in the antisymmetric square of the adjoint $\mathfrak{g}$:

$$\wedge^2 \mathfrak{g} = \mathfrak{g} \oplus X_2$$

We derived the following universal dimension (also quantum dimension) formula:

$$X(x,k,n,\alpha,\beta,\gamma)=\prod_{i=0}^{k-1}\frac{(\alpha(i-2)-2\beta)^2(\alpha(i-2)-2\gamma)^2(\beta+\gamma+\alpha(-(i-2))^2}{(\alpha(i+1))^2(\beta-\alpha(i-1))^2(\gamma-\alpha(i-1))^2}\times$$

$$\times\prod_{i=0}^{n}\frac{(\alpha(i+k-2)-2\beta)(\alpha(i+k-2)-2\gamma)(\beta+\gamma+\alpha(-(i+k-2))}{(\alpha(i+k+1))(\beta-\alpha(i+k-1))(\gamma-\alpha(i+k-1))}\times$$

$$\times\prod_{i=1}^{2k+n}\frac{(-\beta-2\gamma+\alpha(i-3))(-2\beta-\gamma+\alpha(i-3))(\alpha(i-5)-2(\beta+\gamma))}{(\alpha(i-2)-2\beta)(\alpha(i-2)-2\gamma)(\beta+\gamma-\alpha(i-2))}\times$$

$$\times\frac{(\alpha+\beta)(\alpha+\gamma)(\alpha(n+1))}{(2\alpha+2\beta)(2\alpha+2\gamma)(2\alpha+\beta+\gamma)}\times\frac{(\alpha(3k+n-4)-2(\beta+\gamma))(\alpha(3k+2n-3)-2(\beta+\gamma))}{(3\alpha+2\beta+2\gamma)(4\alpha+2\beta+2\gamma)}$$

As we see, this formula is far more “cumbersome” compared to the Vogel dimension formula:

$$dim\mathfrak{g} = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma}, t = \alpha + \beta + \gamma$$

Can one find another, more “simple-looking” formula with the same outputs at the points from Vogel’s table?

or

Or, generally, are the known universal formulae **unique** ?

Observe, that:

Universal dimension formula =

ratio of products of the same number of linear factors of Vogel's parameters with integer coefficients

We are interested in the existence of two universal formulae, which have this structure and yield the same outputs at the points from Vogel's table.

Let Q_1 and Q_2 be two universal formulae which have the mentioned structure and yield the same reasonable outputs at the points corresponding to simple Lie algebras.

Then the function $Q(\alpha, \beta, \gamma) = \frac{Q_1}{Q_2}$ obviously shares the desired structure and yields 1 at the associated points.

In general Q writes as:

$$Q = \prod_{i=1}^{k} \frac{n_i\alpha + x_i\beta + y_i\gamma}{m_i\alpha + z_i\beta + t_i\gamma}$$

Where a change of coordinates is made, so that the ***sl, so*** and ***exc*** lines become:

$$\alpha = 0; \beta = 0; \gamma = 0,$$

We impose Q to be equivalent to one - $Q \equiv 1$ - on each of the 3 ***(sl, so, exc)*** lines in the Vogel plane:

$$\alpha = 0,$$
$$\beta = 0,$$
$$\gamma = 0$$

And obtain the following result:

$$Q = \prod_{i=1}^{k} \frac{n_i\alpha + x_i\beta + y_i\gamma}{k_i n_{s(i)}\alpha + x_i\beta + y_i\gamma} = \prod_{i=1}^{k} \frac{n_i\alpha + x_i\beta + y_i\gamma}{c_i n_{p(i)}\alpha + x_i\beta + y_i\gamma}$$

$$x_i = c_i x_{p(i)}; y_i = k_i y_{s(i)}; k_i n_{s(i)} = c_i n_{p(i)};$$

$$c_1 c_2 \dots c_k = 1; k_1 k_2 \dots k_k = 1;$$

For some permutations $s(i), p(i), i = 1{,}2...k$

The structure of universal dimension formulae allows an interesting geometrical interpretation

$$F = \prod_{i=1}^{l} \frac{n_i\alpha + x_i\beta + y_i\gamma}{m_i\alpha + z_i\beta + t_i\gamma}$$

Each of the linear factors in Q is in one-to-one correspondence with some line in the Vogel projective plane:

