Yerevan 12/20

From Yang-Mills in de Sitter space to electromagnetic knots

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- Description of de Sitter space
- Reduction of Yang–Mills equations
- Yang–Mills configurations on de Sitter space
- Conformal equivalence to Minkowski space
- Maxwell solutions in Minkowski space
- All electromagnetic solutions
- Examples
- Gravitational backreaction (for the YM solutions)

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Description of de Sitter space

Four-dimensional de Sitter space dS_4 is a one-sheeted hyperboloid in $\mathbb{R}^{1,4}$ via

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 \;=\; \ell^2$$

metric: $$\mathrm{d}s^2 \;=\; -\mathrm{d}Z_0^2 + \mathrm{d}Z_1^2 + \mathrm{d}Z_2^2 + \mathrm{d}Z_3^2 + \mathrm{d}Z_4^2$$

Topologically, $\mathrm{dS}_4 \simeq \mathbb{R} \times S^3$. Embed unit S^3 with metric $\mathrm{d}\Omega_3^2$ into $\mathbb{R}^4$:

$$(\chi, \theta, \phi) \;\mapsto\; \omega_A(\chi, \theta, \phi) \quad \text{with} \quad A = 1, 2, 3, 4 \quad \text{and} \quad \omega_A \omega_A = 1$$

Closed-slicing global coordinates (T, χ, θ, ϕ):

$$Z_0 = \ell \sinh T \qquad \text{and} \qquad Z_A = \ell\, \omega_A \cosh T \qquad \text{with} \quad T \in \mathbb{R}$$

$$\text{metric:} \qquad \mathrm{d}s^2 \;=\; \ell^2 \left(-\mathrm{d}T^2 + \cosh^2 T\, \mathrm{d}\Omega_3^2\right)$$

Switch to conformal time coordinates via

$$\sinh T = -\cot \tau \qquad \Leftrightarrow \qquad \frac{\mathrm{d}T}{\mathrm{d}\tau} \;=\; \cosh T \;=\; \frac{1}{\sin \tau}$$

Range: $T \in \mathbb{R} \quad \Leftrightarrow \quad \tau \in \mathcal{I} = (0, \pi)$ open interval

$$\text{metric:} \qquad \mathrm{d}s^2 \;=\; \frac{\ell^2}{\sin^2 \tau} \left(-\mathrm{d}\tau^2 + \mathrm{d}\Omega_3^2\right) \;=\; \frac{\ell^2}{\sin^2 \tau}\, \mathrm{d}s^2_{\mathrm{cyl}}$$

finite Lorentzian cylinder $\mathcal{I} \times S^3$

Reduction of Yang–Mills to matrix equations

Gauge potential $\mathcal{A}$ and gauge field $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$ with values in a Lie algebra $\mathfrak{g}$

General form of the gauge potential in temporal gauge $\mathcal{A}_\tau = 0$ is

$$\mathcal{A} \;=\; \sum_{a=1}^{3} X_a(\tau,\omega)\; e^a \qquad \text{on} \quad \mathcal{I}\times S^3$$

where $X_a \in \mathfrak{g}$ and $\{e^a\}$ is a basis of left-invariant one-forms on $S^3 \simeq \mathrm{SU}(2)$, with

$$\mathrm{d}e^a + \varepsilon^a_{\ bc}\, e^b \wedge e^c \;=\; 0 \qquad \text{and} \qquad e^a e^a \;=\; \mathrm{d}\Omega_3^2$$

In terms of S^3 coordinates $(a,i,j,k = 1,2,3)$:

$$e^a \;=\; -\eta^a_{BC}\,\omega_B\,\mathrm{d}\omega_C \qquad \text{where} \quad \eta^i_{jk} = \varepsilon^i_{\ jk} \quad \text{and} \quad \eta^i_{j4} = -\eta^i_{4j} = \delta^i_j$$

Left-invariant right multiplication: $\quad R_a = -\eta^a_{BC}\,\omega_B \dfrac{\partial}{\partial\omega_C} \quad \Rightarrow \quad [R_a, R_b] = 2\,\varepsilon_{abc}\, R_c$

An arbitrary function Φ on S^3 obeys $\qquad \mathrm{d}\Phi \;=\; e^a\, R_a \Phi$

Resulting gauge field: $\dot{X}_a := \mathrm{d}X_a/\mathrm{d}\tau$

$$\mathcal{F} = \mathcal{F}_{\tau a}\, e^\tau \wedge e^a + \tfrac{1}{2}\mathcal{F}_{bc}\, e^b \wedge e^c$$

$$= \dot{X}_a\, e^\tau \wedge e^a + \tfrac{1}{2}\Big(R_{[b}X_{c]} - 2\varepsilon^a_{bc}X_a + [X_b, X_c]\Big)\, e^b \wedge e^c$$

Yang–Mills Lagrangian: $D_a X_b := R_a X_b + [X_a, X_b]$

$$\mathcal{L} = \tfrac{1}{8}\,\mathrm{tr}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} = -\tfrac{1}{4}\,\mathrm{tr}\mathcal{F}_{\tau a}\mathcal{F}_{\tau a} + \tfrac{1}{8}\,\mathrm{tr}\mathcal{F}_{ab}\mathcal{F}_{ab}$$

$$= -\tfrac{1}{2}\mathrm{tr}\Big\{\tfrac{1}{2}\dot{X}_a\dot{X}_a - 2X_aX_a + \varepsilon_{abc}X_aD_bX_c - \tfrac{1}{2}(D_aX_b)(D_aX_b)\Big\}$$

Yang–Mills equations:

$$\ddot{X}_a = -4\,X_a + 2\,\varepsilon_{abc}R_bX_c + R_bR_{[b}X_{a]} + 3\,\varepsilon_{abc}\,[X_b, X_c]$$

$$+\,2[X_b, R_bX_a] - [X_b, R_aX_b] - [X_a, R_bX_b] - \Big[X_b, [X_a, X_b]\Big]$$

and $$0 = R_a\dot{X}_a + [X_a, \dot{X}_a]$$

Too hard to solve analytically?

Impose O(4) symmetry $\Rightarrow$ $X_a(\tau,\omega) = X_a(\tau)$

YM equations become ordinary matrix differential equations:

$$\ddot{X}_a \;=\; -4\,X_a + 3\,\varepsilon_{abc}\,[X_b, X_c] - \Big[X_b, [X_a, X_b]\Big] \qquad \text{and} \qquad [\dot{X}_a, X_a] \;=\; 0$$

three coupled ordinary differential equations for three matrix functions $X_a(\tau)$

Still too complicated...?

Choose gauge group SU(2), hence $\mathfrak{g} = su(2)$ and spin-j representation

The three SU(2) generators T_a are normalized to $C(\frac{1}{2}) = \frac{1}{2}, \quad C(1) = 2$

$[T_b, T_c] = 2\,\varepsilon^a_{bc} T_a$ and $\mathrm{tr}(T_a T_b) = -4C(j)\,\delta_{ab}$ for $C(j) = \frac{1}{3}\,j(j{+}1)(2j{+}1)$

Simplest ansatz for the matrices X_a:

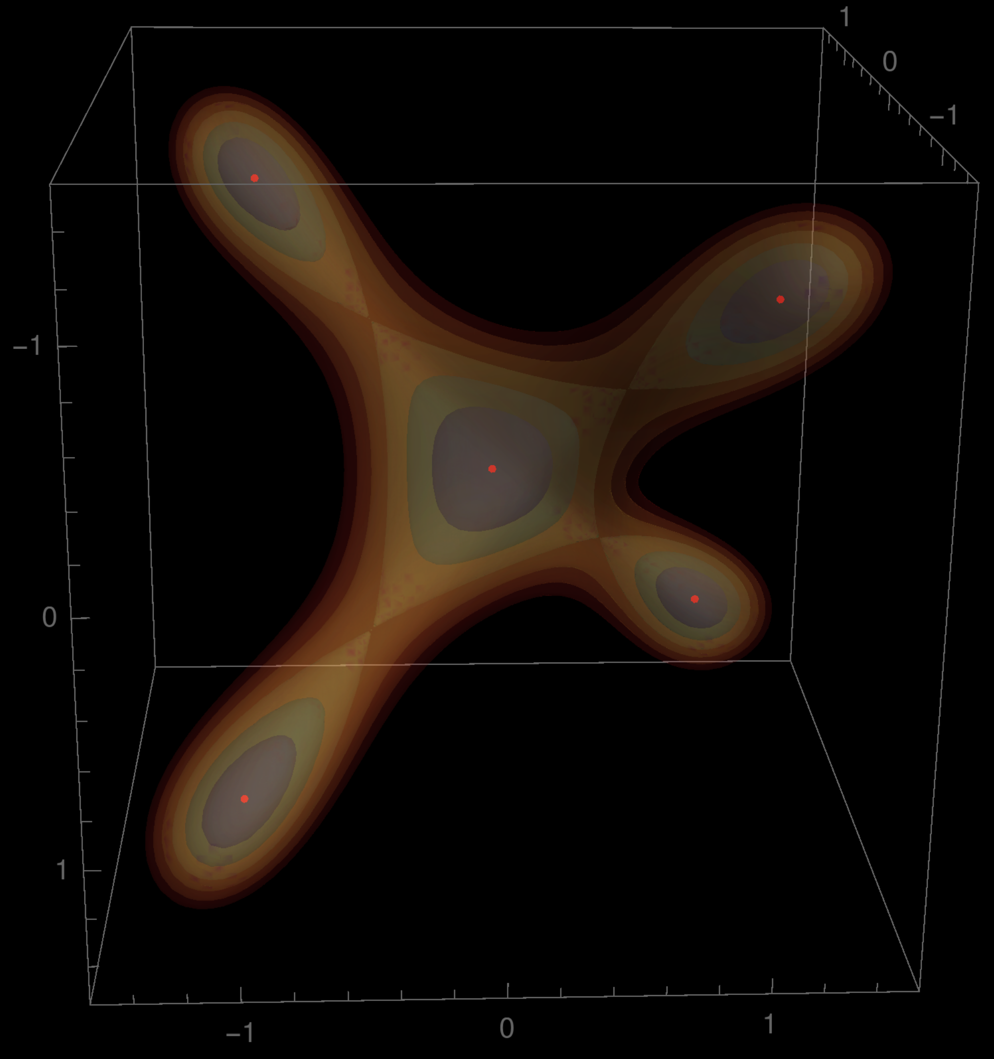
$$X_1 = \Psi_1 T_1\,, \quad X_2 = \Psi_2 T_2\,, \quad X_3 = \Psi_3 T_3 \qquad \text{with} \quad \Psi_a = \Psi_a(\tau) \in \mathbb{R}$$

