New approach to N=2 supersymmetric Ruijsenaars-Schneider model

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Ruijsenaars-Schneider models

In the paper [Annals of Physics 170 (1986), 370-405] Ruijsenaars and Schneider, with hope of describing soliton scattering, developed a new class of integrable systems, defined by the Hamiltonian

$$H_{RS} = mc^2 \sum_i \cosh(\theta_i) \prod_{j\neq i} f(x_i - x_j), \quad \{x_i, \theta_j\} = \delta_{ij},$$

where c is the speed of light. The Hamiltonian H_{RS} forms a Poincare algebra with respect to Poisson brackets together with

$$P = mc \sum_i \sinh(\theta_i) \prod_{j\neq i} f(x_i - x_j), \quad B = -\frac{1}{c} \sum_i x_i,$$

$$\{H_{RS}, P\} = 0, \quad \{H, B\} = P, \quad \{P, B\} = H/c^2$$

if $f^2(x) = a + b\wp(x)$, where $\wp(x)$ is the Weierstrass elliptic function. This condition follows if one just requires $\{H, P\} = 0$. This condition is also sufficient for integrability of the system (we do not discuss integrals there further).
Let us also note that in the limit $c \to \infty$ this system reduces to Calogero-Moser one.

Ruijsenaars-Schneider models

One can alternatively formulate the condition of relativistic invariance as $\{S_+, S_-\} = 0$, with

$$S_\pm = \tfrac{1}{2}(H_{RS} \pm P) = \tfrac{1}{2}\sum_i e^{\pm\theta_i}\prod_{j\neq i} f(x_i - x_j).$$

(for further convenience, let us put $m = 1$, $c = 1$ in what follows). $S_\pm$ were called by Ruijsenaars and Schneider " the lightcone Hamiltonians". One can also use S_+ as an alternative Hamiltonian, resulting in the simple equations of motion

$$\ddot{x}_i = 2\sum_{j\neq i}^{n} \dot{x}_i\dot{x}_j W(x_i - x_j), \quad W(x) = -f'(x)/f(x).$$

Another advantage of using S_+ as a Hamiltonian is that it can be rewritten in form, resembling the usual "free" Hamiltonian, at the cost of deforming Poisson brackets:

$$H = \tfrac{1}{2}\sum_{i=1}^{n} p_i^2, \quad p_i = e^{\theta_i}\prod_{j(\neq i)}^{n}\sqrt{f(x_i - x_j)},$$

$$\{x_i, p_j\} = \delta_{ij}p_j, \quad \{p_i, p_j\} = (1 - \delta_{ij})\, p_i p_j W(x_i - x_j).$$

Attempts of $N=2$ supersymmetrization

The attempts of construction of $N=2$ supersymmetric version of the RS model involved Hamiltonian $H_{bos}=S_+=1/2\sum_i p_i^2$, and a restricted set of functions W, which can be obtained from the general elliptic case in special limits

$$W(x)\;\in\;\{1/x,\;1/\sin(x),\;1/\sinh(x),\;1/\tan(x),\;1/\tanh(x)\}\,.$$

The $N=2$ supersymmetrization involves construction of supercharges Q, $\overline{Q}$ with brackets

$$\{Q,Q\}=0,\;\;\{\overline{Q},\overline{Q}\}=0,\;\;\{Q,\overline{Q}\}=-2\mathrm{i}H,\;\;H=\tfrac{1}{2}\sum_i p_i^2+\text{fermions}.$$

In [JHEP **1804** (2018) 079], Galajinsky introduced the fermions ψ_i, $\bar\psi_j$ with standard brackets and the supercharges at most cubic in the fermions

$$\{\psi_i,\bar\psi_j\}=-\mathrm{i}\delta_{ij},\;\;\{\psi_i,\psi_j\}=0\;\;\{\bar\psi_i,\bar\psi_j\}=0,\;\;Q=\tfrac{1}{2}\sum_i p_i\psi_i+\sum_{ijk}f_{ijk}\psi_i\psi_j\bar\psi_k.$$

and found that $\{Q,Q\}=0$ is possible to achieve for at most 3 particles only.
In [Phys.Lett.B 807 (2020) 135545] Krivonos and Lechtenfeld proposed using simple “free” supercharges with modified brackets between the fermions

$$\mathcal{Q}=\sum_i p_i\psi_i,\;\;\overline{\mathcal{Q}}=\sum_i p_i\bar\psi_i,\;\;\{\psi_i,\psi_j\}=-\psi_i\psi_j W(x_i-x_j),\;\;\{\psi_i,\bar\psi_j\}=-\mathrm{i}\delta_{ij}+\ldots$$