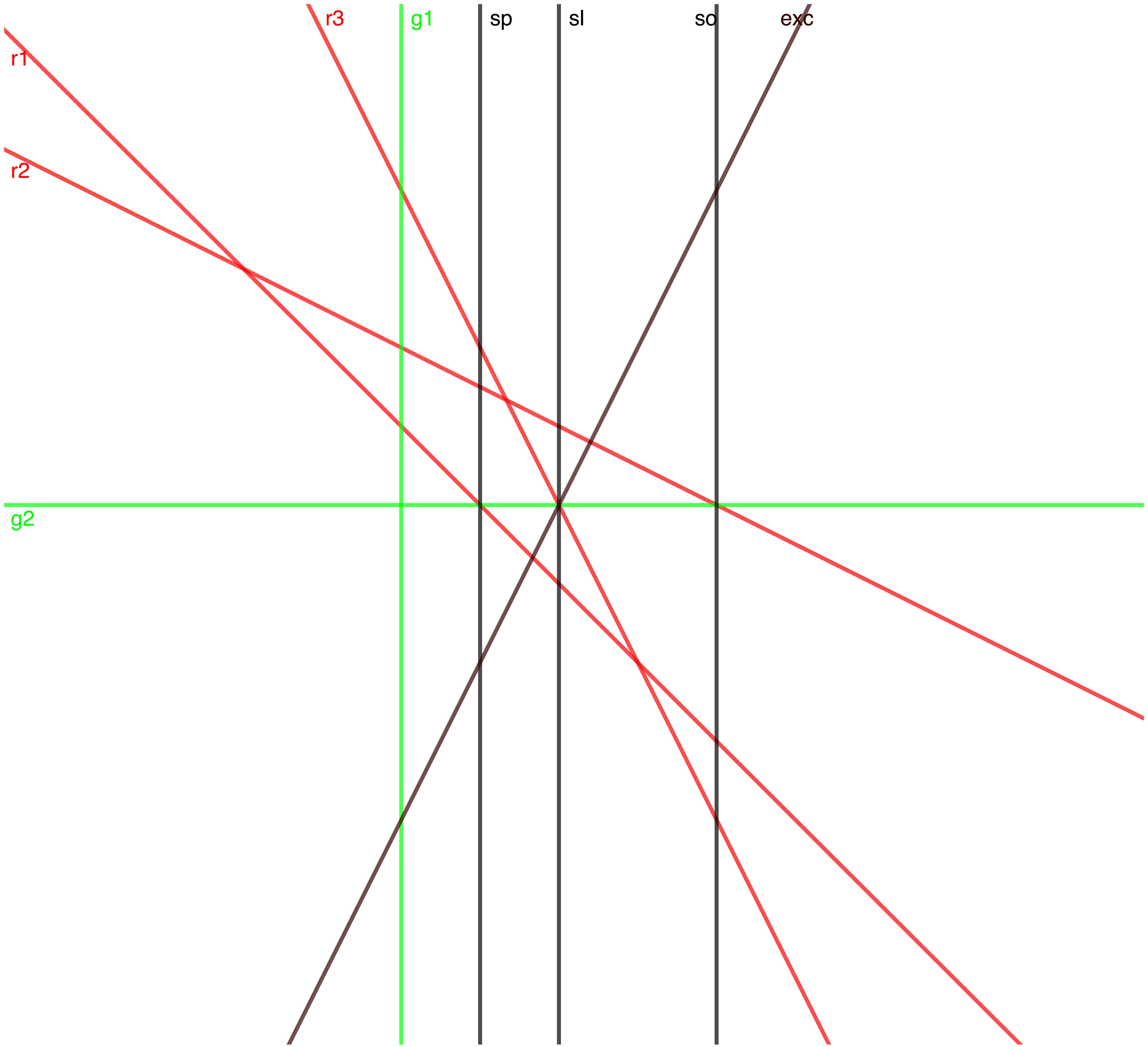
$$x\alpha + y\beta + z\gamma = 0$$

This means that any universal dimension formula can be "depicted" into the projective plane:

Example:

$$dim \mathfrak{g} = \frac{(\alpha + 2\beta + 2\gamma)(2\alpha + \beta + \gamma)(2\alpha + 2\beta + \gamma)}{\alpha\beta\gamma}$$

Here $\alpha = -2$, so that the line $\alpha = 0$ is the ideal line.



Lines from Vogel's dimension formula

Now, let’s turn to Q: obviously, for any Q, written for k multipliers we sketch a unique picture consisting of ***2k*** lines + ***l*** distinguished lines .

Let the distinguished lines from Vogel's table be called **black** lines, the lines corresponding to the **numerator** of Q - **red** lines, and those corresponding to its **denominator** - **green** lines.

Specifically, each of the black lines must contain k points, at which a green and a red line intersect.

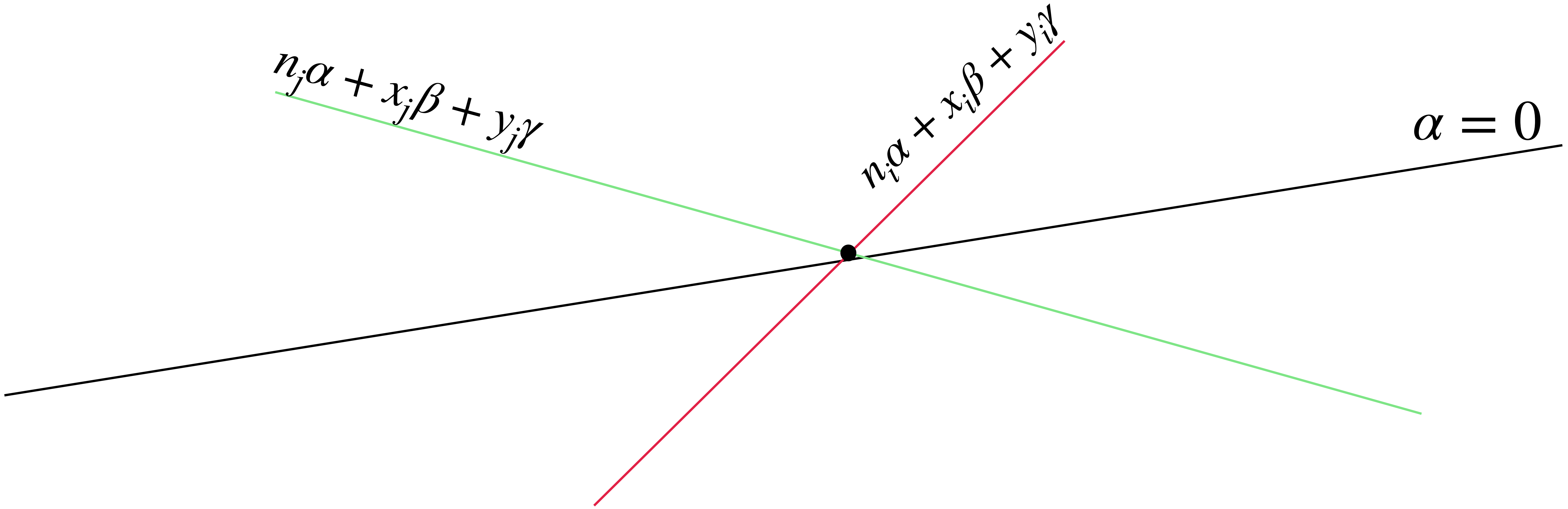
Indeed, on a black, say $\alpha = 0$ line, all factors $n_i\alpha + x_i\beta + y_i\gamma$ take the form:

$$x_i\beta + y_i\gamma$$

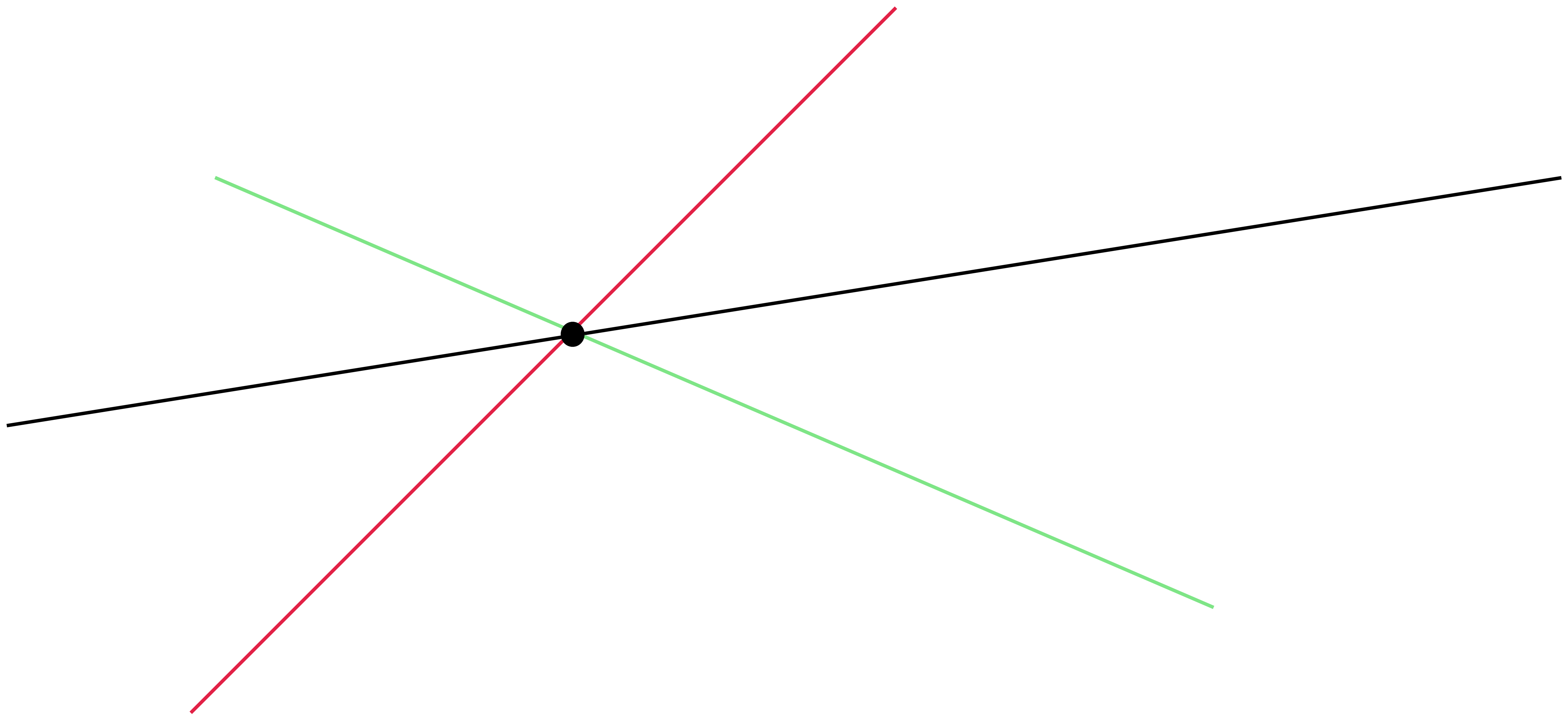
From the $Q \equiv 1$ condition on a black line, it follows that for each of the factors of the **numerator** there is a proportional factor in the **denominator**, that is:

$$x_i\beta + y_i\gamma = z_i(x_j\beta + y_j\gamma)$$

Which means that the corresponding lines pass through the same point



<u>So we have 4k points, at which precisely three lines meet - a black, a green and a red.</u>



If we require, that each of the 2k+l lines contain **precisely** k points, we happen to be dealing with a well-known geometrical notion -
a ***configuration of points and lines:***

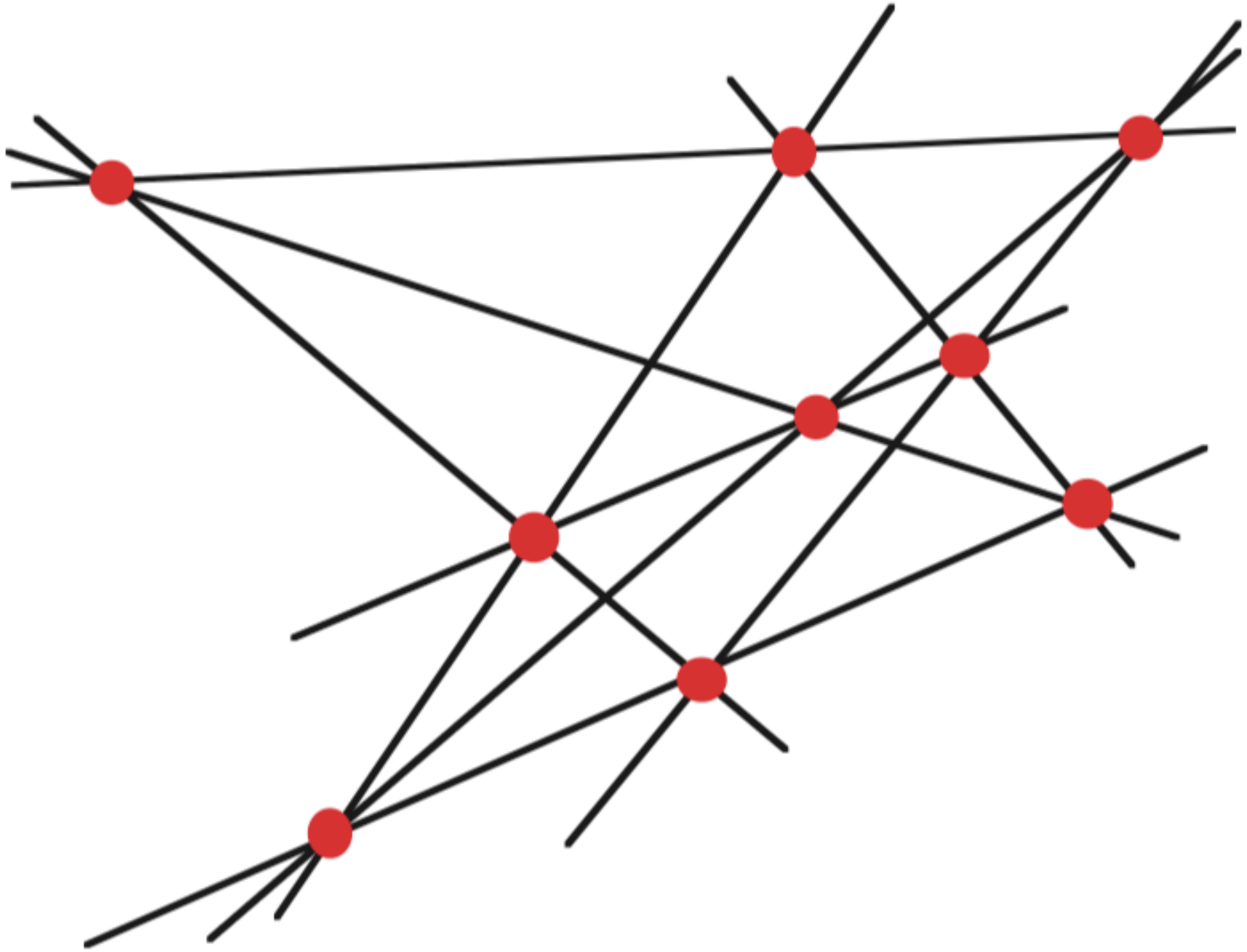
*A configuration (p_γ, l_π) is a set of **p** points and **l** lines, such that at every point **precisely γ** of these lines meet and every line contains **precisely π** of these points*.

