Massless finite and infinite spin representations of Poincaré group in six dimensions

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Plan

Motivation and aspects

℧ Study of the various aspects of field theory in higher dimensions attracts much attention due to the remarkable and sometimes even unexpected properties at classical and quantum levels. Many of such properties are closely related to superstring theory which may be treated as a theory of infinite number of higher spin fields in higher dimensional space-time (see e.g. M.B. Green, J.H. Schwarz, E. Witten, Superstring theory, Cambridge Univ. Press, 1987).

℧ The fundamental space-time background in relativistic theory is Minkowski space where the basic symmetry is described by Poincaré group. Theory of unitary irreducible representations of Poincaré group in four dimensions was constructed in the pioneer papers E.P. Wigner, Annals Math. 40 (1939) 149; E.P. Wigner, Z. Physik 124 (1947) 665; V. Bargmann, E.P. Wigner, Proc. Nat. Acad. Sci. US 34 (1948) 211. Review of the unitary irreducible representations in higher dimensions and their applications for constructing the relativistic field equations is given in lectures X. Bekaert, N. Boulanger, *The unitary representations of the Poincaré group in any spacetime dimension*, `arXiv:0611263`. (see also the recent paper S. Weinberg, *Massless Particles in Higher Dimensions*, Phys. Rev. D102 (2020) 095022, `arXiv:2010.05823`)

Motivation and aspects

℧ Although the generic scheme of constructing the representations of the Poincaré group in any dimension seems can be realized on the base of known method of induced representations (see e.g. A.O. Barut, R. Raczka, Theory of Group Representations and Applications, Polish Scientific Publishing, 1977 and A.P. Isaev, V.A. Rubakov, Theory Of Groups And Symmetries (I): Finite Groups, Lie Groups, And Lie Algebras. World Scientific, 2019 **(IR)**)

℧ Some of such aspects are appropriate only for each concrete dimension and can not be formulated at once for all dimensions. For example, the spinor representations of the Lie algebra of multidimensional Lorenz group are defined independently for each space-time dimension. Therefore one can expect that a structure of relativistic symmetry representations in higher dimensions is much more reacher and more complicated then in the four-dimensional Minkowski space.

Motivation and aspects

℧ In this report one will explane a possible way of constructing the massless finite and infinite spin irreducible representations of the Lie algebra of the Poincaré group in six-dimensional Minkowski space.

℧ Some aspects of such representations are considered in papers L. Mezincescu, A.J. Routh, P.K. Townsend, *Supertwistors and massive particles*, Annals Phys. 346 (2014) 66, `arXiv:1312.2768` **(MRT)**, A.S. Arvanitakis, L. Mezincescu, P.K. Townsend, *Pauli-Lubanski, supertwistors, and the super-spinning particle*, JHEP 1706 (2017) 151, `arXiv:1601.05294` however many issues, especially the infinite spin representations, were not addressed and complete analysis was not done.

℧ Recently there was the paper S.M. Kuzenko, A.E. Pindur, *Massless particles in five and higher dimensions*, `arXiv:2010.07124` **(KP)** , where the unitary irreducible massless representations of the Poincaré group in five-dimensional Minkowski space were constructed and some issues related to representations in arbitrary dimensions were briefly studied and the representations of super Poincaré group were considered. The infinite spin representations were not addressed.

D-dimensional Poincaré algebra

℧ The generators P_m and $M_{mn} = -M_{nm}$ of the Lie algebra $\mathfrak{iso}(1, D-1)$ of the Poincaré group in D-dimensional space-time have the commutators

$$[P_n, P_k] = 0 \ , \qquad [M_{mn}, P_k] = i\,(\eta_{mk} P_n - \eta_{nk} P_m) \ , \tag{0.1}$$

$$[M_{mn}, M_{kl}] = i\,(\eta_{mk} M_{nl} + \eta_{nl} M_{mk} - \eta_{ml} M_{nk} - \eta_{nk} M_{ml}) \ , \tag{0.2}$$

where the D-vector indices run the values $m, n = 0, 1, \ldots, D-1$ and we use the space-time metric $\eta^{mn} = \mathrm{diag}(+1, \underbrace{-1, \ldots, -1}_{D-1})$.

℧ We introduce the third rank tensor W_{mnk} and the vector Υ_m as the elements of the enveloping algebra of $\mathfrak{iso}(1,5)$ (see **MRT**)

$$W_{mnk} = \varepsilon_{mnklpr} P^l M^{pr} \ , \tag{0.3}$$

$$\Upsilon_m = \varepsilon_{mnklpr} P^n M^{kl} M^{pr} \ . \tag{0.4}$$

Casimir operators of 6D Poincaré algebra

℧ The operators (0.3) and (0.4) satisfy the equations

$$P^m W_{mnk} = 0\,, \qquad [P_l, W_{mnk}] = 0\,, \tag{0.5}$$

$$P^m \Upsilon_m = 0\,, \qquad [P_l, \Upsilon_m] = 0\,. \tag{0.6}$$

℧ By using of these equations one can check that the operators

$$C_2 := P^m P_m\,, \tag{0.7}$$

$$C_4 := \frac{1}{24} W^{mnk} W_{mnk}\,, \tag{0.8}$$

$$C_6 := \frac{1}{64} \Upsilon^m \Upsilon_m \tag{0.9}$$

are the Casimir operators of the Poincaré algebra $\mathfrak{iso}(1,5)$.