Attempts of $N=2$ supersymmetrization

The modification of brackets between the fermions allowed to compensate the term $\sum_i p_i p_j \psi_i \psi_j W(x_i - x_j)$ which appeared in the bracket $\{\mathcal{Q}, \mathcal{Q}\}$ due to $\{p_i, p_j\} = p_i p_j W(x_i - x_j)$. To satisfy Jacobi identities, it is required to revise other brackets:

$$\{\psi_i, \bar{\psi}_j\} = -\mathrm{i}\delta_{ij} + \psi_i \bar{\psi}_j W(x_i - x_j), \quad \{x_i, \psi_j\} = \{x_i, \bar{\psi}_j\} = 0,$$
$$\{p_i, \psi_j\} = \mathrm{i}/2\delta_{ij} p_i \psi_i \sum_{k\neq i} W'(x_i - x_k)\psi_k \bar{\psi}_k - \mathrm{i}/2 p_i \psi_j \psi_i \bar{\psi}_i W'(x_i - x_j),$$
$$\{p_i, \bar{\psi}_j\} = -\mathrm{i}/2\delta_{ij} p_i \bar{\psi}_i \sum_{k\neq i} W'(x_i - x_k)\psi_k \bar{\psi}_k + \mathrm{i}/2 p_i \bar{\psi}_j \psi_i \bar{\psi}_i W'(x_i - x_j)$$

Jacobi identities and $\{\mathcal{Q}, \mathcal{Q}\} = 0$ hold for any $W(x)$.
Note that one can also define fermions with standard brackets:

$$\xi_i = \psi_i \exp\Big(\mathrm{i}/2 \sum_{k\neq i} W(x_i - x_k)\psi_k \bar{\psi}_k\Big) \;\Leftrightarrow\; \psi_i = \xi_i \exp\Big(-\mathrm{i}/2 \sum_{k\neq i} W(x_i - x_k)\xi_k \bar{\xi}_k\Big),$$
$$\mathcal{Q} = \sum_i p_i \xi_i \exp\Big(-\mathrm{i}/2 \sum_{k\neq i} W(x_i - x_k)\xi_k \bar{\xi}_k\Big), \quad \{\xi_i, \xi_j\} = \{\xi_i, p_j\} = 0, \quad \{\xi_i, \bar{\xi}_j\} = -\mathrm{i}\delta_{ij}.$$

Supercharges and matrix power series

Let us propose a bit different approach to the $N = 2$ Ruijsenaars-Schneider model. It is based on simple brackets

$$\{x_i, p_j\} = \delta_{ij} p_j, \quad \{p_i, p_j\} = p_i p_j (1 - \delta_{ij}) W(x_i - x_j), \quad \{\xi_i, \bar{\xi}_j\} = -\mathrm{i}\delta_{ij}$$

and the following ansatz for the supercharges:

$$Q = \sum_{i,j} p_i \left(\frac{1}{1-\Pi}\right)_{ij} \xi_j, \quad \overline{Q} = \sum_{i,j} p_i \left(\frac{1}{1-\Pi}\right)_{ij} \bar{\xi}_j, \quad \Pi_{ij} = \frac{\mathrm{i}}{2}(1-\delta_{ij}) W(x_i - x_j)\left(\xi_i \bar{\xi}_j - \xi_j \bar{\xi}_i\right).$$

Here, $(1-\Pi)^{-1}$ is matrix power series $(1-\Pi)^{-1}_{ij} = \delta_{ij} + \Pi_{ij} + \sum_k \Pi_{ik}\Pi_{kj} + \dots$. To make the system $N = 2$ supersymmetric, the supercharges should satisfy $\{Q, Q\} = \{\overline{Q}, \overline{Q}\} = 0$.

