Split-Quaternion Analyticity and (2+1)-Electrodynamics

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PLAN OF THE TALK

- Observable Algebra
- Normed Algebras
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- Quaternions
- Vectors vs Verzors
- Split Quaternions
- Rotations
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Observable Algebra

Usually geometry is thought to be objective without any connection to the way it is observed. However, the properties of space-time (such as dimension, distances, etc.) are just interpretation of the symmetries of physical signals we receive. Analyzing the signals our brain operates via classical associations and numbers. So algebras closely relates to the physical measurements, which consists not only with the actual receiving of signals, but with perceiving them to form the data.

So we can introduce some kind of anthrophic principle in mathematics:

***Investigating the algebras we study the way our brain abstracts physical observations*.**

For example, algebra of rational numbers expresses our experience that there exist some classical objects with which we can exchange signals and these objects do not change much during 'algebraic' operations.

Any observable quantity, which our brain could extract from a single measurement, is a real number (the norm to two-way signals). In general one-way signal can be expressed with any kind of number. Introduction of some distance (norm) always means comparison of two physical objects using one of them as an etalon.

In the algebraic language all these features of our way of thinking mean that to perceive the real world our brain uses division algebras with the unit element over the field of real numbers.

Normed Algebras

According to the theorem of **Adolf Hurwitz** (**1898**), besides of the usual algebra of real numbers, there are three unique division algebras, the algebra of complex numbers, quaternions and octonions.

Essential for normed algebras is the existence of the unit

$$e^2 = 1$$

and a different number of adjoined hyper-complex units

$$e_n^2 = \pm 1 \,.$$

The general element is

$$z = a\,e + b_n\,e^n \,, \qquad z^* = a\,e - b_n\,e^n \,,$$

where a and b are real numbers.

- **Real numbers:** $n = 0$
- **Complex numbers:** $n = 1$
- **Quaternions:** $n = 3$
- **Octonions:** $n = 7$

In physical applications mainly the algebras with the imaginary basis elements (versors), $e_n^2 = -1$, are used. In this case their norms is positively defined

$$N = z\,z^* = a^2 + b_n\,b^n \,.$$

Choosing some basis elements as, $e_n^2 = +1$, means their interpretation as unit vectors. This leads to split algebras, which contain both classes of basis units. Norms of split algebras in general are not positively defined.

Zero Divisors

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In split-algebras two types of **Zero Divisors** can be constructed:

1). Projection operators: $D^2 = D$

Commuting primitive projection operators are:

$$D_n^+ = (1 + e_n)/2 \, ,$$
$$D_n^- = (1 - e_n)/2 \, .$$

In division algebras only projection operator is the identity.

2). Grassmann's numbers: $G^2 = 0$

This kind of numbers can be constructed by coupling of two basis elements (except of unity) with the opposite choice of the sign of their squares

$$e_1^2 = 1 \, , \quad e_2^2 = -1 \, .$$

The examples of primitive **Grassmann** numbers are:

$$G^+ = (e_1 + e_2)/2 \, ,$$
$$G^- = (e_1 - e_2)/2 \, .$$

Separately, the quantities G^+ and G^- are the **Grassmann** numbers, but they do not commute with each other (in contrast with the projection operators D^+ and D^-) and thus do not obey the full **Grassmann** algebra. Instead they are the elements of the algebra, which is some syntheses of the **Grassmann** and **Clifford** algebras with the anti-commutator,

$$G^+ G^- + G^- G^+ = 1 \, .$$

2D Numbers

Complex numbers are convenient due to **Leonhard Euler**'s "*the most remarkable formula in mathematics*

$$e^{i\theta} = \cos\theta + i \sin\theta \, .$$

This is our jewel. We may relate the geometry to the algebra by representing complex numbers in a plane

$$x + i\, y = \rho\, e^{i\theta} \, .$$

This is the unification of algebra and geometry" (**Richard Feynman**). According to **2014** ranking by mathematicians the most beautiful mathematical formula is **Euler**'s identity.

Another possible **2**-dimensional normed algebra is formed by the hyper-numbers $z = t + j\, x$. The hyper-unit j has the properties similar to ordinary unit vector: $j^2 = 1$. A hyper-number does not have an inverse when its norm

$$N = z\, z^* = t^2 - x^2$$

is zero, i.e., when $x = \pm t$, or when $z = t\,(1 \pm j)$. These two lines play the same role as the point $z = 0$ does for the complexes, and provide the essential property of the light cone. **Lorentz** boosts can be succinctly expressed as

$$t + j\, x = (t' + j\, x')\, e^{i\theta} ,$$

where $\tanh\theta = v$.