Casimir operators of 6D Poincaré algebra. Remark

℧ Note that the quantity $\varepsilon^{mnklpr} W_{mnk} W_{lpr}$ could be an additional Casimir operator for $\mathfrak{iso}(1,5)$ algebra. But it is identically equal to zero. This fact is a special case of the property of any rank r antisymmetric tensor $W_{m_1 \dots m_r}$ in $2r$-dimensional space, when r is odd number. Indeed, in this case we have $(W,V)_\varepsilon = (-1)^r (V,W)_\varepsilon$, where $(W,V)_\varepsilon := \varepsilon^{m_1 \dots m_r n_1 \dots n_r} W_{m_1 \dots m_r} V_{n_1 \dots n_r}$ and $\varepsilon^{m_1 \dots m_r n_1 \dots n_r} [W_{m_1 \dots m_r}, V_{n_1 \dots n_r}] = 0$. Thus, for antisymmetric tensor with components

$$W_{m_1 \dots m_r} = \varepsilon_{m_1 \dots m_r n_1 \dots n_r} P^{n_1} M^{n_2 n_3} \dots M^{n_{r-1} n_r} \ ,$$

which is defined only for odd r, we always have $(W,W)_\varepsilon = 0$. In this case a Casimir operator for $\mathfrak{iso}(1, 2r-1)$ algebra, of the second order in W, has the unique form

$$W^2 = \frac{1}{(r+1)!} \, W^{m_1 \dots m_r} W_{m_1 \dots m_r} \, .$$

Whereas for even r we have antisymmetric tensor with components

$$L_{m_1 \dots m_r} = \varepsilon_{m_1 \dots m_r n_1 \dots n_r} M^{n_1 n_2} \dots M^{n_{r-1} n_r}$$

which yields for $\mathfrak{so}(\ell, 2r-\ell)$ algebra additional to $L^2 = L^{m_1 \dots m_r} L_{m_1 \dots m_r}$ Casimir operator $(L,L)_\varepsilon \neq 0$ (see below for the case of $\mathfrak{so}(4)$ algebra).

Casimir operators of 6D Poincaré algebra

℧ Taking into account the expressions (0.3), (0.4) we obtain explicit form of all Casimir operators C_2, C_4, C_6:

$$C_2 = P^m P_m \,, \tag{0.10}$$

$$C_4 = \Pi^m \Pi_m - \frac{1}{2} M^{mn} M_{mn} \, C_2 \,, \tag{0.11}$$

$$\begin{aligned} C_6 = & - \Pi^k M_{km} \, \Pi_l M^{lm} + \frac{1}{2} \Big(M^{mn} M_{mn} - 8 \Big) \, C_4 \\ & + \frac{1}{8} \Big[M^{kl} M_{kl} \Big(M^{mn} M_{mn} - 8 \Big) + 2 M^{mn} M_{nk} M^{kl} M_{lm} \Big] \, C_2 \,, \end{aligned} \tag{0.12}$$

where we introduce a new vector Π with components

$$\Pi_m := P^k \, M_{km} = M_{km} \, P^k - 5i \, P_m \,, \tag{0.13}$$

Standard massless momentum reference frame

℧ Further we consider the massless unitary representations of the algebra $\mathfrak{iso}(1,5)$ when the quadratic Casimir operator (0.10) is fixed as following:

$$C_2 \equiv P^2 = P^m P_m = 0\,. \tag{0.14}$$

℧ Let our Poincaré algebra acts in the representation space $\mathcal{H}$ with basis vectors $|k,\sigma\rangle$, where σ is a set of eigenvalues of all operators commuting with P_m and $P_m|k,\sigma\rangle = k_m|k,\sigma\rangle$. We take the light-cone reference frame for massless particle momentum $k^m = (k^0, k^a, k^5) = (k,0,0,0,0,k)$ in which momentum operator P^m has a standard form

$$P^0 = P^5 = k\,, \qquad P^a = 0\,, \quad a = 1,2,3,4\,. \tag{0.15}$$

℧ We stress that all operator formulas presented in this report (and written in the light-cone frame) should be understood as a result of their action on the subspace $\mathcal{H}_k \subset \mathcal{H}$ spanned by vectors $|k,\sigma\rangle$ with fixed light-cone momentum k_m.

Light-cone coordinates

℧ The transition to this light-cone reference frame is conveniently performed in the light-cone basis where any $6D$ vector $X^m = (X^0, X^a, X^5)$ has the light-cone coordinates $X^m = (X^+, X^-, X^a)$, where

$$X^\pm = \frac{1}{\sqrt{2}}\left(X^0 \pm X^5\right)\,, \qquad X_\pm = \frac{1}{\sqrt{2}}\left(X_0 \pm X_5\right) \qquad \Rightarrow \qquad X^\pm = X_\mp\,.$$

℧ Then, in the light-cone basis the contraction of two $6D$ vectors X^m and Y^m is

$$X^m Y_m = X^+ Y_+ + X^- Y_- + X^a Y_a =$$
$$= \eta^{-+} X_- Y_+ + \eta^{+-} X_+ Y_- + \eta^{ab} X_b Y_a \;=\; X_- Y_+ + X_+ Y_- - X_a Y_a\,,$$

where we use the light-cone metric $\eta^{\pm\mp} = \eta_{\pm\mp} = 1$, $\eta^{\pm\pm} = \eta_{\pm\pm} = 0$, $\eta^{ab} = \eta_{ab} = -\delta_{ab}$.

℧ In the light-cone basis the total antisymmetric tensor ε_{mnklpr} has components

$$\varepsilon_{-+abcd} = -\varepsilon_{+-abcd} = \varepsilon^{+-abcd} = -\varepsilon^{-+abcd} = \varepsilon_{abcd}\,,$$

and we normalize the antisymmetric tensors $\varepsilon_{mnk\ell pr}$ and ε_{abcd} as $\epsilon_{012345} = 1$ and $\epsilon_{1234} = 1$.