Much like the case of [Phys.Lett.B 807 (2020) 135545], the Π_{ij} term in the power series allows to cancel the contribution of $\{p_i, p_j\}$ to $\{Q, Q\}$ in the quadratic approximation in the fermions. Then other term should be chosen to cancel quadric terms, and so on. One can examine a few first terms in the simplest case $W = 1/x$ and obtain that the matrix power series should read

$$\delta_{ij} + \Pi_{ij} + \sum_k \Pi_{ik}\Pi_{kj} + \sum_{k,l} \Pi_{ik}\Pi_{kl}\Pi_{lj} + \dots$$

Constraints on W

Studying the supercharges for general W, one can check that $\{Q, Q\} = 0$ is satisfied in the 2nd and 4th approximation in the fermions, regardless of W. However, unusually for $N = 2$ supersymmetry, the terms of 6th power in the fermions cancel only if a rather nontrivial constraint is satisfied

$$E_{[ik](jl)} = W_{ij} W_{ik} W_{il} - W_{ij} W_{il} W_{jk} + W_{ik} W_{il} W_{jk} - W_{il} W_{jk} W_{jl} - W_{ij} W_{jk} W_{kl} + \\ + W_{ij} W_{il} W_{kl} + W_{ij} W_{jk} W_{kl} - W_{ik} W_{jk} W_{kl} + W_{il} W_{jk} W_{kl} - W_{ij} W_{jl} W_{kl} = 0,$$

where $W_{ij} = W(x_i - x_j)$. One can show, however, that is actually simpler: solving together $E_{[ik](jl)} = 0$ and $E_{[jk](il)} = 0$ w.r.t. to W_{ik} and W_{il}, one finds that

$$W_{ik} = \frac{W_{ij} W_{jk} + W_{jk} W_{jl} - W_{jk} W_{kl} + W_{jl} W_{kl}}{W_{ij} + W_{jk}}, \quad W_{il} = \frac{W_{ij} W_{jl} + W_{jk} W_{jl} - W_{jk} W_{kl} + W_{jl} W_{kl}}{W_{ij} + W_{jl}},$$

$$\text{or} \quad W_{ik} W_{ij} + W_{ik} W_{jk} + W_{ji} W_{jk} = W_{jk} W_{jl} + W_{kj} W_{kl} + W_{jl} W_{kl}$$

One can note that the left hand side of this equation depends on x_i, does not depend on x_l and the right hand side the opposite. They, therefore, should be equal to the same function of x_j, x_k. Moreover, as left hand side is symmetric with respect to mutual interchanges of x_i, x_j, x_k this function should be simply a constant.

Solving equation on W

Equation

$$W(x_i - x_j)W(x_i - x_k) + W(x_k - x_i)W(x_k - x_j) + W(x_j - x_i)W(x_j - x_k) = c = \text{const}$$

strongly restricts domain of acceptable functions W. Surprisingly, its solutions are just a rational function and trigonometric/hyperbolic cotangent, which are among those that are needed to make the model integrable. To show this, let us note that

- The only significant values of c are -1, 0, 1, solutions with others can be obtained by rescaling;
- This equation should restrict functional form of W and should be valid for any x_i, x_j, x_k within the domain of acceptability;
- The only solution smooth at 0 is $W = 0$. Indeed, if W is smooth and odd, $W(0) = 0$. When, if $x_j = x_k = 0$, one finds $W(x_i)^2 = c = 0$.

Therefore, let us consider supposedly smooth $\varphi(x) = 1/W(x)$ and put $x_k = 0$. Then one immediately finds

$$W(x_i - x_j) = \frac{c - W(x_i)W(x_j)}{W(x_i) - W(x_j)} \quad \Rightarrow \quad \varphi(x_i + x_j) = \frac{\varphi(x_i) + \varphi(x_j)}{1 + c\varphi(x_i)\varphi(x_j)}.$$

Substituting this back to the main equation, one finds that it is satisfied identically. Moreover, one can recognize in the property of φ the laws that $\tan(x)$/$\tanh(x)$ satisfy.