It is easy to deduce, that ***pγ=lπ.***

Back to the Q:

we are dealing with the $(kl_3, (2k+l)_k)$ configurations,

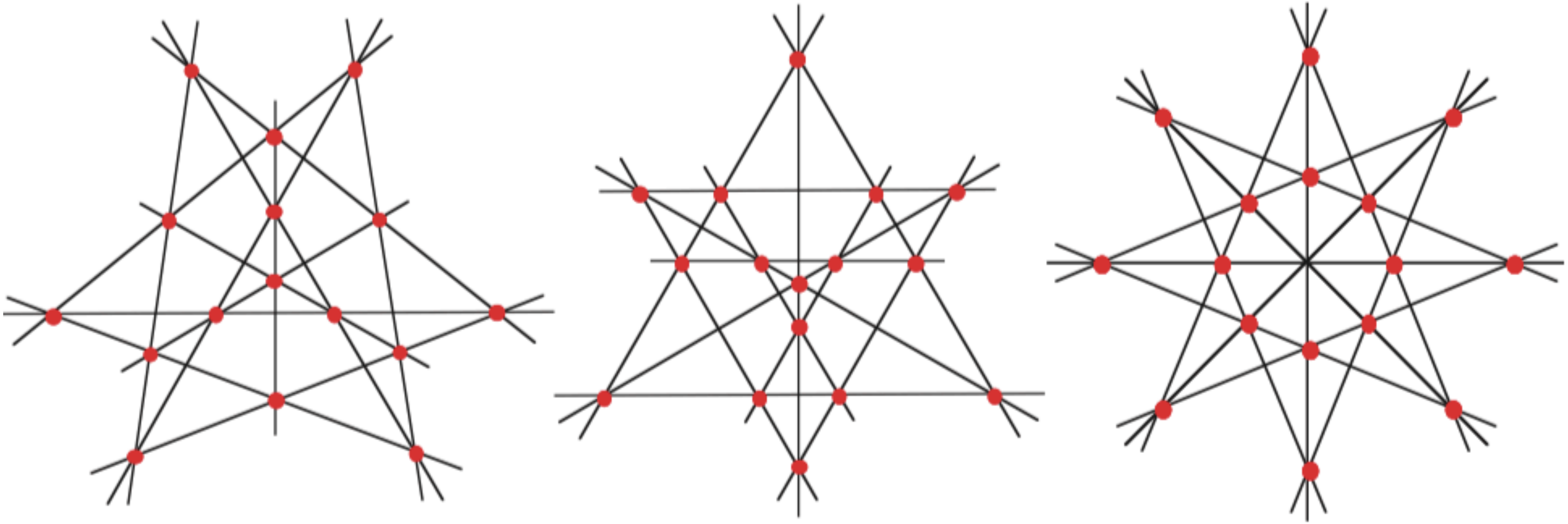
If we impose $Q \equiv 1$ on 3 distinguished lines, we will deal with $(9_3, 9_3)$ configuration [*Pappus configuration, 4-th century]*



The Pappus configuration

If we impose $Q \equiv 1$ on 4 distinguished lines, we will deal with $(16_3, 12_4)$ configuration

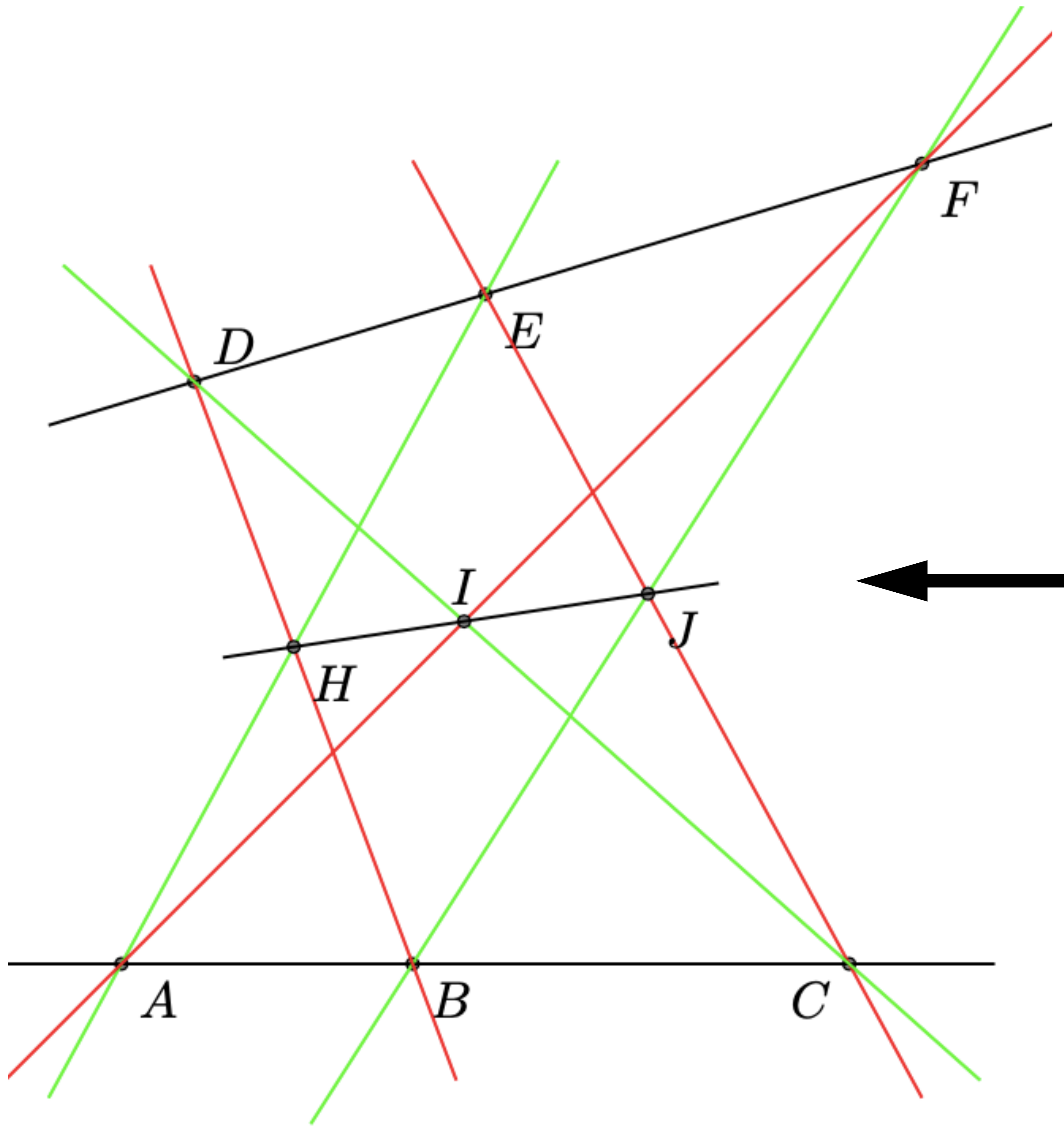
[Grünbaum Branco, Configurations of points and lines, 2009]