Massless Casimir's in light-cone frame

℧ In the light-cone basis the standard momentum has the components

$$P^+ = P_- = \sqrt{2}k\,, \qquad P^- = P_+ = 0\,, \qquad P^a = 0\,, \quad a = 1,2,3,4\,.$$

℧ Casimir operators C_4 and C_6 take the form

$$\hat{C}_4 = -\hat{\Pi}_a \hat{\Pi}_a\,, \tag{0.16}$$

$$\hat{C}_6 = \hat{\Pi}_b M_{ba}\,\hat{\Pi}_c M_{ca} - \frac{1}{2} M_{bc} M_{bc}\,\hat{\Pi}_a \hat{\Pi}_a\,, \tag{0.17}$$

where we introduce Hermitian operators

$$\hat{\Pi}_a := \sqrt{2}kM_{+a}\,. \tag{0.18}$$

℧ The operators $\hat{\Pi}_a$ and M_{ab}, which generate (0.16) and (0.17), form the Lie algebra of $ISO(4)$ group

$$[\hat{\Pi}_a, \hat{\Pi}_b] = 0\,, \qquad [\hat{\Pi}_a, M_{bc}] = i\left(\delta_{ab}\hat{\Pi}_c - \delta_{ac}\hat{\Pi}_b\right)\,, \tag{0.19}$$

$$[M_{ab}, M_{cd}] = i\left(\delta_{bc}M_{ad} - \delta_{bd}M_{ac} + \delta_{ac}M_{db} - \delta_{ad}M_{cb}\right)\,, \tag{0.20}$$

and therefore generate the isometries of the four-dimensional Euclidean space. As a result, the operators $\hat{C}_4$ and $\hat{C}_6$ are the Casimir operators of the $\mathfrak{iso}(4)$ algebra.

Decomposition $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and 't Hooft symbols

℧ Six generators of rotations M_{ab} in four-dimensional Euclidean space are decomposed into the sum

$$M_{ab} = M_{ab}^{(+)} + M_{ab}^{(-)}\,, \tag{0.21}$$

where

$$M_{ab}^{(\pm)} := \frac{1}{2}\left(M_{ab} \pm \frac{1}{2}\,\epsilon_{abcd} M_{cd}\right) \tag{0.22}$$

are (anti-)selfdual parts. They are satisfied the identities

$$M_{ab}^{(\pm)} = \pm\frac{1}{2}\,\epsilon_{abcd} M_{cd}^{(\pm)}\,. \tag{0.23}$$

℧ The generators (0.22) form the algebra

$$[M_{ab}^{(\pm)}, M_{cd}^{(\pm)}] = i\Big(\delta_{bc} M_{ad}^{(\pm)} - \delta_{bd} M_{ac}^{(\pm)} + \delta_{ac} M_{db}^{(\pm)} - \delta_{ad} M_{cb}^{(\pm)}\Big)\,, \tag{0.24}$$

$$[M_{ab}^{(+)}, M_{cd}^{(-)}] = 0\,,$$

which is a direct sum of two algebras with three generators $M_{ab}^{(+)}$ and with three generators $M_{ab}^{(-)}$ respectively.

℧ Each of these algebras, containing three generators $M_{ab}^{(+)}$ or $M_{ab}^{(-)}$, is the $\mathfrak{su}(2)$ algebra.

Decomposition $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and 't Hooft symbols

℧ This becomes clear (see e.g. **IR**) after using the 't Hooft symbols G. 't Hooft, *Computation of the quantum effects due to a four-dimensional pseudoparticle*, Phys. Rev. D14 (1976) 3432.

℧ The 't Hooft symbols $\eta^{\rm i}_{ab} = -\eta^{\rm i}_{ba}$, ${\rm i} = 1, 2, 3$ and $\bar\eta^{\rm i'}_{ab} = -\bar\eta^{\rm i'}_{ba}$, ${\rm i} = 1, 2, 3$ are (anti-)selfdual tensors with respect to the $SO(4)$ indices a, b:

$$\eta^{\rm i}_{ab} = \frac{1}{2}\,\epsilon_{abcd}\eta^{\rm i}_{cd}\,, \qquad \bar\eta^{\rm i'}_{ab} = -\frac{1}{2}\,\epsilon_{abcd}\bar\eta^{\rm i'}_{cd}\,. \tag{0.25}$$

℧ Below we use the following standard representations for the 't Hooft symbols

$$\eta^{\rm i}_{ab} = \begin{cases} \epsilon_{{\rm i}ab} & a, b = 1, 2, 3, \\ \delta_{{\rm i}a} & b = 4, \end{cases} \qquad \bar\eta^{\rm i'}_{ab} = \begin{cases} \epsilon_{{\rm i'}ab} & a, b = 1, 2, 3, \\ -\delta_{{\rm i'}a} & b = 4. \end{cases} \tag{0.26}$$

℧ Due to the properties (0.25) the 't Hooft symbols connect (anti-)selfdual $SO(4)$ tensors $M^{(\pm)}_{ab}$ with the $SO(3)$ vectors $M^{(+)}_{\rm i}$, $M^{(-)}_{\rm i'}$ by means of the following relations

$$M^{(+)}_{ab} = -\eta^{\rm i}_{ab}M^{(+)}_{\rm i}\,, \qquad M^{(-)}_{ab} = -\bar\eta^{\rm i'}_{ab}M^{(-)}_{\rm i'}\,. \tag{0.27}$$

Decomposition $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and 't Hooft symbols

℧ Such defined operators $M^{(+)}_{\mathrm{i}}$ and $M^{(-)}_{\mathrm{i}'}$ form two $\mathfrak{su}(2)$ algebras with standard form of the commutators

$$[M^{(+)}_{\mathrm{i}}, M^{(+)}_{\mathrm{j}}] = i\epsilon_{\mathrm{ijk}} M^{(+)}_{\mathrm{k}}\,, [M^{(-)}_{\mathrm{i}'}, M^{(-)}_{\mathrm{j}'}] = i\epsilon_{\mathrm{i}'\mathrm{j}'\mathrm{k}'} M^{(-)}_{\mathrm{k}'}\,, [M^{(+)}_{\mathrm{i}}, M^{(-)}_{\mathrm{j}'}] = 0\,.$$