Solving equation on W

There are no more solutions, as one can derive the differential equation φ should satisfy. Taking $x_i = x$ and $x_j = \epsilon$ as an infinitesimal parameter, one can write

$$\varphi(x+\epsilon) = \varphi(x) + \epsilon\,\varphi'(x) + O(\epsilon^2).$$

At the same time, the equation on φ implies

$$\varphi(x+\epsilon) = \frac{\varphi(x)+\varphi(\epsilon)}{1+c\,\varphi(x)\varphi(\epsilon)}.$$

Treating ϵ as an infinitesimal parameter, one notes that $\varphi(\epsilon) = \varphi(0) + \epsilon\varphi'(0) + O(\epsilon^2) = a\epsilon + O(\epsilon^2)$. Here, $\varphi(0) = 0$, as $\varphi(x)$ is odd, and $a = \varphi'(0)$ is some constant.

$$\varphi(x+\epsilon) - \varphi(x) = \frac{\varphi(x)+\varphi(\epsilon)-\varphi(x)-c\varphi(x)^2\varphi(\epsilon)}{1+c\,\varphi(x)\varphi(\epsilon)} = a\epsilon(1-c\,\varphi^2(x)) + O(\epsilon^2).$$

Therefore, $\varphi(x)$ satisfies differential equation with easily obtained odd solutions

$$\varphi'(x) = a(1-c\,\varphi^2(x)) \;\Rightarrow\; \begin{cases} c=0 \;\Rightarrow \varphi(x) = a\,x \\ c=-1 \;\Rightarrow \varphi(x) = \tan(a\,x) \\ c=1 \;\Rightarrow \varphi(x) = \tanh(a\,x) \end{cases}$$

How different are new supercharges?

The cubic condition

$$E_{[ik](jl)} = W_{ij}W_{ik}W_{il} - W_{ij}W_{il}W_{jk} + W_{ik}W_{il}W_{jk} - W_{il}W_{jk}W_{jl} - W_{ij}W_{jk}W_{kl} + \\ + W_{ij}W_{il}W_{kl} + W_{ij}W_{jk}W_{kl} - W_{ik}W_{jk}W_{kl} + W_{il}W_{jk}W_{kl} - W_{ij}W_{jl}W_{kl} = 0$$

that we already solved ensures that $\{Q, Q\} = 0$ is satisfied in the 6th approximation in the fermions, but in the case of 5 and more particles this is not enough. At the same time, we learned that domain of acceptable W's is rather restricted, and thus should be taken into account while proving that $\{Q, Q\} = 0$. We, therefore, adopt a different approach and, instead of directly proving that $\{Q, Q\} = 0$ for any number of particles, try to relate our supercharges

$$Q = \sum_{i,j} p_i \left(\frac{1}{1-\Pi}\right)_{ij} \xi_j, \quad \Pi_{ij} = \frac{\mathrm{i}}{2}(1-\delta_{ij})W(x_i - x_j)\big(\xi_i\bar{\xi}_j - \xi_j\bar{\xi}_i\big)$$

to ones found in [Phys.Lett.B 807 (2020) 135545]

$$\mathcal{Q} = \sum_i p_i \xi_i \exp\big(-\mathrm{i}/2 \sum_{k\neq i} W(x_i - x_k)\xi_k\bar{\xi}_k\big).$$

Let us show that in the case of rational W they are identical, and in the case of trigonometric/hyperbolic W acquire simple fermionic modification.

Property of matrix Π_{ij}

Connection between different supercharges can be established if one notes that the matrix $\Pi_{ij} = \mathrm{i}/2(1-\delta_{ij})W(x_i - x_j)\big(\xi_i\bar{\xi}_j - \xi_j\bar{\xi}_i\big)$ satisfies $\xi_i \Pi_{ij}\xi_j = 0$. Moreover, it could be proven by induction that

$$\xi_i\big(\Pi^\alpha\big)_{ij}\xi_j = \frac{\mathrm{i}}{2}\xi_i\bar{\xi}_i\sum_k W(x_i - x_k)\big(\Pi^{\alpha-1}\big)_{kj}\xi_k\xi_j \;\;\Rightarrow\;\; \xi_i\big(\Pi^\alpha\big)_{ij}\xi_j = 0.$$

Therefore, the matrix power series in the supercharge

$$\sum_j \big(\Pi^\alpha\big)_{ij}\xi_j = \frac{\mathrm{i}}{2}\sum_{j,k} W(x_i - x_k)\big(\xi_i\bar{\xi}_k - \xi_k\bar{\xi}_i\big)\big(\Pi^{\alpha-1}\big)_{kj}\xi_j = -\xi_i\frac{\mathrm{i}}{2}\sum_{j,k} W(x_i - x_k)\big(\Pi^{\alpha-1}\big)_{kj}\xi_j\bar{\xi}_k$$