Quaternions

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William Hamilton's discovery of quaternions (**1843**) was the first time in history when the concept of **2**-dimensional numbers was successfully generalized. The general element of the algebra,

$$q = a + b\,i + c\,j + d\,k\,,$$

can be written only by i and j. The third element

$$k = (ij)$$

is similar to a pseudo-vector.

The conjugated quaternion q^* can be constructed using the properties of the basis units under the conjugation

$$i^* = -\,i\,, \qquad j^* = -\,j\,, \qquad k^* = -\,k\,.$$

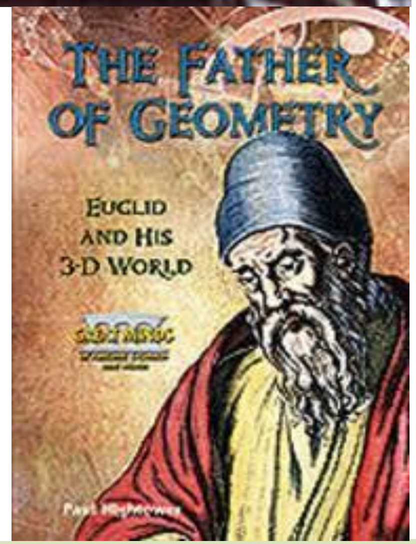
When i and j are pure imaginary, i.e.

$$i^2 = j^2 = -\,1$$

(like the complex unit), we have **Hamilton**'s quaternion with the **Euclidean** norm:

$$N = q\,q^* = a^2 + b^2 + c^2 + d^2\,.$$

The quaternion algebra is associative and therefore can be represented by matrices. We get the simplest representation of the quaternion basis units by the complex **Pauli** matrices accompanied by the unit matrix.



Vectors vs Verzors

Hamilton interpreted quaternion as made up of a '*scalar*' plus a '*vector*' part. Later he use also word '*versors*' (meaning '*rotators*'), since he understand that his units ***i***, ***j***, ***k*** can be interpreted as representing **π/2** rotation, instead of something that corresponds to a vector.

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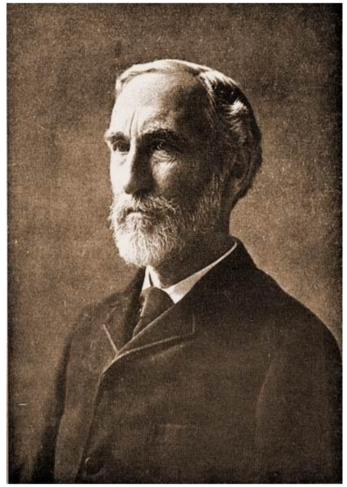

Vector algebra was developed in **1880**is by **Oliver Heaviside** and **Willard Gibbs** from quaternions. They removed scalar part of quaternions and kept **Hamilton**'s term '*vector*' represented a pure quaternion. Instead of the whole quaternion product they defined two types of vector products. They also changed the sign from minus to plus in the scalar product, resulting to positive squares of there basis units $e_n^2 = +1$, corresponds to a shift from '*versors*' to unit '*vectors*'. The product of quaternions is associative, while the both type of vector products are not

$$[A\cdot(A+B)\cdot B]=\begin{matrix}[(A\cdot A+A\cdot B)\cdot B]=B\\ [A\cdot(A\cdot B+B\cdot B)]=A\end{matrix}\qquad [A\times(A+B)\times B]=\begin{matrix}[(A\times A+A\times B)\times B]=A\\ [A\times(A\times B+B\times B)]=B\end{matrix}$$

Division operation is not defined for vectors. There was lost also the property of '*versors*' that they are rotation generators and express not only the final state, but the direction in which this rotation has been performed.

Split Quaternions

Using of the elements with positive squares,

$$i^2 = j^2 = 1 \,,$$

leads to split quaternions, which norm has **(2+2)**-signature,

$$N = q\,q^* = a^2 - b^2 - c^2 + d^2 \,.$$

Now the third unit element **(*ij*)** of split quaternions differs from other two and can not serve as a geometrical object in the usual sense. It behaves like a pure imaginary object

$$(ij)^2 = -\,1 \,.$$

In this fashion one can justify the origins of complex numbers without introducing them *ad-hoc*.