℧ In term of the operators $M^{(+)}_{\mathrm{i}}$ and $M^{(-)}_{\mathrm{i}'}$ the Casimir $\hat{C}_6$ takes the form (we use the equalities $\eta^{\mathrm{i}}_{ab}\eta^{\mathrm{j}}_{ab} = 4\delta^{\mathrm{ij}}$, $\bar{\eta}^{\mathrm{i}'}_{ab}\bar{\eta}^{\mathrm{j}'}_{ab} = 4\delta^{\mathrm{i}'\mathrm{j}'}$ and $\eta^{\mathrm{i}}_{ab}\bar{\eta}^{\mathrm{j}'}_{ab} = 0$)

$$\hat{C}_6 = 2M^{(+)}_{\mathrm{i}} M^{(-)}_{\mathrm{j}'}\, \eta^{\mathrm{i}}_{ab}\bar{\eta}^{\mathrm{j}'}_{ac}\, \hat{\Pi}_b\hat{\Pi}_c - \left(M^{(+)}_{\mathrm{i}} M^{(+)}_{\mathrm{i}} + M^{(-)}_{\mathrm{i}'} M^{(-)}_{\mathrm{i}'}\right) \hat{\Pi}_a\hat{\Pi}_a\,.$$

Classification orbits

℧ Thus, in the massless case $C_2 \equiv P^m P_m = 0$ the unitary irreducible representations are defined by the eigenvalues of the $\mathfrak{iso}(4)$ Casimir operators $\hat{C}_4$ and $\hat{C}_6$.
℧ For this noncompact symmetry there are two different cases defined the value of Casimir operator $\hat{C}_4$, i.e. square of "four-translation" generator $\hat{\Pi}_a$. So, in next slides we consider following unitary massless representations:

- **Finite spin (helicity) representations**.
  In these cases the $SO(4)$ four-vector $\hat{\Pi}_a$ has zero norm:

$$\hat{\Pi}_a \hat{\Pi}_a = 0 \,. \tag{0.28}$$

- **Infinite (continuous) spin representations**.
  In case of these representations the Euclidean four-vector $\hat{\Pi}_a$ has nonzero norm:

$$\hat{\Pi}_a \hat{\Pi}_a = \mu^2 \neq 0 \,. \tag{0.29}$$

Massless finite spin representations

℧ This case is characterized by the fulfillment of condition (0.28), which implies that all components $\hat{\Pi}_a$ (since they are Hermitian operators) of the Euclidean vector are zero:

$$\hat{\Pi}_a = 0 \qquad \text{at all} \qquad a = 1, 2, 3, 4 \,. \tag{0.30}$$

As a result, the Casimir operators $\hat{C}_4$ and $\hat{C}_6$ are vanish in this case: $\hat{C}_4 = 0$ and $\hat{C}_6 = 0$. Passing from this light-cone reference frame to an arbitrary frame, we get that all Casimir operators on the massless finite spin states take zero values (see also **MRT**)

$$C_4 = 0, \qquad C_6 = 0 \,, \tag{0.31}$$

℧ Due to (0.30) the Euclidean four-translations are realized trivially for these representations. As a result such representations of $ISO(1,5)$ are finite dimensional. Each such massless representation defines some $6D$ standard massless representation with finite number of massless particle states. As we saw above, such representations are induced from irreducible $SO(4)$ representations. Let us show below that the Casimir operators of the stability subgroup $SO(4)$ define the $6D$ helicity operators.

$6D$ helicity operators

℧ First, let us consider the vector Υ_m defined in (0.4). In the case $C_6 = 0$, we have $\Upsilon_m \Upsilon^m = 0$ and, in the light-cone reference frame the components of $6D$ vector Υ are

$$\Upsilon^+ = \Lambda_1 P^+ , \qquad \Upsilon^- = \Upsilon_a = 0 , \tag{0.32}$$

where we have

$$\Lambda_1 \ := \ \epsilon_{abcd} M_{ab} M_{cd} . \tag{0.33}$$

This operator is the Casimir operator of the $\mathfrak{so}(4)$ algebra.
℧ The conditions (0.32) demonstrate that vectors Υ and P are collinear in the light-cone reference frame and this property is conserved in any reference frame. Namely, the relations

$$P^m \Upsilon_m = 0 , \ \ [P_l, \Upsilon_m] = 0$$

show that the light like vector Υ is transverse to the vector P and its components Υ_m commute with P_k.

Therefore, the vector Υ_m is proportional to the vector P_m:

$$\Upsilon_m = \Lambda_1 P_m \,, \tag{0.34}$$

where the operator (0.33) is represented in the form

$$\Lambda_1 := \frac{\Upsilon_0}{P_0} \,. \tag{0.35}$$

℧ This expression appears for the 4D helicity operator when Υ_m is replaced by W_m (see also analogous consideration in **MRT**). Due to the relations

$$[M_{0i}, \Lambda_1] = \frac{i}{P_0}\big(\Upsilon_i - \Lambda_1 P_i\big) = 0 \,, \quad [M_{ik}, \Lambda_1] = 0 = [P_k, \Lambda_1] \,, \tag{0.36}$$

$$(i, k = 1, \ldots, 5) \,,$$

we conclude that the operator (0.35) is invariant with respect to the 6D Poincare symmetry. Therefore, the operator Λ_1, defined in (0.35), is a 6D analog of the helicity operator and it coincides with one of $\mathfrak{so}(4)$ Casimir operators in the light-cone reference frame.