are proportional to ξ_i and some function of x_i and the fermions, just as in the exponential case, and one can present $\sum_j (1-\Pi)^{-1}_{ij}\xi_j = \xi_i + \lambda_i\xi_i$ for any W. The function λ_i, in turn, is determined by the relation

$$\xi_i = \sum_{j,k}(1-\Pi)_{ij}\left(\frac{1}{1-\Pi}\right)_{jk}\xi_k \;\Rightarrow\; \xi_i\big(\lambda_i + \frac{\mathrm{i}}{2}\sum_j W(x_i - x_j)\xi_j\bar{\xi}_j + \frac{\mathrm{i}}{2}\sum_j W(x_i - x_j)\xi_j\bar{\xi}_j\lambda_j\big) = 0.$$

Equation on λ_i

The relation for λ_i can be written in more clear notation with evident formal solution

$$\sum_j (\delta_{ij} - Z_{ij})\lambda_j = T_i, \quad Z_{ij} = -\frac{\mathrm{i}}{2} W(x_i - x_j)\xi_j\bar{\xi}_j, \quad T_i = \sum_j Z_{ij} \;\Rightarrow \lambda_i = \sum_{\alpha=0}^{\infty} \sum_j \left(Z^{\alpha}\right)_{ij} T_j.$$

Thus λ_i, in general, is not simply a function of $\mathrm{i}/2 \sum_k W(x_i - x_k)\xi_k\bar{\xi}_k$, but still a matrix power series. Let us recall, however, that we are interested in W's that satisfy the equation

$$W(x_i - x_j)W(x_i - x_k) + W(x_k - x_i)W(x_k - x_j) + W(x_j - x_i)W(x_j - x_k) = c$$

As a result, Z_{ij} satisfies the relation that can be used to simplify the power series in Z_{ij}:

$$\sum_j Z_{ij}Z_{jk} = \left(-\frac{\mathrm{i}}{2}\right)^2 \sum_j W(x_i - x_j)W(x_j - x_k)\xi_j\bar{\xi}_j\, \xi_k\bar{\xi}_k =$$

$$= \left(-\frac{\mathrm{i}}{2}\right)^2 \sum_j \left(W(x_i - x_j)W(x_i - x_k) - W(x_i - x_k)W(x_k - x_j) - c\right)\xi_j\bar{\xi}_j\, \xi_k\bar{\xi}_k \;\Rightarrow$$

$$\sum_j Z_{ij}Z_{jk} = (T_i - T_k)Z_{ik} + \frac{1}{4}c\, \xi_k\bar{\xi}_k\, J, \quad J = \sum_m \xi_m\bar{\xi}_m.$$

Let us use it to resum $\sum_{\alpha=0}^{\infty} \sum_j \left(Z^{\alpha}\right)_{ij} T_j$ in the cases $c = 0$, $c = \pm 1$ separately.

Power series and supercharge for $c = 0$

In the simpler case $c = 0$, $\sum_j Z_{ij} Z_{jk} = \left(T_i - T_k\right) Z_{ik}$ and one can note that a few first terms in series $\sum_{\alpha=0}^{\infty} \sum_j \left(Z^\alpha\right)_{ij} T_j$ can be presented as functions of T_i:

$$\sum_j Z_{ij} T_j = \sum_{j,k} Z_{ij} Z_{jk} = \left(T_i\right)^2 - \sum_k Z_{ik} T_k \;\Rightarrow\; \sum_j Z_{ij} T_j = \frac{1}{2}\left(T_i\right)^2,$$

$$\sum_j \left(Z^2\right)_{ij} T_j = \frac{1}{2} \sum_j Z_{ij} \left(T_j\right)^2 = T_i \sum_k Z_{ik} T_k - \sum_k Z_{ik} \left(T_k\right)^2 \;\Rightarrow$$

$$\sum_j Z_{ij} \left(T_j\right)^2 = \frac{1}{3}\left(T_i\right)^3, \; \sum_j \left(Z^2\right)_{ij} T_j = \frac{1}{6}\left(T_i\right)^3.$$