Resulting simplification of Yang–Mills Lagrangian density:

$$\mathcal{L} \;=\; 4\,C(j)\left\{\tfrac{1}{4}\dot\Psi_a\dot\Psi_a - (\Psi_1 - \Psi_2\Psi_3)^2 - (\Psi_2 - \Psi_3\Psi_1)^2 - (\Psi_3 - \Psi_1\Psi_2)^2\right\}$$

Interpretation: $\{\Psi_a\} =$ particle coordinates in $\mathbb{R}^3$ $\Rightarrow$ Newtonian dynamics with

potential $\quad \frac{1}{2}\mathcal{V}(\Psi) \;=\; (\Psi_1{-}\Psi_2\Psi_3)^2 + (\Psi_2{-}\Psi_3\Psi_1)^2 + (\Psi_3{-}\Psi_1\Psi_2)^2$


1
0
−1
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1
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1

Yang–Mills configurations on de Sitter space

Only analytic nonabelian solutions:

$$\Psi_1 = \Psi_2 = \Psi_3 =: \Psi \qquad \text{with} \qquad \ddot{\Psi} = 16\,\Psi\,(\Psi-1)(2\Psi-1)$$

$$\Rightarrow \qquad \mathcal{A} \;=\; \Psi(\tau)\,g^{-1}\mathrm{d}g \qquad \text{for} \quad g : S^3 \to \mathrm{SU}(2)$$

double well: $\Psi(\tau) = 0 \text{ or } 1\,, \quad \Psi(\tau) = \frac{1}{2}\,, \quad \Psi(\tau) = \text{bounce}$

Substitute solution $\Psi(\tau)$ into $\mathcal{F}$ and get SU(2) color electric and magnetic fields:

$$\mathcal{E}_a \;=\; \mathcal{F}_{\tau a} \;=\; \dot{\Psi}\,T_a \qquad \text{and} \qquad \mathcal{B}_a \;=\; \tfrac{1}{2}\varepsilon_{abc}\mathcal{F}_{bc} \;=\; 2\,\Psi\,(\Psi-1)\,T_a$$

Their total de Sitter energy and action is finite and proportional to double-well energy V_0

Related to solutions found by de Alfaro, Fubini, Furlan (1976) and Lüscher (1977)

Only other analytic solutions are abelian, generally: $X_a(\tau) = \widetilde{X}_a(\tau)\,T_3$

Conformal equivalence to Minkowski space

The $Z_0+Z_4<0$ half of dS_4 is also conformally equivalent to future Minkowski space:

$$Z_0 = \frac{t^2-r^2-\ell^2}{2\,t}\,,\quad Z_1 = \ell\,\frac{x}{t}\,,\quad Z_2 = \ell\,\frac{y}{t}\,,\quad Z_3 = \ell\,\frac{z}{t}\,,\quad Z_4 = \frac{r^2-t^2-\ell^2}{2\,t}$$

with $\quad x,y,z\in\mathbb{R}\quad$ and $\quad r^2=x^2+y^2+z^2\quad$ but $\quad t\in\mathbb{R}_+$

since $\quad t\in[0,\infty]\quad$ corresponds to $\quad Z_0\in[-\infty,\infty]\quad$ but $\quad Z_0+Z_4<0$

metric: $$\mathrm{d}s^2 = \frac{\ell^2}{t^2}\left(-\mathrm{d}t^2+\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right)$$

Can cover whole $\mathbb{R}^{1,3}$ by gluing a second dS_4 copy and using patch $Z_0+Z_4>0$

Direct relation between cylinder (τ,χ,θ,ϕ) and Minkowski (t,r,θ,ϕ) coordinates:

$$\cot\tau = \frac{r^2-t^2+\ell^2}{2\,\ell\,t}\,,\quad \omega_1 = \gamma\,\frac{x}{\ell}\,,\quad \omega_2 = \gamma\,\frac{y}{\ell}\,,\quad \omega_3 = \gamma\,\frac{z}{\ell}\,,\quad \omega_4 = \gamma\frac{r^2-t^2-\ell^2}{2\,\ell^2}$$

$$\Longleftrightarrow \qquad \cos\tau = \gamma\,\frac{r^2-t^2+\ell^2}{2\,\ell^2}\,,\quad \cos\chi = \gamma\,\frac{r^2-t^2-\ell^2}{2\,\ell^2} \qquad \text{and} \qquad S^2_{(\tau,\chi)} = S^2_{(t,r)}$$

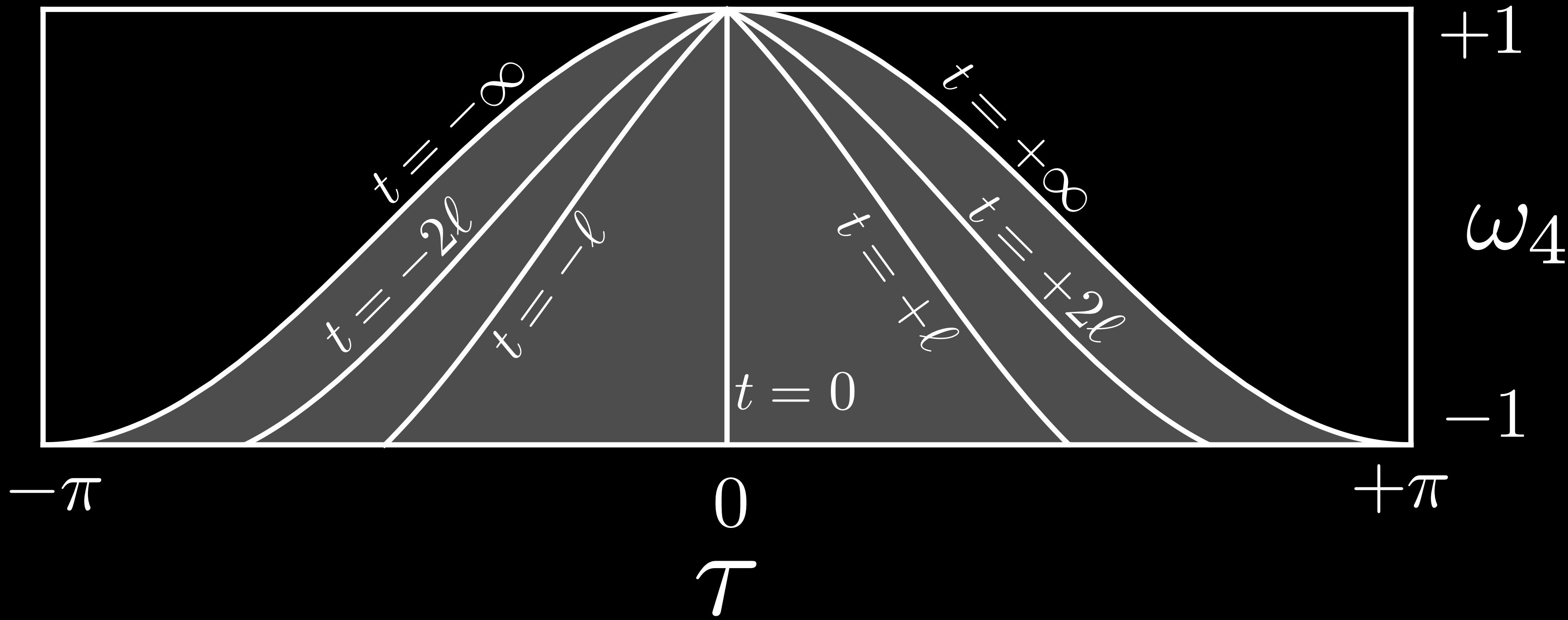
with the convenient abbreviation $\quad \gamma = \dfrac{2\,\ell^2}{\sqrt{4\,\ell^2t^2+(r^2-t^2+\ell^2)^2}}$