$6D$ helicity operators

℧ We note that irreducible $\mathfrak{so}(4)$ representations are characterized by two quadratic Casimir operators. Another Casimir operator is appeared as a helicity operator if we use the construction proposed in **KP**. Indeed, by using the prescription of **KP**, one can construct another (third order in generators of $\mathfrak{iso}(1,5)$) vector with components

$$S_m := 3M^{nk}P_{[m}M_{nk]} = M^{nk}M_{nk}P_m - 2M^{kn}M_{mn}P_k \,. \qquad (0.37)$$

℧ The square of this $6D$ vector is

$$S^m S_m = M^4 P^2 + 4\Big[\Pi^k M_{km}\Pi_l M^{lm} - M^2(\Pi^2 + P^2) + \Pi^2\Big] \,, \qquad (0.38)$$

while its contraction with $6D$ vector momentum P_m gives

$$P^m S_m = M^{mn}M_{mn}P^2 - 2\Pi^m\Pi_m \equiv -2\, C_4 \,, \qquad (0.39)$$

and the commutators of S_m and P_n are

$$[S_m,\ P_n] = 2iM_{mn}P^2 + 4i\Pi_{[m}P_{n]} \,. \qquad (0.40)$$

$6D$ helicity operators

℧ For the massless finite spin representations previous equations (0.38), (0.39) and (0.40) are reduced to

$$S^m S_m = 0\,, \qquad P^m S_m = 0\,, \qquad [S_m, P_n] = 0\,, \tag{0.41}$$

which are the same as conditions for light-like vectors Υ and P.
℧ So, in the case of massless finite spin representations, the vectors P_m and S_m are also proportional to each other. One can check this in the light-cone reference frame, the components of the $6D$ vector S_m are equal to

$$S^+ = \Lambda_2 P^+\,, \qquad S^- = S_a = 0\,, \tag{0.42}$$

where the operator

$$\Lambda_2 \;:=\; M_{ab} M_{ab} \tag{0.43}$$

is second $\mathfrak{so}(4)$ Casimir operator.

$6D$ helicity operators

℧ Due to the relations (0.41) for the general frame the relations (0.42) take the form

$$S_m = \Lambda_2 P_m \,, \tag{0.44}$$

where the operator Λ_2 defines second helicity operator and has equivalent "covariant" form

$$\Lambda_2 := \frac{S_0}{P_0} \,. \tag{0.45}$$

℧ So these massless representations of finite spin are characterized by the pair (λ_1, λ_2), where real numbers $\lambda_{1,2}$ define the eigenvalue of the Casimir operators Λ_1 and Λ_2, respectively.

℧ Using the formula for the decomposition of generators M_{ab} on (anti-)selfdual parts and relations of these parts with t' Hooft symbols we represent helicity operators Λ_1 and Λ_2 in the form

$$\Lambda_1 = 2\left(M^{(+)}_{ab} M^{(+)}_{ab} - M^{(-)}_{ab} M^{(-)}_{ab}\right) = 8\left(M^{(+)}_{\mathrm{i}} M^{(+)}_{\mathrm{i}} - M^{(-)}_{\mathrm{i}'} M^{(-)}_{\mathrm{i}'}\right) \,,$$

$$\Lambda_2 = M^{(+)}_{ab} M^{(+)}_{ab} + M^{(-)}_{ab} M^{(-)}_{ab} = 4\left(M^{(+)}_{\mathrm{i}} M^{(+)}_{\mathrm{i}} + M^{(-)}_{\mathrm{i}'} M^{(-)}_{\mathrm{i}'}\right) \,.$$

$6D$ helicity operators

℧ In case of unitary representations, the operators $M^{(+)}_{\mathrm{i}} M^{(+)}_{\mathrm{i}}$ and $M^{(-)}_{\mathrm{i}'} M^{(-)}_{\mathrm{i}'}$ are equal to $j_+(j_+{+}1)$ and $j_-(j_-{+}1)$ respectively. Therefore, the eigenvalues of the helicity operators take the values

$$\lambda_1 = 8j_+(j_+ + 1) - 8j_-(j_- + 1)\,, \quad (0.46)$$

$$\lambda_2 = 4j_+(j_+ + 1) + 4j_-(j_- + 1)\,, \quad (0.47)$$

where $j_\pm$ are integer or half-integer numbers in the case of the unitary representations.

℧ Note that the standard $4D$ helicity operator is invariant under proper $SO(1,3)$ rotations but changes its sign under improper $O(1,3)$ rotations. We have the same property for Λ_1 but it is not the case for Λ_2.

Examples

℧ Here we will demonstrate the use of the obtained formulas for determining the helicities on the examples of some massless finite spin fields. To clarity and avoid technical complications, we will consider only bosonic integer-spin fields.

℧ Since the irreducible massless representations of the $6D$ Poincaré group are induced by the irreducible $SO(4)$ representations in the light-cone reference frame, we will use the following procedure.

℧ Below, in all examples of this section, we first consider a fixed irreducible $SO(4)$ representation and determine the values of the helicities. Here we will use the defining representation for the $\mathfrak{so}(4)$ generators

$$(\mathcal{M}_{ab})_{eg} = i(\delta_{ae}\delta_{bg} - \delta_{ag}\delta_{be})\,. \tag{0.48}$$

℧ Then we reconstruct the corresponding $6D$ field, for which the equations of motion and gauge fixing show that the independent components are exactly those $SO(4)$ fields which were considered earlier in the Euclidean four-dimensional picture.