Therefore, one can assume that $\sum_j \left(Z^\alpha\right)_{ij} T_j = f(\alpha) T_i^{\alpha+1}$ and substitute this into the relation, which follows from $\sum_j Z_{ij} Z_{jk} = \left(T_i - T_k\right) Z_{ik}$:

$$\sum_j \left(Z^\alpha\right)_{ij} T_j = \sum_{j,k} \left(Z^2\right)_{ik} \left(Z^{\alpha-2}\right)_{kj} T_j = T_i \sum_j \left(Z^{\alpha-1}\right)_{ij} T_j - \sum_k Z_{ik} T_k \sum_j \left(Z^{\alpha-2}\right)_{kj} T_j$$

to find that $\sum_j Z_{ij} \left(T_j\right)^\alpha$ also should be known. It is not difficult to establish relation for it,

$$\sum_{j,k} Z_{ij} Z_{jk} \left(T_k\right)^\alpha = T_i \sum_k Z_{ik} \left(T_k\right)^\alpha - \sum_k Z_{ik} \left(T_k\right)^{\alpha+1}.$$

Power series and supercharge for $c = 0$

It is also safe to assume that $\sum_j Z_{ij}\left(T_j\right)^\alpha = g(\alpha)T_i^{\alpha+1}$. Then, substituting this into

$$\sum_{j,k} Z_{ij}Z_{jk}\left(T_k\right)^\alpha = T_i \sum_k Z_{ik}\left(T_k\right)^\alpha - \sum_k Z_{ik}\left(T_k\right)^{\alpha+1},$$

one finds self-sufficient iterative relation

$$g(\alpha)g(\alpha+1) = g(\alpha) - g(\alpha+1) \;\Rightarrow (1/g)\,(\alpha+1) - (1/g)\,(\alpha) = 1,$$
$$(1/g)\,(\alpha) = \alpha + \text{const or } g(\alpha) = 1/(\alpha+1) \text{ for } g(1) = 1/2.$$

Then the relation on $\sum_j \left(Z^\alpha\right)_{ij} T_j$ can be reduced to

$$(\alpha+1)f(\alpha) = (\alpha+1)f(\alpha-1) - f(\alpha-2) \;\Rightarrow\; f(\alpha) = \frac{1}{(1+\alpha)!}.$$

Therefore, we find the exponential solution for λ_i:

$$\lambda_i = \sum_{\alpha=0}^{\infty}\sum_j \left(Z^\alpha\right)_{ij} T_j = \sum_{\alpha=0}^{\infty} \frac{\left(T_i\right)^{\alpha+1}}{(1+\alpha)!} = e^{T_i} - 1 \text{ and, therefore,}$$

$$Q = \sum_{i,j} p_i \left(\frac{1}{1-\Pi}\right)_{ij} \xi_j = \sum_i p_i \xi_i e^{-\frac{i}{2}\sum_k W(x_i - x_k)\xi_k\bar{\xi}_k} = \mathcal{Q} \text{ for } W(x) = \frac{1}{x}.$$

Power series and supercharge for $c = \pm 1$

Rewriting supercharge in the case $c = \pm 1$ is somewhat more difficult, as the relation on $\left(Z^2\right)_{ij}$ contains, aside of Z and T, also combination of fermions $J = \sum_m \xi_m \bar{\xi}_m$:

$$\sum_j Z_{ij} Z_{jk} = \left(T_i - T_k\right) Z_{ik} + \frac{1}{4} c\, J\, \xi_k \bar{\xi}_k.$$

Keeping in mind the previous experience, we write down the equation for $\sum_j Z_{ij} \left(T_j\right)^{\alpha}$

$$\sum_{j,k} Z_{ij} Z_{jk} \left(T_k\right)^{\alpha} = T_i \sum_j Z_{ij} \left(T_j\right)^{\alpha} - \sum_j Z_{ij} \left(T_j\right)^{\alpha+1} + \frac{c}{4} J \sum_k \xi_k \bar{\xi}_k \left(T_k\right)^{\alpha} \Rightarrow$$

$$\sum_j Z_{ij} \left(T_j\right)^{\alpha} = g(\alpha) \left(T_i\right)^{\alpha+1} + h(\alpha) \frac{c}{4} J \sum_k \xi_k \bar{\xi}_k \left(T_k\right)^{\alpha-1} \Rightarrow g(\alpha) = \frac{1}{\alpha+1},\ h(\alpha) = \frac{\alpha}{\alpha+1}$$