Inversion: $\quad \dfrac{t}{\ell} = \dfrac{\sin\tau}{\cos\tau-\cos\chi}\,,\quad \dfrac{r}{\ell} = \dfrac{\sin\chi}{\cos\tau-\cos\chi} \qquad \Rightarrow \qquad \dfrac{r}{t} = \dfrac{\sin\chi}{\sin\tau}$

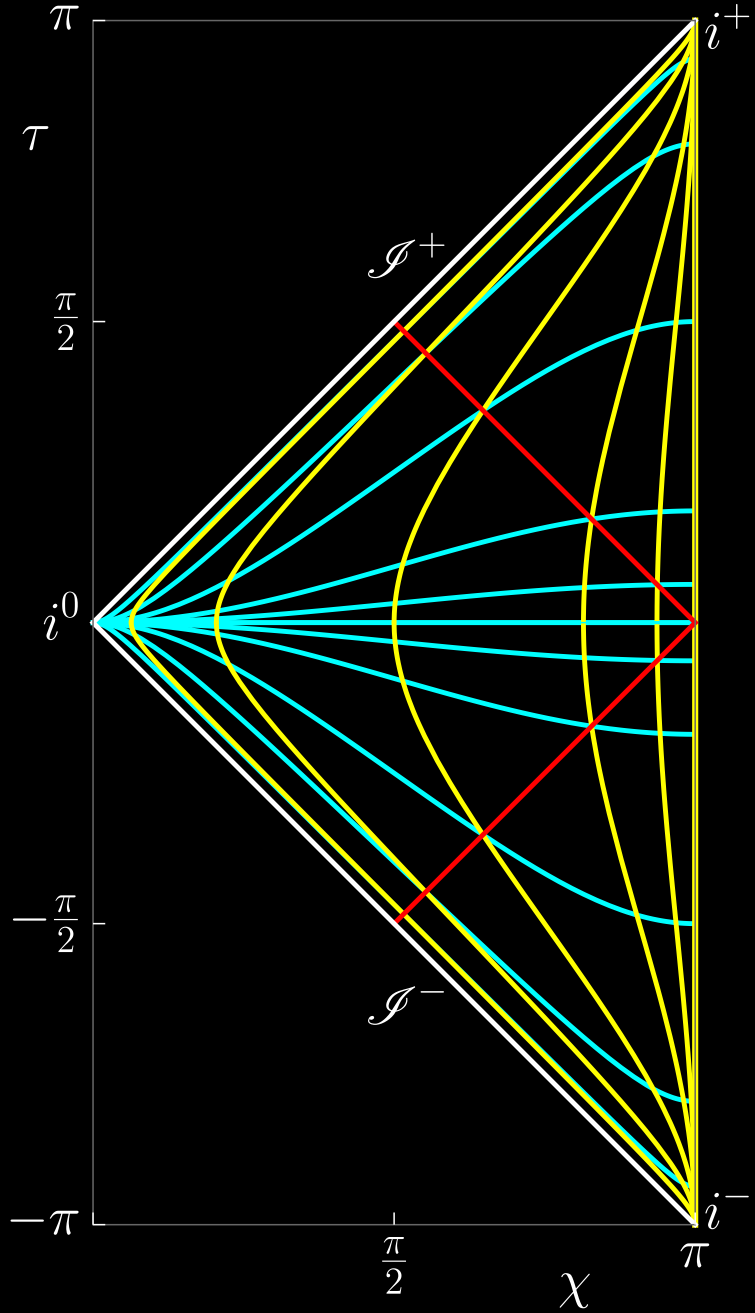
$t=-\infty,0,\infty$ corresponds to $\tau=-\pi,0,\pi$ so the cylinder is doubled to $2\mathcal{I}\times S^3$

Full Minkowski space is covered by the cylinder patch $\quad \cos\chi\equiv\omega_4\le\cos\tau$

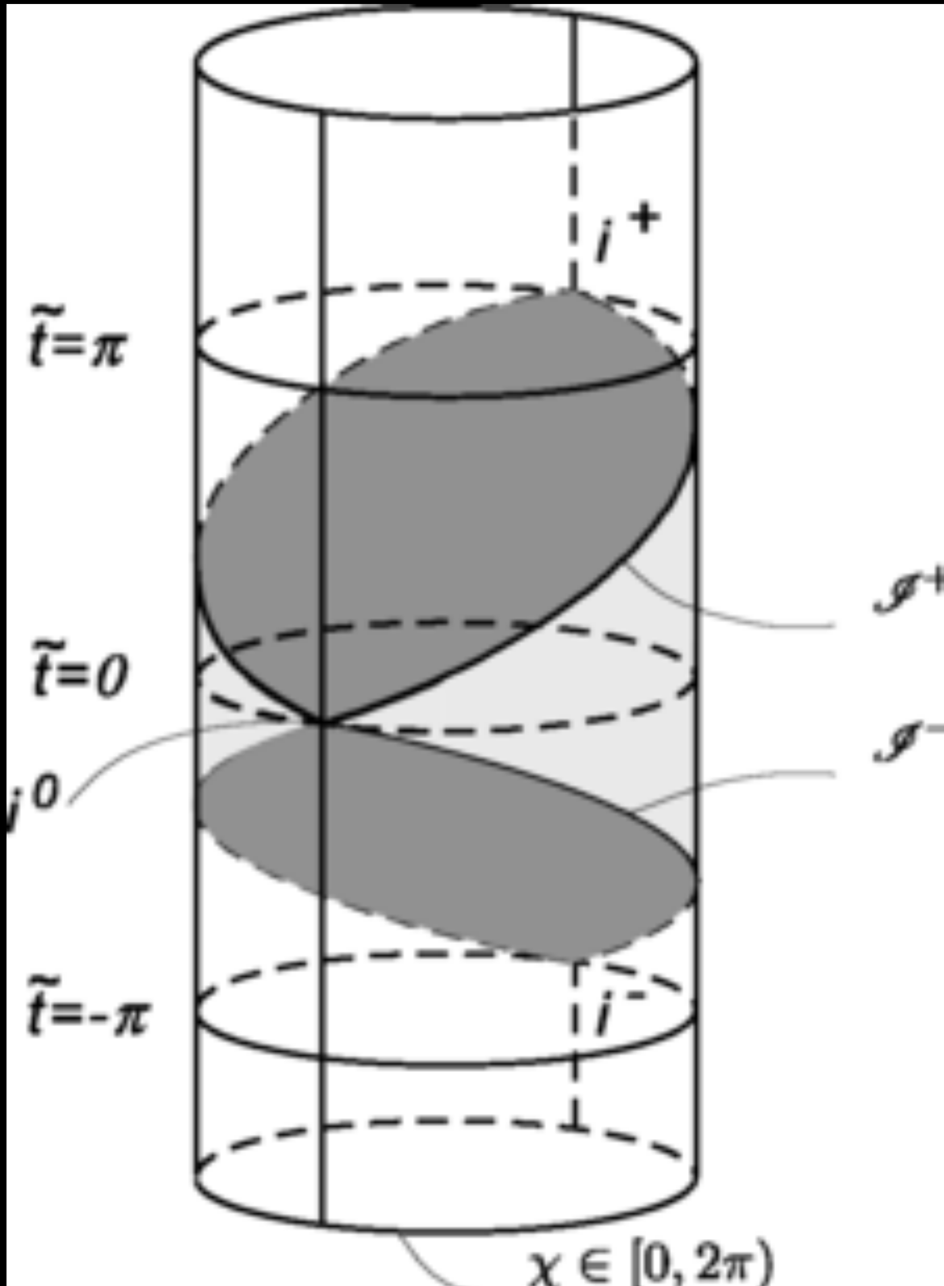
$\tau=\tau(t,r)$ $\qquad$ but more useful is $\qquad \exp(2\mathrm{i}\,\tau) = \dfrac{[(\ell+\mathrm{i}t)^2+r^2]^2}{4\,\ell^2t^2+(r^2-t^2+\ell^2)^2}$



$t = -\infty$
$t = -2\ell$
$t = -\ell$
$t = 0$
$t = +\ell$
$t = +2\ell$
$t = +\infty$
$+1$
-1
ω_4
$-\pi$
0
$+\pi$
τ



π
τ
$\mathscr{I}^+$
$\frac{\pi}{2}$
i^+
i^0
$-\frac{\pi}{2}$
$\mathscr{I}^-$
$-\pi$
i^-
$\frac{\pi}{2}$
π
χ



i^+
$\tilde{t}=\pi$
$\mathscr{I}^+$
$\tilde{t}=0$
$\mathscr{I}^-$
i^0
$\tilde{t}=-\pi$
i^-
$\chi \in [0, 2\pi)$

Maxwell solutions in Minkowski space

Yang–Mills and Maxwell are conformally invariant in four spacetime dimensions

$\Rightarrow$ may solve on cylinder $2\mathcal{I} \times S^3$ rather than directly on Minkowski space $\mathbb{R}^{1,3}$

Why? S^3 enables manifestly O(4)-covariant formalism!