(16_3, 12_4) Configurations

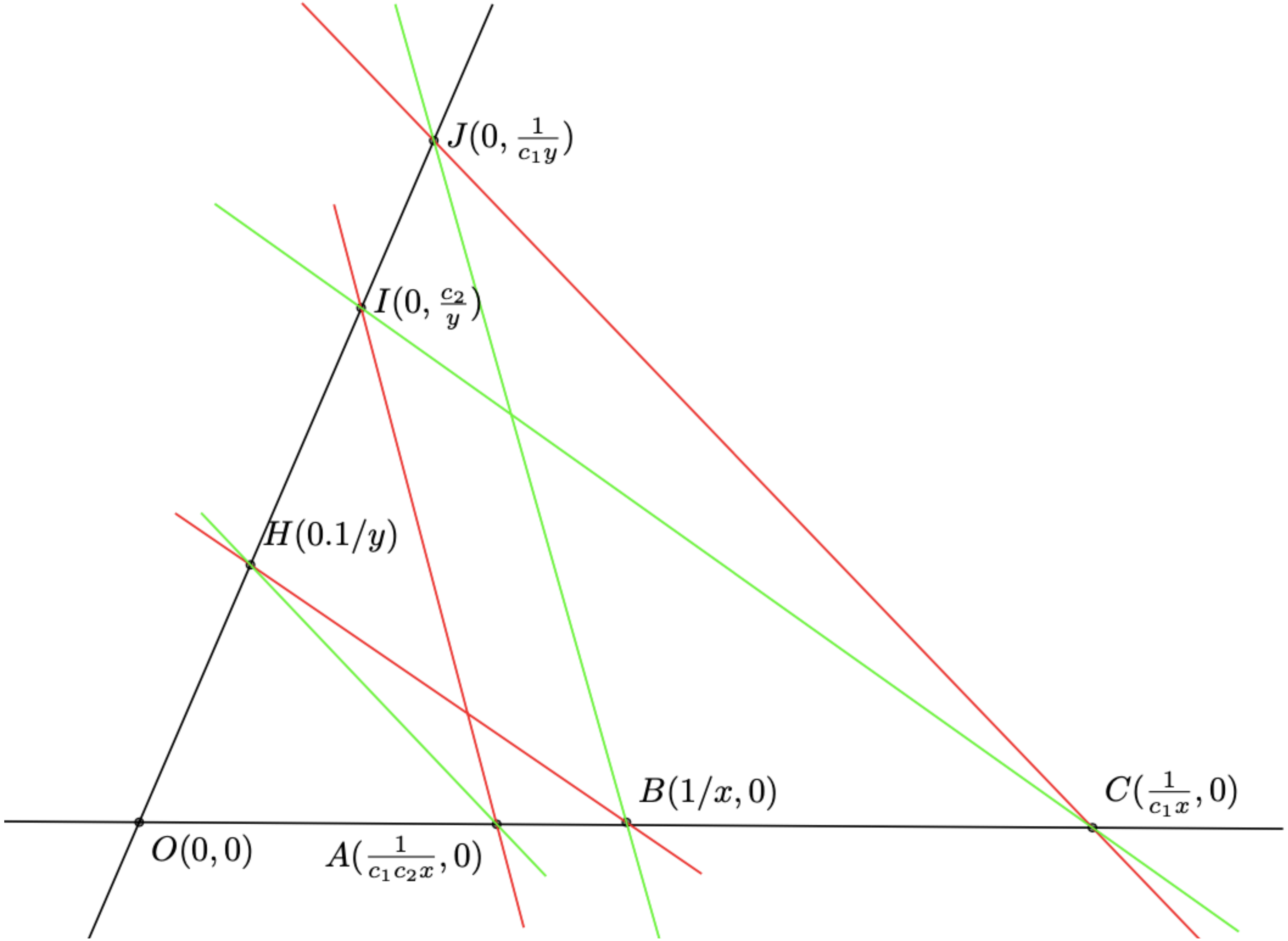
3 black lines

If we restrict ourselves with only 3 black lines,-**sl, sp** and **exc,**- instead of 4, we happen to find a Q, which corresponds to the famous ***Pappus configuration***:



$$\longleftrightarrow \quad \frac{(\alpha+\beta x+\gamma y)(\alpha c_2+\beta c_1 c_2 x+\gamma y)(\alpha+\beta c_1 c_2 x+c_1\gamma y)}{(\alpha+\beta c_1 x+c_1\gamma y)(\alpha c_2+\beta c_2 x+\gamma y)(\alpha+\beta c_1 c_2 x+\gamma y)}$$

To make sense of the free parameters let's present the same configuration after a projective transformation, which takes the sl ($\alpha = 0$) line into the ideal line.