As $\sum_k \xi_k \bar{\xi}_k \left(T_k\right)^{\alpha} = -\frac{\mathrm{i}}{2} \sum_{k,l} \xi_k \bar{\xi}_k W(x_k - x_l) \xi_l \bar{\xi}_l \left(T_k\right)^{\alpha-1} = -\sum_{k,l} \xi_l \bar{\xi}_l Z_{lk} \left(T_k\right)^{\alpha-1}$ and we can obtain:

$$\sum_k \xi_k \bar{\xi}_k \left(T_k\right)^{\alpha} = -\frac{1}{\alpha} \sum_l \xi_l \bar{\xi}_l \left(T_l\right)^{\alpha} - \frac{\alpha-1}{\alpha} \frac{cJ}{4} \sum_l \xi_l \bar{\xi}_l \left(T_l\right)^{\alpha-2} \Rightarrow$$

$$\sum_k \xi_k \bar{\xi}_k \left(T_k\right)^{\alpha} = \begin{cases} 0, & \alpha = 2\mathbb{N}+1, \\ \frac{J}{1+\alpha} \left(-\frac{cJ^2}{4}\right)^{\alpha/2}, & \alpha = 2\mathbb{N}. \end{cases}$$

Power series and supercharge for $c = \pm 1$

With these results, one can expect that $\sum_j \left(Z^\alpha\right)_{ij} T_j$ can be represented as a function of T_i and J only:

$$\sum_j \left(Z^\alpha\right)_{ij} T_j = \sum_{\beta=0}^{\alpha+1} f(\alpha,\beta) \left(T_i\right)^{\alpha+1-\beta} J^\beta .$$

Substituting this into equation for $\left(Z^\alpha\right)_{ij} T_j$, one finds equation for $f(\alpha,\beta)$

$$\sum_j \left(Z^\alpha\right)_{ij} T_j = T_i \sum_k \left(Z^{\alpha-1}\right)_{ij} T_j - \sum_k Z_{ik} T_k \sum_m \left(Z^{\alpha-2}\right)_{km} T_m + \frac{c}{4} J \sum_{k,m} \xi_k \bar{\xi}_k \left(Z^{\alpha-2}\right)_{km} T_m,$$

Indeed, one finds an equation for $f(\alpha,\beta)$ as

$$\sum_{\beta=0}^{\alpha+1} f(\alpha,\beta) \left(T_i\right)^{\alpha+1-\beta} J^\beta = \sum_{\beta=0}^{\alpha} f(\alpha-1,\beta) \left(T_i\right)^{\alpha+1-\beta} J^\beta -$$

$$- \sum_{\beta=0}^{\alpha-1} \frac{f(\alpha-2,\beta)}{\alpha-\beta+1} \left(T_i\right)^{\alpha+1-\beta} J^\beta + \frac{cJ}{4} \sum_{\beta=0}^{\alpha-1} \frac{f(\alpha-2,\beta)}{\alpha-\beta+1} \sum_k \xi_k \bar{\xi}_k \left(T_k\right)^{\alpha-1-\beta} J^\beta .$$

Counting powers of J, one can note that the last term is proportional to $J^{\alpha+1}$. This power of J can be found also only in the first term. Only first and second terms contain J^α. Thus one should consider separately terms with $J^{\alpha+1}$, J^α and J^γ, $\gamma < \alpha$.

Power series and supercharge for $c = \pm 1$

Examining the terms with J^α and J^γ, $\gamma < \alpha$, one finds

$$f(\alpha,\beta) = f(\alpha-1,\beta) - \frac{f(\alpha-2,\beta)}{\alpha-\beta+1}, \quad f(\alpha,\alpha) = f(\alpha-1,\alpha) \quad \Rightarrow \quad f(\alpha,\beta) = \frac{a(\beta)}{(\alpha-\beta+1)!}.$$

The function $a(\beta)$ could only be determined by considering terms $\sim J^{\alpha+1}$. Careful analysis shows that $a(\beta) = 0$ for odd β, while for even β they are related by an equation

$$a(\beta+2 = 2\mathbb{N}) = \sum_{\gamma=0}^{\beta/2} \frac{(-1)^{(\beta-2\gamma)/2} a(2\gamma)}{(\beta-2\gamma+2)!} \left(\frac{c}{4}\right)^{(\beta+2-2\gamma)/2}.$$

For any β, this allows to find $a(\beta)$ in terms of $a(\gamma < \beta)$, and, as $a(0) = 1$, find all of them. A few first of $a(\beta)$ read

$$a(0) = 1,\ a(2) = \frac{c}{8},\ a(4) = \frac{5c^2}{384},\ a(6) = \frac{61c^3}{46\,080},\ a(8) = \frac{277c^4}{2\,064\,384},\ \ldots$$