Recall general form of the gauge potential in the $\mathcal{A}_\tau = 0$ gauge,

$$\mathcal{A} = \sum_{a=1}^{3} X_a(\tau, \chi, \theta, \phi)\, e^a \qquad \text{on} \quad 2\mathcal{I} \times S^3$$

where $X_a \in \mathfrak{g}$ and $\{e^a\}$ are left-invariant one-forms on S^3

Translate YM or Maxwell solutions from $2\mathcal{I} \times S^3$ to $\mathbb{R}^{1,3}$ simply by coordinate change

$$\tau = \tau(t,r) \qquad \text{and} \qquad \chi = \chi(t,r)$$

Helpful: $\exp(2\mathrm{i}\tau)$ is a rational function of t and r

Behavior at the boundary $\cos\tau = \omega_4$ yields fall-off properties at $t \to \pm\infty$

Need left-invariant one-forms in terms of Minkowski coordinates (calculation!):

$$e^\tau \;\equiv\; \mathrm{d}\tau \;=\; \frac{\gamma^2}{\ell^3}\Big(\tfrac{1}{2}(t^2+r^2+\ell^2)\,\mathrm{d}t - t\,x^k\mathrm{d}x^k\Big)$$

$$e^a \;=\; -\eta^a_{BC}\omega_B\mathrm{d}\omega_C \;=\; \frac{\gamma^2}{\ell^3}\Big(t\,x^a\mathrm{d}t - \big(\tfrac{1}{2}(t^2-r^2+\ell^2)\,\delta^a_k + x^a x^k + \ell\,\varepsilon^a{}_{jk}x^j\big)\,\mathrm{d}x^k\Big)$$

with notation $(x^i) = (x,y,z)$ and (for later) $(x^\mu) = (x^0, x^i) = (t,x,y,z)$

Specialize to Maxwell theory, i.e. $\mathfrak{g} = \mathbb{R}$ and $X_a(\tau,\omega)$ are real functions $\Rightarrow$

$$\mathcal{A} \;=\; X_a\, e^a \quad , \qquad \mathcal{F} \;=\; \dot{X}_a\, e^\tau \wedge e^a + \tfrac{1}{2}\Big(R_{[b}X_{c]} - 2\varepsilon^{a}_{bc} X_a\Big)\, e^b \wedge e^c$$

$$\mathcal{L} \;=\; \tfrac{1}{4}\dot{X}_a\dot{X}_a - X_a X_a + \tfrac{1}{2}\varepsilon_{abc} X_a R_b X_c - \tfrac{1}{4}(R_a X_b)(R_a X_b)$$

$$\ddot{X}_a \;=\; -4\, X_a + 2\, \varepsilon_{abc} R_b X_c + R_b R_{[b} X_{a]} \qquad \text{and} \qquad R_a \dot{X}_a \;=\; 0$$

Start with O(4)-symmetric case $\Rightarrow$ $X_a(\tau,\omega) = X_a(\tau)$ $\Rightarrow$ $R_a X_b = 0$

$$\mathcal{L} \;=\; \tfrac{1}{4}\dot{X}_a\dot{X}_a - X_a X_a \qquad \Rightarrow \qquad \ddot{X}_a \;=\; -4\, X_a \qquad \Rightarrow$$ harmonic motion

Oscillatory solutions $X_a(\tau) \;=\; c_a\,\mathrm{cos}\big(2(\tau{-}\tau_a)\big)$ for $\tau \in (-\pi, +\pi)$

Conversion to Minkowski solutions (with $x \equiv \{x^\mu\}$):

$$\mathcal{A} \;=\; X_a(\tau(x))\, e^a(x) \;=\; A_\mu(x)\,\mathrm{d}x^\mu \qquad \text{yields } A_\mu(x) \qquad A_t \neq 0$$

$$\mathrm{d}\mathcal{A} \;=\; \dot{X}_a\, e^\tau \wedge e^a - \varepsilon^a{}_{bc} X_a\, e^b \wedge e^c \;=\; \tfrac12 F_{\mu\nu}(x)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \qquad \text{yields } F_{\mu\nu}(x)$$

and hence electric and magnetic fields $E_i \;=\; F_{it}$ and $B_i \;=\; \tfrac12 \varepsilon_{ijk} F_{jk}$

Finite energy and zero action on de Sitter and on Minkowski space

An example (putting $\ell = 1$):

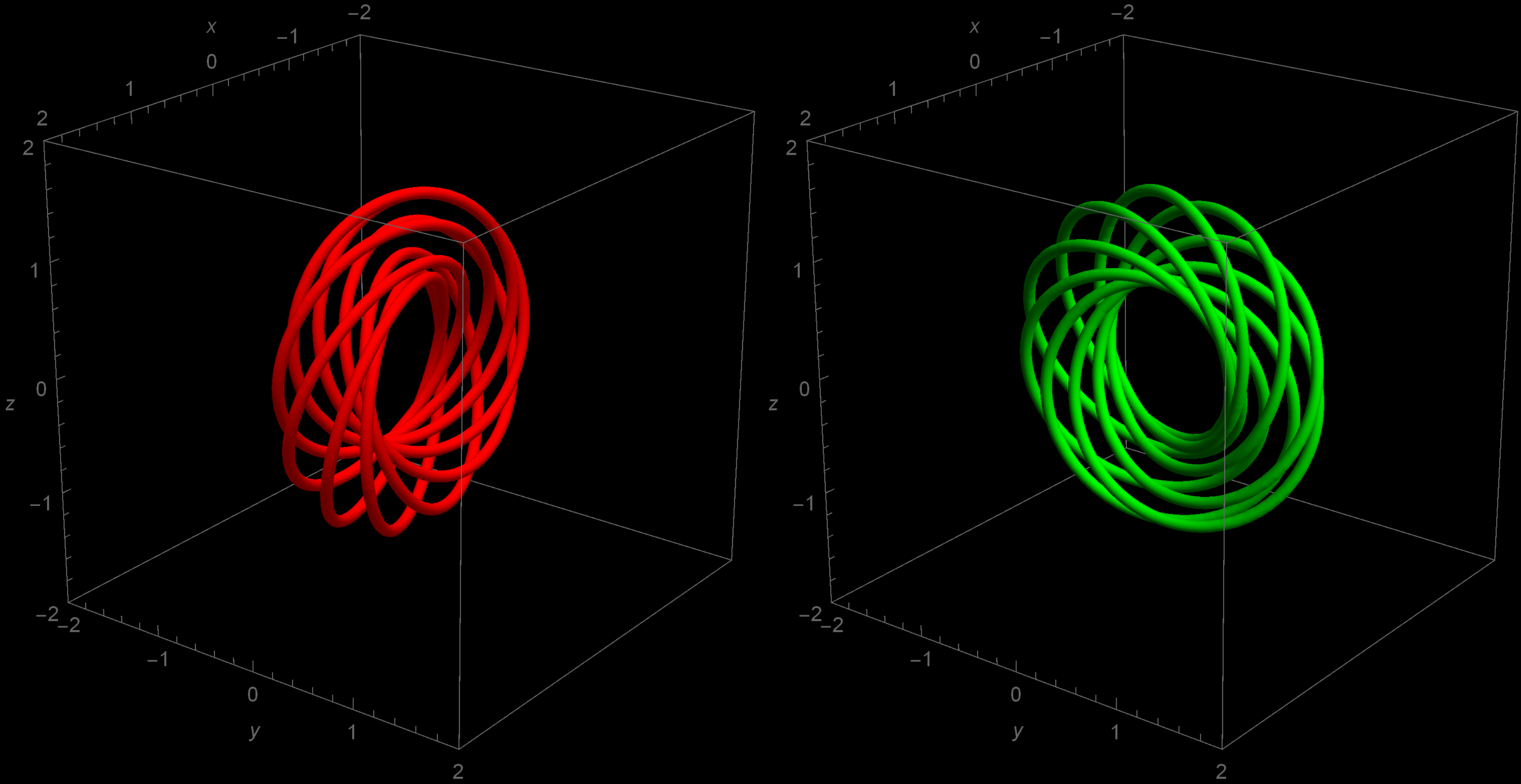
$$X_1(\tau) \;=\; -\tfrac{1}{8}\sin 2\tau\,, \qquad X_2(\tau) \;=\; -\tfrac{1}{8}\cos 2\tau\,, \qquad X_3(\tau) \;=\; 0$$