Vector field

℧ Let us consider the $SO(4)$ vector field A_a. In this case the $\mathfrak{so}(4)$ generators coincide with (0.48):

$$(M_{ab})_{eg} = (\mathcal{M}_{ab})_{eg}\,. \tag{0.49}$$

℧ Then, the $SO(4)$ Casimir operators take the form

$$\begin{aligned}(\Lambda_1)_{eg} &= \epsilon_{abcd}(M_{ab}M_{cd})_{eg} = 0\,,\\ (\Lambda_2)_{eg} &= (M_{ab}M_{ab})_{eg} = 6\delta_{eg}\,.\end{aligned} \tag{0.50}$$

℧ When acting on the $SO(4)$ vector field A_a, the operators (0.50) give the following values of helicities:

$$\lambda_1 = 0\,, \quad \lambda_2 = 6\,; \qquad j_+ = j_- = \frac{1}{2}\,. \tag{0.51}$$

Vector field

℧ This Euclidean vector field A_a describes physical components of the $6D$ vector gauge field A_m. In the momentum representation the $U(1)$ massless gauge field A_m is described by the equations of motion

$$P^m F_{mn} = 0\,, \tag{0.52}$$

where $F_{mn} = i(P_m A_n - P_n A_m)$ is the field strength, and determined up to gauge transformations

$$\delta A_m = iP_m \varphi\ . \tag{0.53}$$

One of the possible gauge fixing for transformations (0.53) is the light-cone gauge (W. Siegel, *Fields*, `arXiv:hep-th/9912205` **(WS)**)

$$A^+ = 0\,. \tag{0.54}$$

Then in the light-cone frame, the equations of motion (0.52) give $A^- = 0$ and an independent field is given by the transverse part A_a of the $6D$ gauge field A_m.

Second rank symmetric tensor field

℧ Now we consider the $SO(4)$ second rank tensors. In this case the $\mathfrak{so}(4)$ generators take the matrix form

$$(M_{ab})_{e_1e_2,g_1g_2} = \big((\mathcal{M}_{ab})_1 + (\mathcal{M}_{ab})_2\big)_{e_1e_2,g_1g_2} = (\mathcal{M}_{ab})_{e_1g_1}\delta_{e_2g_2} + \delta_{e_1g_1}(\mathcal{M}_{ab})_{e_2g_2}$$

and the $SO(4)$ Casimir operators are

$$(\Lambda_1)_{e_1e_2,g_1g_2} = \epsilon_{abcd}\big(M_{ab}M_{cd}\big)_{e_1e_2,g_1g_2} = 2\,\epsilon_{abcd}\big((M_{ab})_1(M_{cd})_2\big)_{e_1e_2,g_1g_2}$$
$$= \; 8\,\epsilon_{e_1e_2g_1g_2}\,,$$
$$(\Lambda_2)_{e_1e_2,g_1g_2} = (M_{ab}M_{ab})_{e_1e_2,g_1g_2} = \big((M^2_{ab})_1 + (M^2_{ab})_2 + 2(M_{ab})_1(M_{ab})_2\big)_{e_1e_2,g_1g_2}$$
$$= 12\,\delta_{e_1g_1}\delta_{e_2g_2} + 4(\delta_{e_1g_2}\delta_{e_2g_1} - \delta_{e_1e_2}\delta_{g_1g_2})\,.$$

First, we consider the $SO(4)$ second rank tensor $\hat{h}_{ab}$, which is symmetric $\hat{h}_{ab} = \hat{h}_{ba}$ and traceless $\hat{h}_{aa}$. On this field the helicity operators take the values

$$\lambda_1 = 0\,, \quad \lambda_2 = 16\,; \qquad j_+ = j_- = 1\,. \tag{0.55}$$

Let us show that this field $\hat{h}_{ab}$ describes the physical components of the $6D$ linearized gravitational field.

Second rank symmetric tensor field

℧ Let us show that this field $\hat{h}_{ab}$ describes the physical components of the $6D$ linearized gravitational field. The $6D$ linearized gravitational field $h^{mn} = h^{nm}$ is determined by the well known equations of motion

$$P^2 h^{mn} - P^m P_k h^{nk} - P^n P_k h^{mk} + P^m P^n h_k{}^k = 0\,, \qquad (0.56)$$

and has gauge invariance

$$\delta h^{mn} = iP^{(m}\varphi^{n)}\,. \qquad (0.57)$$

For the transformations (0.57) we can put again the light-cone gauge (see also **WS**)

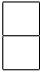
$$h^{+m} = 0\,. \qquad (0.58)$$

The equations of motion (0.56) produce $h^{-m} = 0$, $h_a{}^a = 0$ in the light-cone frame. As a result, nonvanishing physical components of the $6D$ gravity field h_{mn} are given by the traceless part $\hat{h}_{ab}$ of its transverse components h_{ab}.

Third rank (anti-)selfdual antisymmetric tensor fields

℧ Now we consider the $SO(4)$ antisymmetric tensors of the second rank $B^{(\pm)}_{ab} = -B^{(\pm)}_{ba}$, which are (anti-)selfdual

$$B^{(\pm)}_{ab} = \pm \frac{1}{2}\, \epsilon_{abcd} B^{(\pm)}_{cd} \,. \tag{0.59}$$

These tensors form the spaces of two $SO(4)$ irreducible representations which make up the $SO(4)$ reducible representation in the space of all antisymmetric rank 2 tensors associated to Young diagram $[1^2] \equiv$ .

℧ In this case the $\mathfrak{so}(4)$ generators M_{ab} and helicity operators Λ_1, Λ_2 have the same expressions. Then the eigenvalues of the operators Λ_1, Λ_2 are given by numbers

$$\lambda_1 = 16\,, \quad \lambda_2 = 8\,; \qquad j_+ = 1\,, \quad j_- = 0 \tag{0.60}$$

on the space of the selfdual fields $B^{(+)}_{ab}$, and by

$$\lambda_1 = -16\,, \quad \lambda_2 = 8\,; \qquad j_+ = 0\,, \quad j_- = 1 \tag{0.61}$$

on the space of the anti-selfdual fields $B^{(-)}_{ab}$.