It was checked up to 20th order in $T \cdot J$ that

$$\lambda_i = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\alpha+1} f(\alpha,\beta) \big(T_i\big)^{\alpha-\beta+1} J^\beta = e^{T_i} \cos^{-1}\left(\frac{\sqrt{c}J}{2}\right) - 1, \text{where}$$

$$\sum \big(1-\Pi\big)^{-1}_{ij} \xi_j = \xi_i(1+\lambda_i), \quad T_i = -\tfrac{\mathrm{i}}{2} \sum W(x_i - x_k)\xi_k \bar{\xi}_k.$$

Difference in the case $c = \pm 1$

Therefore, we find, comparing with the supercharges $\mathcal{Q}$ obtained in [Phys.Lett.B 807 (2020) 135545],

$$Q_{rat} = \sum_i p_i\,\xi_i e^{-\frac{i}{2}\sum_k \xi_k \bar{\xi}_k/(x_i - x_k)} = \mathcal{Q}_{rat},$$

$$Q_{\tan} = \frac{1}{\cosh\left(\frac{J}{2}\right)} \sum_i p_i\,\xi_i e^{-\frac{i}{2}\sum_k \cot(x_i - x_k)\xi_k \bar{\xi}_k} = \frac{1}{\cosh\left(\frac{J}{2}\right)} \mathcal{Q}_{\tan},$$

$$Q_{\tanh} = \frac{1}{\cos\left(\frac{J}{2}\right)} \sum_i p_i\,\xi_i e^{-\frac{i}{2}\sum_k \coth(x_i - x_k)\xi_k \bar{\xi}_k} = \frac{1}{\cos\left(\frac{J}{2}\right)} \mathcal{Q}_{\tanh}.$$

It should be noted that appearing functions of J do not spoil supersymmetry, and this, moreover, is valid for any function $f(J)$, not only these particular functions. Indeed, one can note that $\{\xi_k \bar{\xi}_k, \xi_m \bar{\xi}_m\} = 0$. Therefore, for any function $f(J)$

$$\{\mathcal{Q}, f(J)\} = f'(J) \sum_{i,j} p_i e^{-\frac{i}{2}\sum_k W(x_i - x_k)\xi_k \bar{\xi}_k} \{\xi_i, \xi_j \bar{\xi}_j\} = \mathrm{i} f'(J)\mathcal{Q} \quad \text{and}$$

$$\{Q, Q\} = \{f(J)\mathcal{Q}, f(J)\mathcal{Q}\} = -2f(J)\mathcal{Q}\{\mathcal{Q}, f(J)\} = -2\mathrm{i} f(J) f'(J)\mathcal{Q}^2 = 0.$$

Therefore, the modified supercharges $Q = f(J)\mathcal{Q}$ still form $N = 2$, $d = 1$ Poincaré superalgebra for any $f(J)$.

Conclusion

In this talk we discussed a version of $N=2$ supersymmetric Ruijenaars-Schneider model, based on the supercharges

$$Q=\sum_{i,j} p_i \left(\frac{1}{1-\Pi}\right)_{ij} \xi_j, \quad \overline{Q}=\sum_{i,j} p_i \left(\frac{1}{1-\Pi}\right)_{ij} \bar{\xi}_j, \quad \Pi_{ij}=\frac{\mathrm{i}}{2}(1-\delta_{ij})W(x_i-x_j)\left(\xi_i\bar{\xi}_j-\xi_j\bar{\xi}_i\right).$$

We showed that if the $N=2$, $d=1$ superalgebra conditions $\{Q,Q\}=\{\overline{Q},\overline{Q}\}=0$ are satisfied, the only acceptable functions W are

$$W(x) \in \{1/x,\ 1/\tan(x),\ 1/\tanh(x)\},$$

which are among those that are needed to make the system integrable. For these particular functions $W(x)$, the supercharges coincide with those found in [Phys.Lett.B 807 (2020) 135545], modified by functions of $J=\sum_k \xi_k\bar{\xi}_k$.

It would be interesting to find the constants of motion of this system and find how the structure of obtained supercharges affects them. Another question to study is the $N=4$ supersymmetric extension of this system.