Result of short computation:

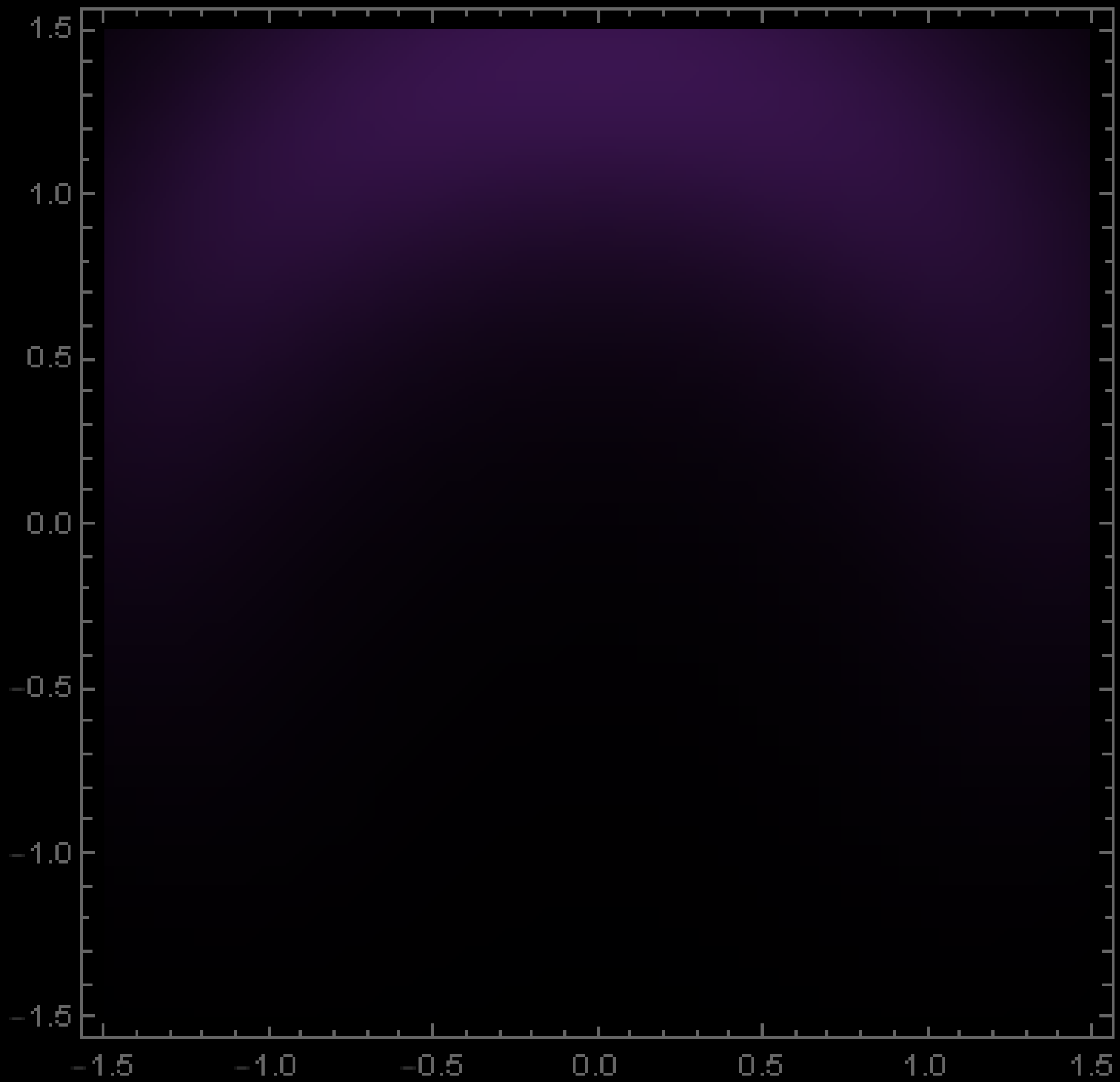
$$\vec{E} + \mathrm{i}\vec{B} \;=\; \frac{1}{\big((t-\mathrm{i})^2 - r^2\big)^3}\begin{pmatrix} (x-\mathrm{i}y)^2 - (t-\mathrm{i}-z)^2 \\ \mathrm{i}(x-\mathrm{i}y)^2 + \mathrm{i}(t-\mathrm{i}-z)^2 \\ -2\,(x-\mathrm{i}y)\,(t-\mathrm{i}-z) \end{pmatrix}$$

This is the celebrated Hopf–Rañada electromagnetic knot.

Our approach also yields its gauge potential.



Some magnetic (red) and electric (green) field lines at $t=0$



Energy density in $y=0$ plane, changing with time

All electromagnetic solutions

Admit arbitrary O(4)-non-symmetric solutions $\Rightarrow$ $X_a = X_a(\tau, \omega)$

But capture the ω-dependence in an O(4)-covariant fashion!

Choose Coulomb gauge $R_a X_a = 0$ $\Rightarrow$ coupled wave equations:

$$\ddot{X}_a \;=\; (R^2 - 4)\, X_a + 2\, \varepsilon_{abc} R_b X_c$$

Expand $X_a(\tau,\omega) \;=\; \sum_{jmn} c_{j;m,n}\, Z_a^{j;m,n}(\omega)\; \mathrm{e}^{\mathrm{i}\Omega^j \tau}$ in hyperspherical harmonics

$Y_{j;m,n}(\omega)$ with $m,n = -j, -j{+}1, \ldots, +j$ and $2j = 0, 1, 2, \ldots$

subject to $-\frac{1}{4} R^2\, Y_{j;m,n} = j(j{+}1)\, Y_{j;m,n}$ and $\frac{\mathrm{i}}{2}\, R_3\, Y_{j;m,n} = n\, Y_{j;m,n}$

Two types of basis solutions $\quad (Z_\pm = (Z_1 \pm \mathrm{i} Z_2)/\sqrt{2})$:

- type I : $\quad j \geq 0\ , \quad m = -j, \ldots, +j\ , \quad n = -j-1, \ldots, j+1\ , \quad \Omega^j = \pm 2(j+1)$

$$Z_+^{j;m,n} = \sqrt{(j-n)(j-n+1)/2}\, Y_{j;m,n+1}$$

$$Z_3^{j;m,n} = \sqrt{(j+1)^2 - n^2}\, Y_{j;m,n}$$

$$Z_-^{j;m,n} = -\sqrt{(j+n)(j+n+1)/2}\, Y_{j;m,n-1}$$

- type II : $\quad j \geq 1\ , \quad m = -j, \ldots, +j\ , \quad n = -j+1, \ldots, j-1\ , \quad \Omega^j = \pm 2j$

$$Z_+^{j;m,n} = -\sqrt{(j+n)(j+n+1)/2}\, Y_{j;m,n+1}$$

$$Z_3^{j;m,n} = \sqrt{j^2 - n^2}\, Y_{j;m,n}$$

$$Z_-^{j;m,n} = \sqrt{(j-n)(j-n+1)/2}\, Y_{j;m,n-1}$$

Each complex solution yields two real ones (real part and imaginary part)

Count for j fixed:

$2(2j{+}1)(2j{+}3)$ type-I solutions and $2(2j{+}1)(2j{-}1)$ type-II solutions ($j{>}0$)

together: $4(2j{+}1)^2$ solutions for $j{>}0$ and 6 solutions for $j{=}0$

Constant solutions ($\Omega = 0$) are not allowed; simplest are $j{=}0$ type I (Hopf–Rañada)

Spin j type I $\longleftarrow$ parity ($L \leftrightarrow R, m \leftrightarrow n$) $\longrightarrow$ spin $j{+}1$ type II

Electromagnetic duality: shifting $|\Omega^j|\tau$ by $\pm\frac{\pi}{2}$ yields a dual solution A_D

Main technical task:

transform a chosen solution on $2\mathcal{I} \times S^3$ to Minkowski coordinates (t, x, y, z),

straightforward due to explicit formulæ for all ingredients $\Rightarrow$ only rational functions

Helicity $\quad h \;=\; \frac{1}{2}\int\limits_{\mathbb{R}^3} \big(A\wedge F + A_D \wedge F_D\big)$ is conserved

Energy $\quad E \;=\; \frac{1}{2}\int\limits_{\mathbb{R}^3} \mathrm{d}^3x\, \big(\vec{E}^2 + \vec{B}^2\big)$ is conserved

Best computed in "sphere frame" at $t=\tau=0$: $\quad \mathcal{F} \;=\; -\mathcal{E}_a\, e^a \wedge e^\tau + \frac{1}{2}\mathcal{B}_a\, \varepsilon^a{}_{bc}\, e^b \wedge e^c$

j fix: $\quad \mathcal{E}_a \;=\; -\mathrm{i}\Omega^j \sum_{mn} c_{m,n}\, Z_a^{m,n}\, \mathrm{e}^{\mathrm{i}\Omega^j \tau} + \text{c.c.}$ and $\quad \mathcal{B}_a \;=\; -\Omega^j \sum_{mn} c_{m,n}\, Z_a^{m,n}\, \mathrm{e}^{\mathrm{i}\Omega^j \tau} + \text{c.c.}$