Third rank (anti-)selfdual antisymmetric tensor fields

℧ It is clear that these $SO(4)$ (anti-)selfdual fields $B^{(\pm)}_{[ab]}$ are independent components of the $6D$ massless (anti-)selfdual 3-rank fields $B^{(\pm)}_{mnk}$ which satisfy the identities

$$B^{(\pm)}_{mnk} = \pm\frac{1}{3!}\,\varepsilon_{mnklpr}B^{(\pm)\,lpr}\,. \tag{0.62}$$

So, the equations of motion of the $6D$ massless fields $B^{(\pm)}_{mnk}$ are

$$\text{a)}\ \ P^m B^{(\pm)}_{mnk} = 0\,, \qquad \text{b)}\ \ P_{[m}B^{(\pm)}_{nkl]} = 0\,, \qquad \text{c)}\ \ P^2 B^{(\pm)}_{nkl} = 0\,. \tag{0.63}$$

℧ Then in the light-cone frame the equations (0.63a) give $B^{(\pm)-mn} = 0$ whereas the equations (0.63b) produce $B^{(\pm)abc} = 0$. As a result, independent fields of the $6D$ tensors $B^{(\pm)}_{mnk}$ are the $SO(4)$ (anti-)selfdual fields $B^{(\pm)-ab} \equiv B^{(\pm)ab}$ which are subjected the $SO(4)$ (anti-)selfdual conditions (0.59) due to the $6D$ (anti-)selfdual conditions (0.62).

Massless infinite (continuous) spin representations

℧ In this case, when the condition $\hat{\Pi}_a\hat{\Pi}_a = \mu^2 \neq 0$ is satisfied and the Euclidean four-vector $\hat{\Pi}_a$ is nonzero. Then the representations of the $ISO(4)$ group, which induce the $6D$ relativistic massless representations, are infinite dimensional.
In case of these representations the Casimir operator $\hat{C}_4$ has nonvanishing eigenvalue

$$C_4 \;=\; \hat{C}_4 \;=\; -\mu^2\,, \qquad \mu \neq 0\,. \tag{0.64}$$

Moreover, we can take the basis with nonzero only the fourth component:

$$\hat{\Pi}_1 = \hat{\Pi}_2 = \hat{\Pi}_3 = 0\,, \qquad \hat{\Pi}_4 = \mu\,. \tag{0.65}$$

Then taking into account $\eta^{\mathrm{i}}_{a4} = \delta_{\mathrm{i}a}$ and $\bar{\eta}^{\mathrm{i}'}_{a4} = -\delta_{\mathrm{i}'a}$ in previous formula for $\hat{C}_6$ (in terms of t' Hoofts symbols and generators $\mathfrak{su}(2)$ algebras $M^{(\pm)}_i$), we obtain that the value of the Casimir operator $\hat{C}_6$:

$$\hat{C}_6 \;=\; -\mu^2\, J_{\mathrm{i}} J_{\mathrm{i}}\,, \tag{0.66}$$

where

$$J_{\mathrm{i}} \;:=\; M^{(+)}_{\mathrm{i}} + M^{(-)}_{\mathrm{i}} \tag{0.67}$$

are the generators of the diagonal $su(2)$ subalgebra of the stability algebra $so(4) = su(2) \oplus su(2)$.

Massless infinite (continuous) spin representations

℧ Using explicit expressions of the 't Hooft symbols (see e.g. Sect. 3.3.3 in **IR**) we find

$$J_{\rm i} \;=\; -\frac{1}{2}\,\epsilon_{\rm ijk}M_{\rm jk}\,, \qquad {\rm i}=1,2,3\,. \tag{0.68}$$

So the operators (0.67) are in fact the generators of the $\mathrm{SO}(3)$ subgroup of the $\mathrm{SO}(4)$ stability group. Therefore, in case of the unitary representations it is necessary to satisfy the equality

$$J^2 \;=\; s(s+1)\,, \tag{0.69}$$

where s is a fixed integer or a half-integer number. So, in the case of the irreducible representations of infinite (continuous) spin, the Casimir operator C_6 takes the value

$$C_6 \;=\; \hat{C}_6 \;=\; -\mu^2\,s(s+1)\,, \tag{0.70}$$

Such irreducible representations describe a tower of infinite number of massless states.

℧ As a result, the massless infinite spin representations are characterized by the pair (μ, s), where the real parameter μ defines the eigenvalue of the Casimir operator C_4 and the (half-)integer number s defines the eigenvalue of the Casimir operator (0.70).

Massless infinite (continuous) spin representations

℧ Let us examine in our consideration the $D = 6$ infinite integer spin system X. Bekaert, J. Mourad; arXiv:hep-th/0509092 **(BM)** which is a higher dimension generalization of the $D = 4$ model E.P. Wigner, Annals Math. 40 (1939) 149; E.P. Wigner, Z. Physik 124 (1947) 665; V. Bargmann, E.P. Wigner, Proc. Nat. Acad. Sci. US 34 (1948) 211; **BM** model is described by the pair of the space-time phase operators

$$x^m \,, p_m \,, \quad [x^m, p_k] = i\delta_k^m \tag{0.71}$$

and two pairs of the additional bosonic phase vectors

$$w^m \,, \xi_m \,, \quad [w^m, \xi_k] = i\delta_k^m \,; \qquad u^m \,, \zeta_m \,, \quad [u^m, \zeta_k] = i\delta_k^m \,. \tag{0.72}$$

Massless infinite (continuous) spin representations

℧ These two pairs of vectors (0.72) are responsible for spinning degrees of freedom. Infinite integer spin field Ψ in **BM** is described by the $D=6$ generalization of the Wigner-Bargmann equations

$$p^2\,\Psi = 0\,, \quad \xi\cdot p\,\Psi = 0\,, \quad (w\cdot p - \mu)\,\Psi = 0\,, \quad (\xi\cdot\xi + 1)\,\Psi = 0\,, \quad (0.73)$$

℧ And additional equations with vector operators from the second pair

$$u{\cdot}p\,\Psi = 0\,,\ \zeta\cdot p\,\Psi = 0\,,\ \zeta\cdot\xi\,\Psi = 0\,,\ \zeta\cdot\zeta\,\Psi = 0\,,\ (u\cdot\zeta - s)\,\Psi = 0\,, \quad (0.74)$$

where $\xi\cdot p := \xi^m p_m$, *etc.*
℧ Note that, in contrast to the four-dimensional case with one pair of auxiliary variables w^m, ξ_m[1], in the six-dimensional case it is necessary to use the second pair of auxiliary vector variables u^m, ζ_m to describe arbitrary infinite spin representations.