In the algebra of split-quaternions we have two classes (totally four) projection operators

$$D_i^{\pm} = (1 \pm i)/2 \,,$$
$$D_j^{\pm} = (1 \pm j)/2 \,,$$

and two classes of **Grassmann** numbers

$$G_i^{\pm} = (i \pm ij)/2 \,,$$
$$G_j^{\pm} = (j \pm ij)/2 \,.$$

Simplest non-trivial representation of the split-quaternion basis units can be done by the real **Pauli** matrices accompanied by the unit matrix.

Rotations

Rotations in **3D Euclidean** and **Minkowski** spaces can be represented by **Hamilton**'s and split quaternions. However, a quaternion gives rotation in two independent planes at once. For example, consider rotation of

$$q = I + x\,i + y\,j + z\,k$$

around z-axis by the unit quaternion

$$\alpha = \cos\theta + k \sin\theta\,.$$

The result of the left product,

$$q' = \alpha q = I\cos\theta - z\sin\theta + (x\cos\theta + y\sin\theta)I + (y\cos\theta - x\sin\theta)j + (z\cos\theta + I\sin\theta)k\,,$$

gives simultaneous rotations in $(I - z)$ and $(x - y)$ planes. The right product, $q\alpha$, reverses the direction of rotation in the plane that does not contain the real axis. To get a single-plane rotation, we have to apply by the half-angle, $\theta/2$, quaternions twice,

$$q' = \alpha(\theta/2)\, q\, \alpha^*(\theta/2)\,.$$

This type of rotations does not affect the scalar part of quaternion, I, and its geometrical interpretation caused difficulties. **Hamilton** himself tried without notable success to interpret scalar part of quaternion as an extra-spatial unit.

Quantum Properties

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The 'double cover' property of rotations is a problem for quaternion intervals,

$$s = \lambda + x\,i + y\,j + ct\,k\,,$$

since the positivity of norms with **2+2** signature for split quaternions,

$$s^2 = \lambda^2 + c^2t^2 - x^2 - y^2 > 0\,,$$

should be satisfied for each rotation. For the boosts along x-direction,

$$s' = \lambda\cosh\theta - y\sinh\theta + (x\cosh\theta + t\sinh\theta)i + (y\cosh\theta - \lambda\sinh\theta)j + (t\cosh\theta + x\sinh\theta)k,$$

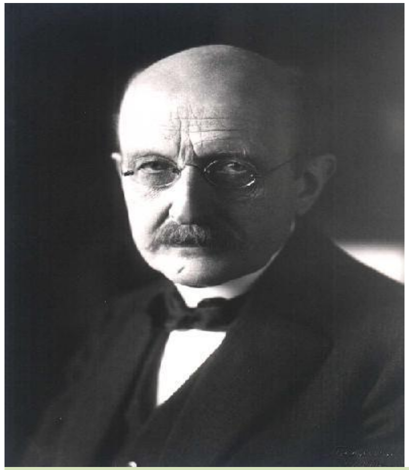
i.e. when p_x increases (p_y decreases), λ decreases, and vice versa. The quantity of dimension x, inversely proportional to p, is wavelength. So λ can be interpreted as the wavelength. Noncommutativity and fundamental spinor representation also are encoded in properties of split quaternions.

In this approach two fundamental constants, light spped c and **Planck**'s constant $\hbar$, have similar geometrical meanings and appear from the positive definiteness of norms,

$$v^2 \leq c^2\,,$$

$$\Delta y/\Delta\lambda \leq 1\,, \quad \Delta\lambda = -\hbar/\Delta p_y\,.$$

So uncertainty principle probably has the same geometrical meaning as the existence of the maximal velocity.