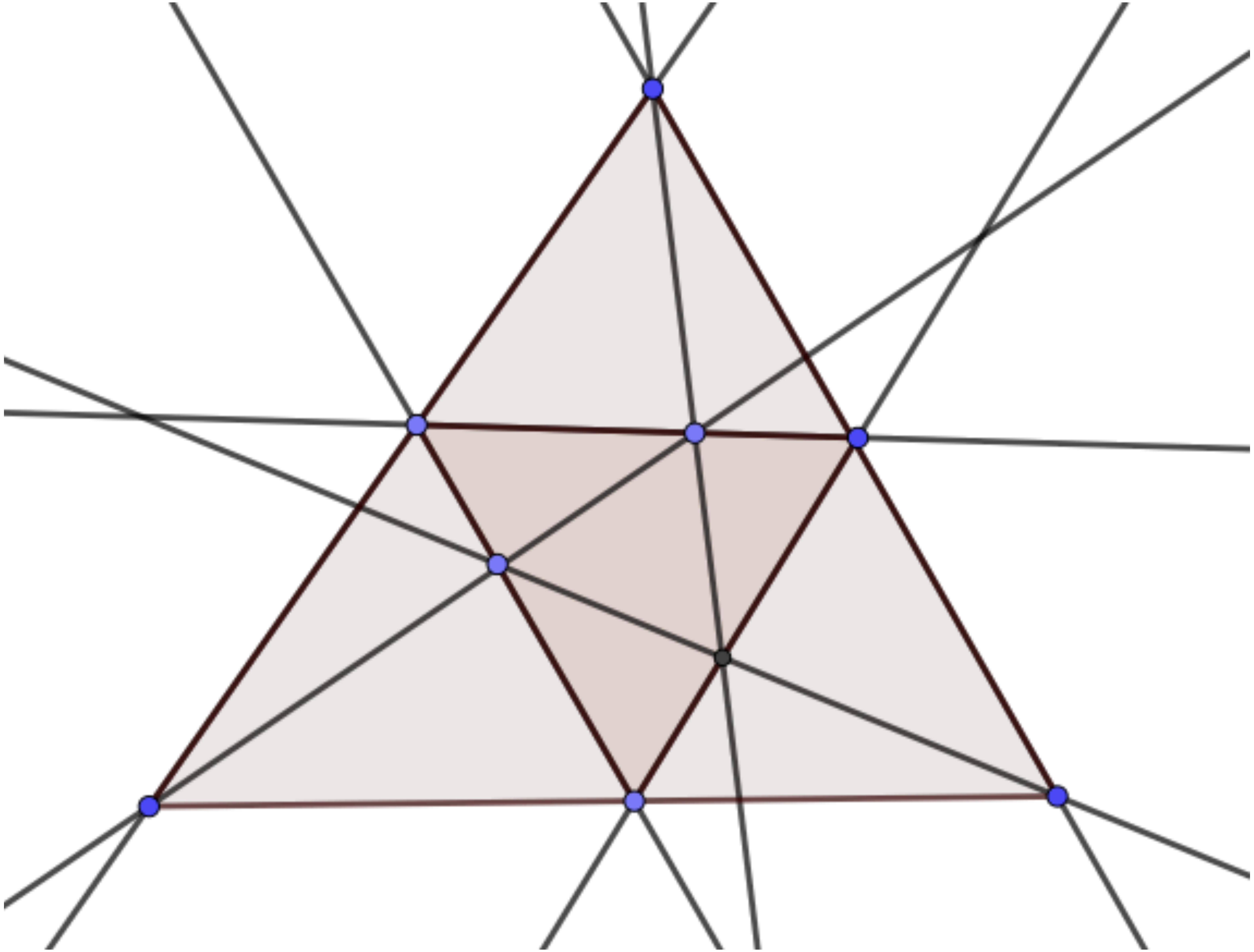


The Pappus configuration after a projective transformation

Thus having a geometrical configuration we derived a function, which meets the desired features for producing a universal dimension formula

Let's show, that this correspondence is ***not*** one-to-one.

Indeed, take a look at this following (9_3) configuration:

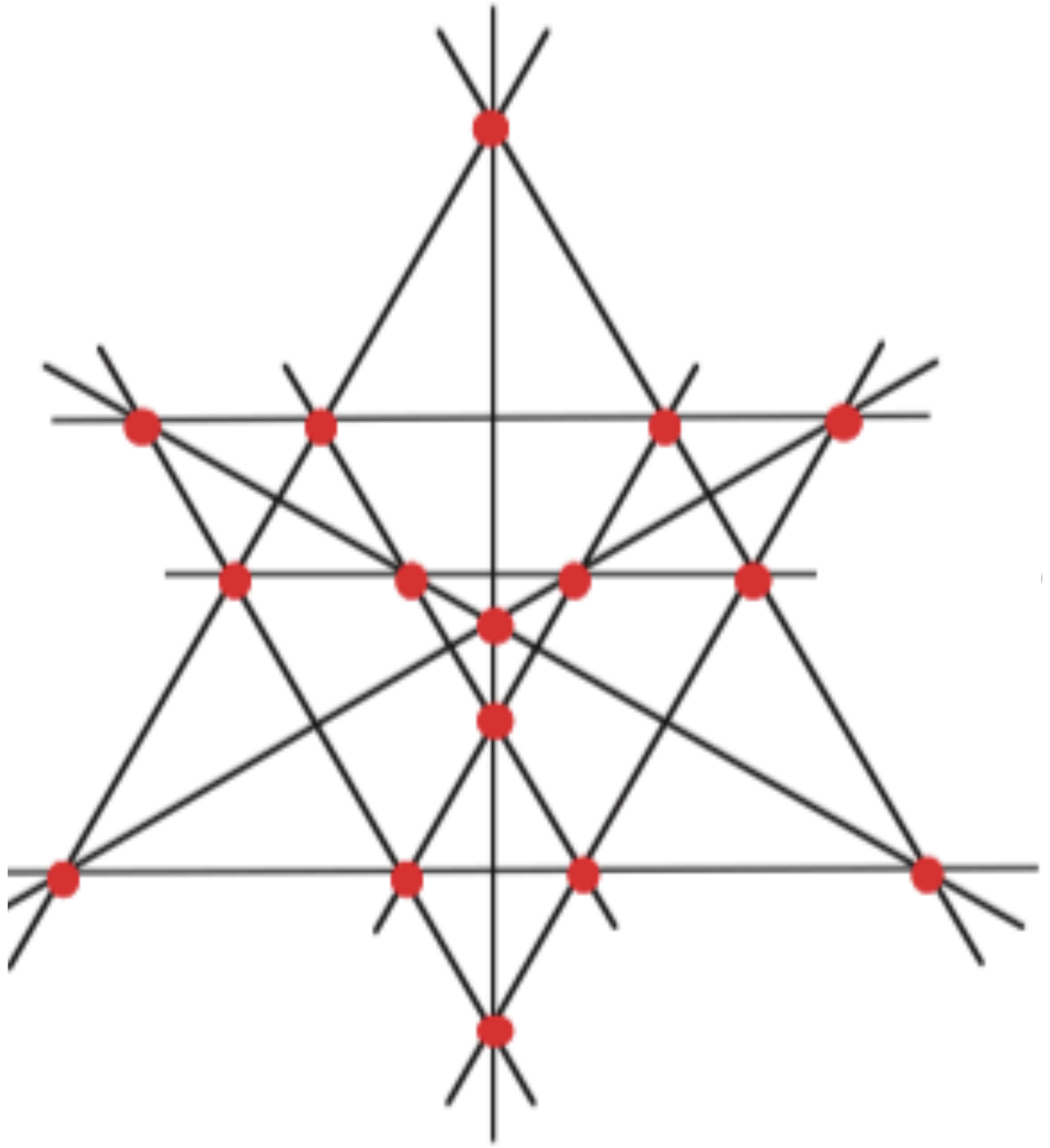


A (9_3) configuration, which does not allow “coloring”

But after some work, we see that it is impossible to “color” the lines of this particular configuration, so that lines of **different colors** meet at each of its point.

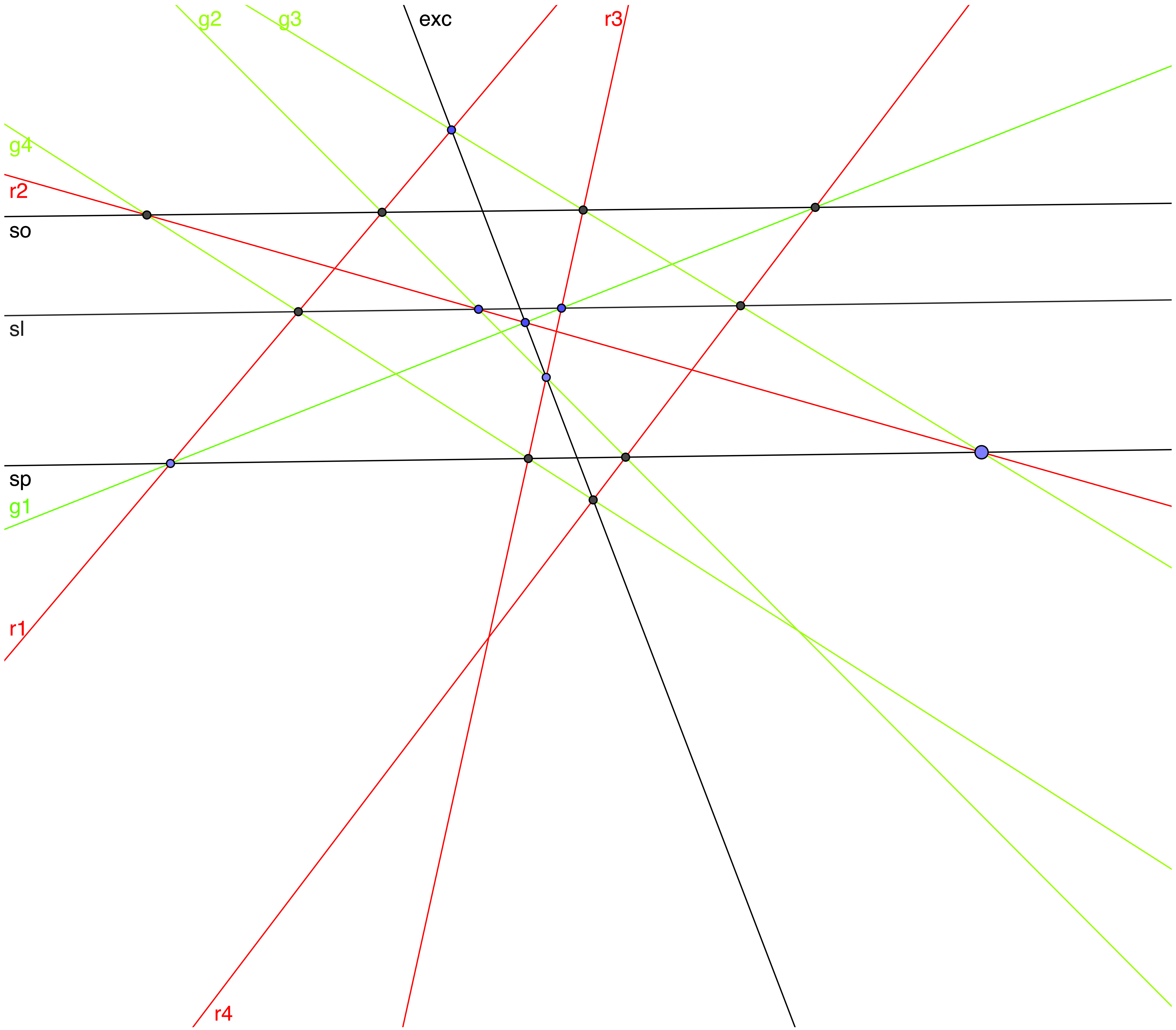
4 black lines

Take the following configuration:



(16_3, 12_4)

And “color” the lines, so that at each of the point lines of different colors meet:



We track the patterns of cancellation in Q, by just tracing the intersection of them.