$\int\limits_{\mathbb{R}^3} \mathrm{d}^3x\; \vec{E}^2 \;=\; \frac{1}{\ell}\int\limits_{S^3} \mathrm{d}^3\Omega_3\; (1-\omega_4)\, \mathcal{E}_a\mathcal{E}_a$ and $\int\limits_{\mathbb{R}^3} \mathrm{d}^3x\; \vec{B}^2 \;=\; \frac{1}{\ell}\int\limits_{S^3} \mathrm{d}^3\Omega_3\; (1-\omega_4)\, \mathcal{B}_a\mathcal{B}_a$

exploiting orthogonality properties of the $Y_{j;m,n}$ $\quad\Rightarrow\quad E^{(j)} = |\Omega^j|\, h^{(j)}/\ell$

Null fields: $\vec{E}^2 - \vec{B}^2 = 0 = \vec{E}\cdot\vec{B} \quad \Leftrightarrow \quad (\vec{E} \pm \mathrm{i}\vec{B})^2 = 0 \quad \Leftrightarrow \quad \sum_a (\mathcal{E}_a \pm \mathrm{i}\mathcal{B}_a)^2 = 0$

fix type I and spin j $\quad \Rightarrow \quad \mathcal{E}_a + \mathrm{i}\mathcal{B}_a = -2\mathrm{i}\,\Omega \sum_{mn} c_{m,n}\, Z_a^{m,n}(\omega)\, \mathrm{e}^{\mathrm{i}\Omega\tau}$

hence $F_{\mu\nu}$ null $\quad \Leftrightarrow \quad \sum_a \Big(\sum_{mn} c_{m,n}\, Z_a^{m,n}(\omega)\Big)^2 = 0$

$\frac{1}{6}(4j+1)(4j+2)(4j+3)$ homog. quadratic eqs. for $(2j+1)(2j+3)$ param's $c_{m,n} \in \mathbb{C}$

vastly overdetermined but solvable $\quad \Rightarrow \quad$ $\dim_{\mathbb{C}}$ (solution manifold) $= 2j+2$

$$c_{m,n} = \sqrt{\tbinom{2j+2}{j+1-n}}\, w^{\frac{j+1-n}{2j+2}}\, \mathrm{e}^{2\pi \mathrm{i} k_m \frac{j+1-n}{2j+2}}\, z_m \quad \text{with} \quad w \in \mathbb{C}^*,\ z_m \in \mathbb{C},\ k_m \in \{0,\dots,2j+1\}$$

complete-intersection projective variety of $\dim_{\mathbb{C}} = 2j+1$ inside $\mathbb{C}P^{(2j+1)(2j+3)}$

spin $j{=}0$: $\quad c_{0,0}^2 = 2\, c_{0,-1} c_{0,1} \quad$ a generic rank-3 quadric in $\mathbb{C}P^2$ or $C(\mathbb{C}P^1) \subset \mathbb{C}^3$

Examples

Example 1: $(j;m,n)=(1,0,0)$, type I, combine $\mathrm{e}^{4\mathrm{i}\tau}+\mathrm{e}^{-4\mathrm{i}\tau}=2\cos 4\tau$

$$X_\pm \;=\; -\tfrac{\sqrt{3}}{\pi}(\omega_1\pm\mathrm{i}\omega_2)(\omega_3\pm\mathrm{i}\omega_4)\;\cos 4\tau\;, \qquad X_3 \;=\; -\tfrac{\sqrt{6}}{\pi}(\omega_1^2+\omega_2^2-\omega_3^2-\omega_4^2)\;\cos 4\tau$$

$\Rightarrow \quad h \;=\; 12$ and $E \;=\; 48/\ell$

Example 2: $(j;m,n)=(2;1,-1)$, type I $\qquad E=6\,h/\ell$

Example 3: $(j;m,n)=(\frac{5}{2};\frac{3}{2},\frac{1}{2})$, type I $\qquad E=7\,h/\ell$

Example 1

$$(E+\mathrm{i}B)_x = \frac{-2\mathrm{i}}{((t-\mathrm{i})^2 - x^2 - y^2 - z^2)^5} \times$$

$$\times \Big\{2y + 3\mathrm{i}ty - xz + 2t^2y + 2\mathrm{i}txz - 8x^2y - 8y^3 + 4yz^2$$

$$+\ 4\mathrm{i}t^3y - 6t^2xz - 8\mathrm{i}tx^2y - 8\mathrm{i}ty^3 + 4\mathrm{i}tyz^2 + 10x^3z + 10xy^2z - 2xz^3$$

$$+\ 2(\mathrm{i}txz + x^2y + y^3 + yz^2)(-t^2 + x^2 + y^2 + z^2) + (\mathrm{i}ty - xz)(-t^2 + x^2 + y^2 + z^2)^2\Big\}$$

$$(E+\mathrm{i}B)_y = \frac{2\mathrm{i}}{((t-\mathrm{i})^2 - x^2 - y^2 - z^2)^5} \times$$

$$\times \Big\{2x + 3\mathrm{i}tx + yz + 2t^2x - 2\mathrm{i}tyz - 8x^3 - 8xy^2 + 4xz^2$$

$$+\ 4\mathrm{i}t^3x + 6t^2yz - 8\mathrm{i}tx^3 - 8\mathrm{i}txy^2 + 4\mathrm{i}txz^2 - 10x^2yz - 10y^3z + 2yz^3$$

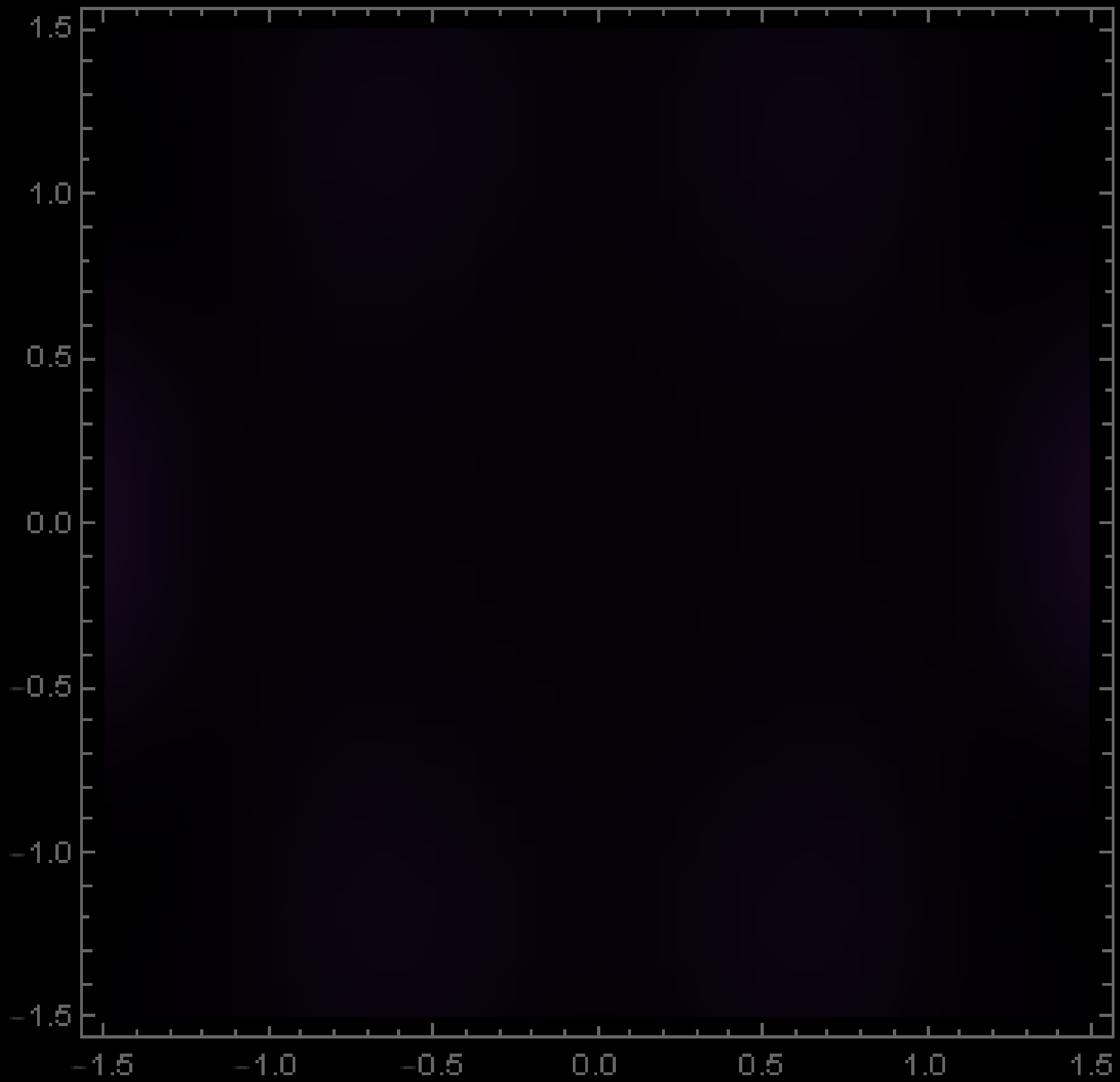
$$+\ 2(-\mathrm{i}tyz + x^3 + xy^2 + xz^2)(-t^2 + x^2 + y^2 + z^2) + (\mathrm{i}tx + yz)(-t^2 + x^2 + y^2 + z^2)^2\Big\}$$

$$(E+\mathrm{i}B)_z = \frac{\mathrm{i}}{((t-\mathrm{i})^2 - x^2 - y^2 - z^2)^5} \times$$