Triality

There is one-to-one correspondence between **Trialities** and division algebras. Distinguishing feature of trialities is that in **1**, **2**, **4**, **8** dimensions, vector, spinor and conjugate spinor representations have the same dimensions. The triality algebra for the case of split quaternions, **p** and **q**, can be defined as:

$$\boldsymbol{\alpha_1(pq) = \alpha_2(p)q + p\alpha_3(q)},$$

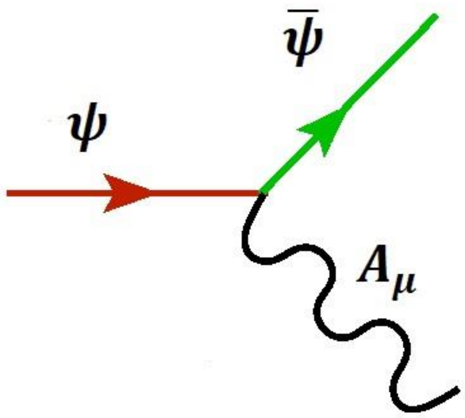
where $\boldsymbol{\alpha_1}$, $\boldsymbol{\alpha_2}$, $\boldsymbol{\alpha_3}$ are unit split quaternions ($\mathbf{det}\,\boldsymbol{\alpha} = \mathbf{1}$). By split quaternions we can describe spinor and vector transformation rules

$$\chi' = \chi\,\alpha^*, \qquad \xi' = \alpha\,\xi, \qquad A' = \alpha\,A\,\alpha^*.$$

Then the triality construction: $\mathbf{Tri} = \xi\,A\,\chi$, is **SO(2,1)** invariant.

Covariant and contravariant spinors, ξ and χ, and a vector, A, can be made from each other: $A = \xi\,\chi^*$, $\chi = A^*\xi$, $\xi = A\,\chi$.

The construction of division algebras from trialities has tantalizing links to physics. In the **Standard Model** of particle physics the interaction is described by a trilinear map involving two spinors and one vector. Moreover, split normed algebras naturally introduce pseudo-**Euclidean** spaces which are associated to **Lorentz**-type groups.



Analyticity

From the condition $ds/ds = 1$ we can define quaternionic gradient operator:

$$d/ds = [d/d\lambda + i\, d/dx + j\, d/dy + k\, d/dt]/2\,,$$

which gives zero if applied to s^*, while d/ds^* annihilates s, i.e. $ds^*/ds = ds/ds^* = 0$. The quaternionic interval (norm), $s^2 = ss^* = s^*s$, is a constant function for the restricted left quaternionic gradient operators,

$$d(ss^*)/ds = d(s^*s)/ds^* = 0\,.$$

So invariance of split quaternionic intervals can be also an algebraic property. To define derivative of a split quaternionic function,

$$\Phi(s,s^*) = \varphi_\lambda + i\,\varphi_x + j\,\varphi_y + k\,\varphi_t\,,$$

we need the condition of analyticity of functions of a split quaternion variable,

$$d\Phi(s,s^*)/ds^* = 0\,,$$

which is similar to the **Cauchy-Riemann** equations from complex analysis.

Analyticity of the triality invariant form:

$$\frac{d(\chi A\,\xi)}{ds^*} = \frac{d\chi}{ds^*}A\xi + \chi\frac{dA}{ds^*}\xi + \chi A\frac{d\xi}{ds^*} = 0$$

is equivalent to the system of three equations for quaternionic vector and spinors,

$$\frac{dA}{ds^*} = 0, \quad \frac{d\chi}{ds^*} = 0, \quad \frac{d\xi}{ds^*} = 0.$$

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Dirac's Equation

It can be shown that the algebraic **Cauchy-Riemann-Fueter** condition for the covariant spinors, $d\xi/ds^* = 0$, is equivalent to the **Dirac** equation.

For a covariant split quaternion ξ we making the replacement,

$$\xi(\lambda, x, y, t) \rightarrow e^{m\lambda}\, \xi(x, y, t) \;,$$

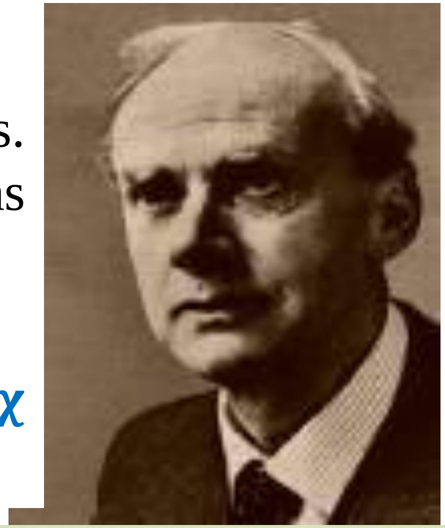
were $\mathbf{m}$ is a real parameter. Also we assume that derivatives of ξ by the extra time-like coordinate λ generates the supersymmetric (**triality**) transformation:

$$\partial_\lambda \xi = \; B\, \chi^* \;,$$

where $B = B\,(x, y, t)$ is a vector-type split quaternion, while $\chi^* = \chi^*(x, y, t)$ is the conjugated contravariant quaternionic spinor.