[1] Note that in the twistor formulation of the infinite spin particle I.L. Buchbinder, S. Fedoruk, A.P. Isaev, *Twistorial and space-time descriptions of massless infinite spin (super)particles and fields*, Nucl. Phys. B 945 (2019) 114660, arXiv:1903.07947, it was more convenient for us to use dimensional additional variables $y^m = w^m/\mu$, $q_m = \mu\xi_m$.

Massless infinite (continuous) spin representations

℧ In the light-cone frame and in the representation $\xi_m = -i\partial/\partial w^m$, $\zeta_m = -i\partial/\partial u^m$ the equations (0.73) give the conditions

$$\frac{\partial}{\partial w^+}\,\Psi = 0\,,\ (p^+ w^- - \mu)\,\Psi = 0\,,\ \left(\frac{\partial}{\partial w_a}\frac{\partial}{\partial w_a} + 1\right)\Psi = 0\,, \qquad (0.75)$$

whereas (0.74) yield

$$p^+ u^-\,\Psi = 0\,,\ \frac{\partial}{\partial u^+}\,\Psi = 0\,,\ \frac{\partial}{\partial u_a}\frac{\partial}{\partial w_a}\,\Psi = 0\,, \qquad (0.76)$$

$$\frac{\partial}{\partial u_a}\frac{\partial}{\partial u_a}\,\Psi = 0\,,\ \left(u_a\frac{\partial}{\partial u_a} - s\right)\Psi = 0\,. \qquad (0.77)$$

℧ The solution of this equations is the field

$$\Psi = \delta(p^+ w^- - \mu)\,\delta(p^+ u^-)\,\Phi(w_a, u_a)\,, \qquad (0.78)$$

where $\Phi(w_a, u_a)$ has series expansions presented in **BM**.

℧ Now we can determine the values of the Casimir operators $\hat{C}_4$ and $\hat{C}_6$ on the field (0.78). For the field (0.78) the generators of the *iso*(4) algebra have the form

$$M_{ab} = i\left(w_a\,\frac{\partial}{\partial w_b} - w_b\,\frac{\partial}{\partial w_a} + u_a\,\frac{\partial}{\partial u_b} - u_b\,\frac{\partial}{\partial u_a}\right)\,,\ \hat{\Pi}_a = -i\mu\frac{\partial}{\partial w_a}\,.$$

As a result, due to third equation from (0.75), we obtain the fulfillment of the condition for the Casimir operator C_4: $C_4 = \hat{C}_4 = -\mu^2$.

Massless infinite (continuous) spin representations

℧ Moreover, this representations lead to the expression

$$\hat{C}_6 = \mu^2\, u_a \frac{\partial}{\partial u_a}\left(u_b \frac{\partial}{\partial u_b} + 1 \right) \frac{\partial}{\partial w_c}\frac{\partial}{\partial w_c} \qquad (0.79)$$
$$+ \mu^2 \left(u_a \frac{\partial}{\partial w_a}\, u_b \frac{\partial}{\partial w_b} \;-\; u_a u_a \frac{\partial}{\partial w_b}\frac{\partial}{\partial w_b} \right) \frac{\partial}{\partial u_c}\frac{\partial}{\partial u_c}$$
$$+ \mu^2 \left(u_a u_a \frac{\partial}{\partial u_b}\frac{\partial}{\partial w_b} \;-\; 2u_a \frac{\partial}{\partial u_a}\, u_b \frac{\partial}{\partial w_b} \right) \frac{\partial}{\partial u_c}\frac{\partial}{\partial w_c}$$

for the sixth order Casimir operator. So, due to the equations (0.75), (0.76) the operator $\hat{C}_6$ takes the value $\hat{C}_6 = -\mu^2 s(s+1)$ on the field Ψ.

℧ Thus, the infinite spin field with only one additional vector variables and obeying the Wigner-Bargmann equations and additional equations describes the irreducible (μ, s) infinite spin representation. The system with only one pair of auxiliary variables w^m, ξ_m and with only the equations of motion (0.73) describe the infinite spin representations at $s = 0$ (see **BM**).

Conclusions

We have studied the massless irreducible representations of the Poincaré group in six-dimensional Minkowski space and give full classification of all massless representations including infinite integer spin case.

- The representations are described by three Casimir operators written in the form (0.7), (0.8), (0.9) or in the equivalent form (0.10), (0.11), (0.12).
- The properties of these operators are explored in the standard massless momentum reference frame, where it is seen that the unitary representations of $ISO(1,5)$ group are induced from representations of $SO(4)$ and $ISO(4)$ groups and correspondingly are divided into finite spin (helicity) and infinite spin representations. Both these representations are studied in details.
- It is proved that the finite spin representation is described by two integer or half-integer numbers while the infinite spin representation is described by one real parameter and one integer or half-integer number.
- In case of half-integer spin we should introduce an additional spinor or twistor variables like in **BM**.

Continuation

- As a continuation of this research it would be interesting to describe the massless representations with half-integer spin and massive irreducible representations of six-dimensional Poincaré group with both integer and half-integer spin.
- Another open problem is constructing the representations of the corresponding six-dimensional *super* Poincaré group.
- Also it would be useful to work out the field realizations of the massless representations considered in the paper **arXiv:2011.14725** and develop a Lagrange formulation for these fields in six-dimensional Minkowski space.

℧ We plan to study all these problems in the forthcoming papers.