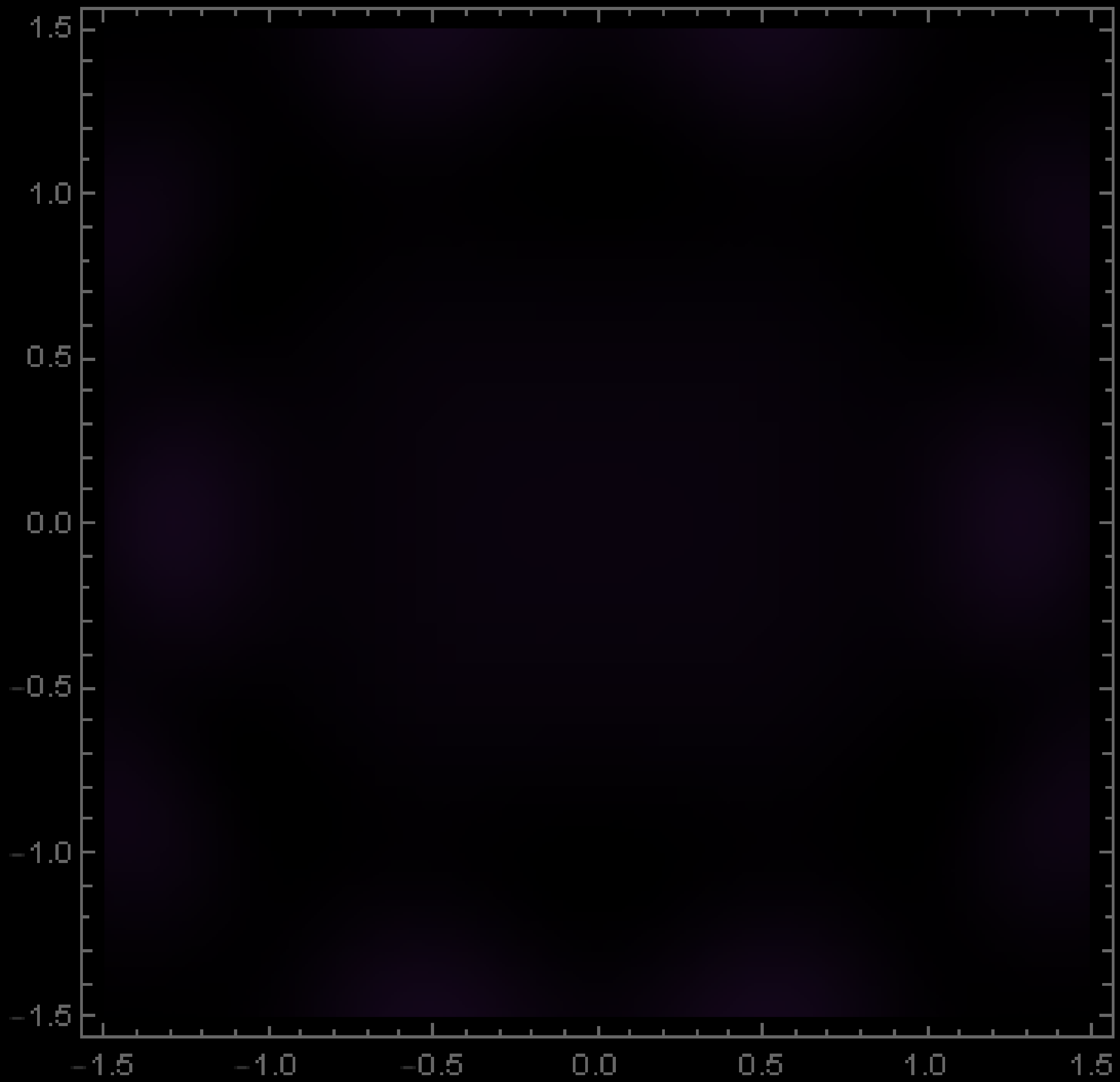
$$\times \Big\{1 + 2\mathrm{i}t + t^2 - 11x^2 - 11y^2 + 3z^2 + 4\mathrm{i}t^3 - 16\mathrm{i}tx^2 - 16\mathrm{i}ty^2 + 4\mathrm{i}tz^2$$

$$-\ t^4 - 2t^2x^2 - 2t^2y^2 - 2t^2z^2 + 11x^4 + 22x^2y^2 + 10x^2z^2 + 11y^4 - 10y^2z^2 + 3z^4$$

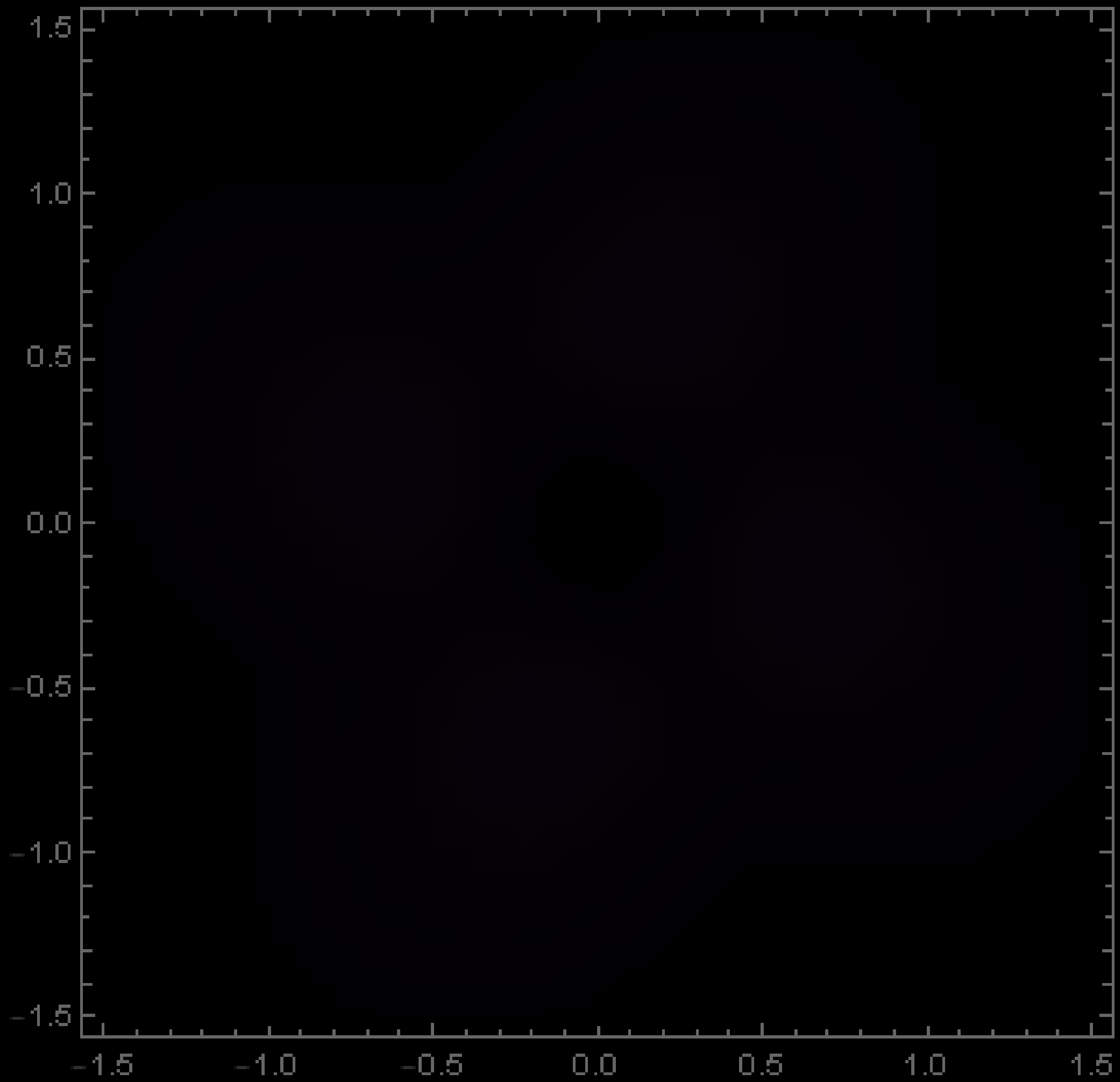
$$+\ 2\mathrm{i}t(t^2 - 3x^2 - 3y^2 - z^2)(t^2 - x^2 - y^2 - z^2) - (t^2 + x^2 + y^2 - z^2)(-t^2 + x^2 + y^2 + z^2)^2\Big\}$$