Then the analyticity condition for the covariant spinor is:

$$\left(\mathrm{e}^{\mathrm{k}}\partial_{\mathrm{k}} - \; \mathrm{m}\;\right)\xi - \; \mathrm{B}\, \chi^* = \; 0,$$

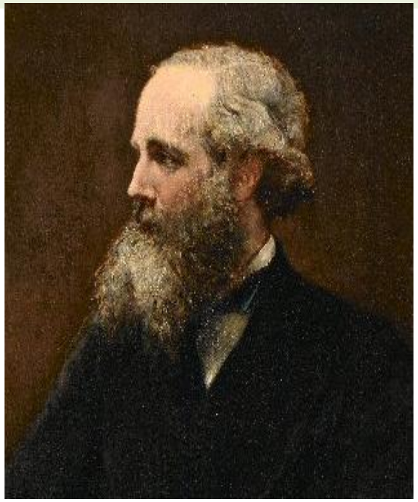


were $\mathbf{e}_k$ $(k = t, x, y)$ are basis elements of split quaternions. When $\mathrm{B_0 = A_t,\ B_1 = A_y,\ B_2 = A_x,\ B_3 = 0}$, this conditions becomes identical to the complex **(2+1)**-Dirac equation

$$\left[\mathrm{i}\gamma^{\mathrm{k}}(\partial_{\mathrm{k}} + \mathrm{i}A_{\mathrm{k}}) - \mathrm{m}\right] \xi(\mathrm{t}, \mathrm{x}, \mathrm{y}) = \mathbf{0}.$$

Analogous logic can be applied to the contravariant spinor χ to obtain **Dirac** equation for conjugated spinors.

Maxwell's Equations



Maxwell's equations in **(2+1)**-space have the form:

$$\partial_x E_x + \partial_y E_y = j_t, \qquad \partial_t E_x - \partial_y H = -j_x,$$

$$\partial_x E_y - \partial_y E_x + \partial_t H = 0, \quad \partial_t E_y + \partial_x H = -j_y,$$

were the relation between the vector potential and fields are

$$H = \partial_x A_y - \partial_y A_x, \;\; E_x = \partial_x A_t - \partial_t A_x, \;\; E_y = \partial_y A_t - \partial_t A_y.$$

The split quaternionic electromagnetic potentials we write as,

$$A = A_0 + e_x A_x + e_y A_y + e_t A_t,$$

were A_k correspond to the **(2+1)**-vector and A_0 is the extra degree of freedom.

The analyticity condition for the vector potential, $dA/ds^* = 0$, is:

$$(\partial_\lambda - e_x\partial_x - e_y\partial_y - e_t\partial_t)A = [\partial_\lambda A - (e_x\partial_x + e_y\partial_y + e_t\partial_t)A_0 - F] = 0.$$

Here F denotes the quaternionic electro-magnetic field,

$$F = e^k \partial_k (e_x A_x + e_y A_y + e_t A_t) = -\partial_k A^k - e_x E_y + e_y E_x + e_t H.$$

We separate the variable $\partial_\lambda A_0 = A_0$ and assume the **triality** transformations,

$$\partial_\lambda (e_x A_x + e_y A_y + e_t A_t) = e_x j_x + e_y j_y - e_t j_t,$$

$j_k \sim \xi\,\chi$ denotes current. Then the analyticity condition gives 1^{st} order system

$$F = A_0, \qquad (e_x\partial_x + e_y\partial_y + e_t\partial_t)A_0 = -e_x j_x - e_y j_y + e_t j_t,$$

which leads to the single second order **Maxwell**'s equation in **(2+1)**-space.