This explicitly dictates the choice of the set of permutations:

$$s(1) = 2, s(2) = 1, s(3) = 4, s(4) = 3$$

$$p(1) = 4, p(2) = 3, p(3) = 2, p(4) = 1$$

$$q(1) = 3, q(2) = 4, q(3) = 1, q(4) = 2$$

Solving the following equations:

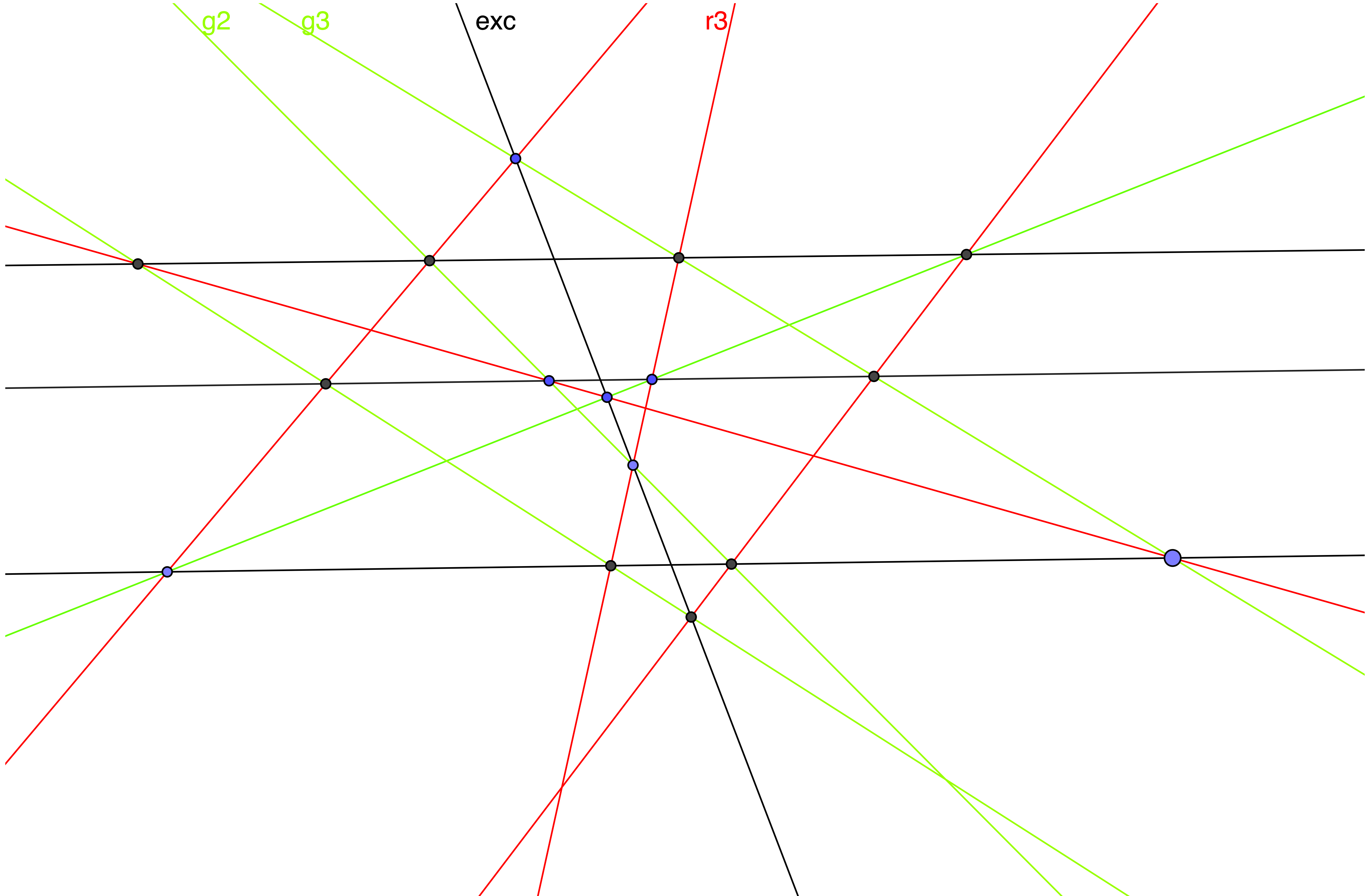
$$x_i = c_i x_{p(i)};\, y_i = k_i y_{s(i)};\, y_i = r_i y_{v(i)};\, k_i n_{s(i)} = c_i n_{p(i)};\, c_i n_{p(i)} + 3x_i = r_i(n_{v(i)} + 3x_{v(i)});$$

$$c_1 c_2 \dots c_k = 1;\, k_1 k_2 \dots k_k = 1;\, r_1 r_2 \dots r_k = 1;$$

$$s(i), p(i), v(i), i = 1,2...k$$

$$Q = \prod_{i=1}^{k} \frac{n_i\alpha + x_i\beta + y_i\gamma}{k_i n_{s(i)}\alpha + x_i\beta + y_i\gamma} =$$

$$= \frac{(\alpha+\beta x+\gamma y)\left(\alpha+\beta\left(-cx-\frac{c}{3}-\frac{1}{3}\right)+c\gamma y\right)\left(\alpha+\beta\left(-\frac{1}{3c}-x-\frac{1}{3}\right)-\gamma y\right)(\alpha+\beta cx-c\gamma y)}{(\alpha+\beta x-\gamma y)(\alpha+\beta cx+c\gamma y)\left(\alpha+\beta\left(-\frac{1}{3c}-x-\frac{1}{3}\right)+\gamma y\right)\left(\alpha+\beta\left(-cx-\frac{c}{3}-\frac{1}{3}\right)-c\gamma y\right)}$$



Open problem

It is shown that all known universal dimension formulae for simple Lie algebras yield reasonable outputs when considering them with permuted Vogel's parameters: [*M.Avetisyan and R.Mkrtchyan, 2018, M.Avetisyan, R.Mkrtchyan, 2019*]

In order to preserve this feature of a universal formula when multiplying it by some Q, impose the latter to be equivalent to 1 on the following **12** lines, in the Vogel plane:

$$\alpha+\beta=0;\alpha+\gamma=0;\beta+\gamma=0;$$

$$2\alpha+\beta=0;\alpha+2\beta=0;2\alpha+\gamma=0;\alpha+2\gamma=0;2\beta+\gamma=0;\beta+2\gamma=0;$$

$$\gamma=2\alpha+2\beta;\alpha=2\beta+2\gamma;\beta=2\alpha+2\gamma\,.$$

From the geometrical point of view this "totally symmetric" function would be corresponding to the $(144_3, 36_{12})$ configuration.

This geometrical configuration has not been studied yet, so that the "totally symmetric" Q can be found either after constructing the $(144_3, 36_{12})$ or by using another method, which is yet to be found.

Ultimately, we set up a correspondence between two classical areas of mathematics:

Lie algebras $\leftrightarrow$ Configurations of points and lines

Thanks!