Conclusions:

- Spinors and vectors in **(2+2)**-space of split quaternions are studied and geometrical applications for **(2+1)-Minkowski** space is considered;
- It is noted that some properties of nature, like the invariance of space-time intervals and non-commutativity, probably are encoded in the structures of normed split algebras. In our approach two fundamental physical constants (light speed and **Planck**'s constant) have similar geometrical meanings and appear from the positive definiteness of norms;
- It is shown that the analyticity condition of split quaternions, applied to the triality invariant, is equivalent to some system of differential equations for quaternionic spinors and vectors, which can be reduced to the exact **Dirac-Maxwell** system in **(2+1)-Minkowski** space.

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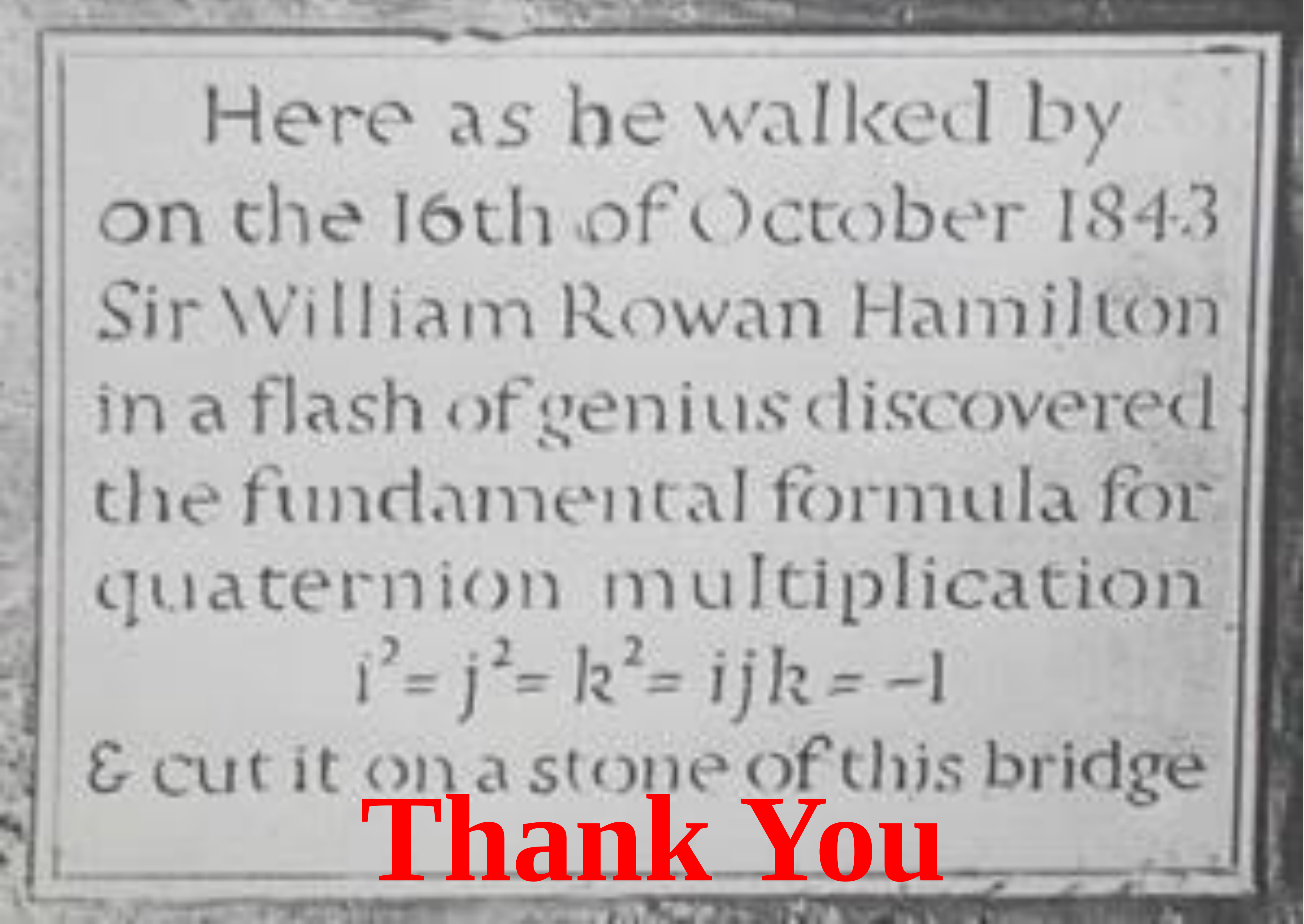
Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

Thank You