$(j; m, n) = (1; 0, 0)$: energy density in y=0 plane, changing with time



$(j; m, n) = (2; 1, -1)$: energy density in y=0 plane, changing with time



$(j;m,n) = (\frac{5}{2};\frac{3}{2},\frac{1}{2})$: energy density at t=0, scanning z = const planes

Gravitational backreaction (for the YM solutions)

In FLRW spacetime $\mathrm{d}s^2 \;=\; -\mathrm{d}T^2 + a(T)^2\,\mathrm{d}\Omega_3^2$

$\;=\; a(T(\tau))^2(-\mathrm{d}\tau^2 + \mathrm{d}\Omega_3^2)$ YM does not feel the geometry

$\Rightarrow$ our cylinder solutions remain valid for any cosmological scale factor $a(T)$

But Einstein's equations see the gauge-field energy-momentum

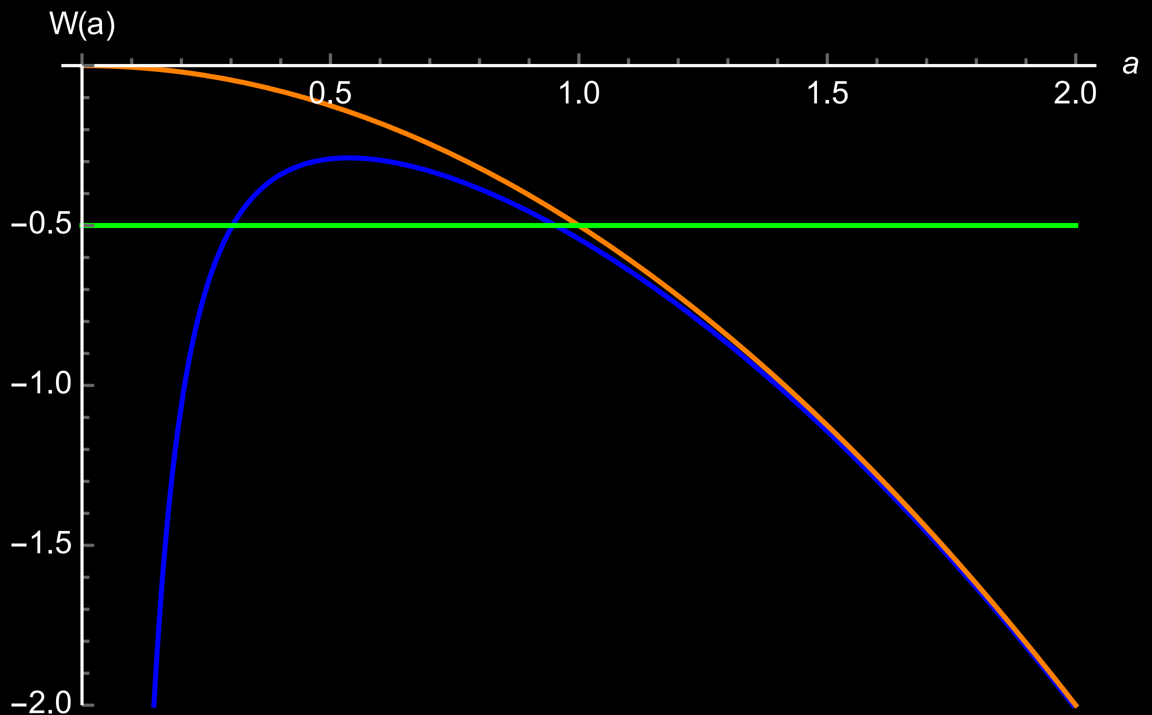
$$T_{TT} \;=\; \gamma\, a^{-4} \;=\; e \;=\; 3\,p \qquad \Rightarrow \qquad \mathrm{tr}\, T \;=\; 0$$

with the "double-well" energy density $\gamma \;=\; 3\,C(j)\,V_0/2g^2$ for YM coupling g

$$R_{\mu\nu} - \tfrac{1}{2}R g_{\mu\nu} + \Lambda\, g_{\mu\nu} \;=\; 8\pi G\, T_{\mu\nu} \qquad \Rightarrow \qquad \left\{ \begin{aligned} -R + 4\,\Lambda \;&=\; 0 \\ R_{TT} + \tfrac{1}{2}R - \Lambda \;&=\; 8\pi G\,\gamma\, a^{-4} \end{aligned} \right\}$$

$$\Rightarrow \quad \left\{ \begin{array}{c} a\,a'' + a'^2 + 1 - \frac{2}{3}\Lambda\, a^2 \;=\; 0 \\ a'^2 + 1 - \frac{1}{3}\Lambda\, a^2 \;=\; 8\pi G\,\frac{1}{3}\gamma\, a^{-2} \end{array} \right\} \quad \Rightarrow \quad \left\{ \begin{array}{c} a'' \;+\; \partial_a W(a) \;=\; 0 \\ \frac{1}{2}a'^2 \;+\; W(a) \;=\; -\frac{1}{2} \end{array} \right\}$$

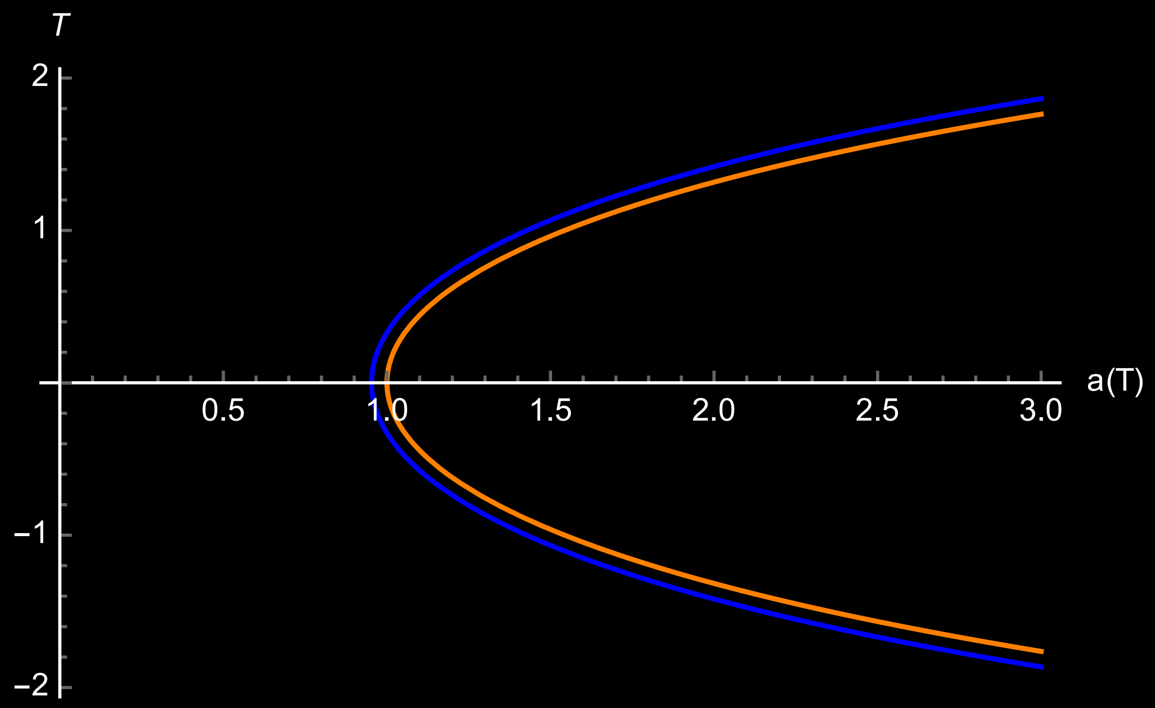
with “cosmological potential” $W(a) \;=\; -\frac{1}{6}\Lambda\, a^2 \;-\; 8\pi G\,\frac{1}{6}\gamma\, a^{-2}$



cosmological potential $W(a)$ and $W(a) = -\frac{1}{2}$ level for $8\pi G = 1$, $\Lambda = 3$ and $\gamma = 0, \frac{1}{4}$

Solution: $a(T)^2 \,=\, \frac{3}{2\Lambda} \,+\, \sqrt{\frac{9}{4\Lambda^2} - 8\pi G\,\frac{\gamma}{\Lambda}}\,\cosh\Big(2\sqrt{\frac{\Lambda}{3}}\,T\Big)$

Asymptotic de Sitter fixes $\Lambda = 3$ $\quad\Rightarrow\quad$ $a(T)^2 \,=\, \frac{1}{2} \,+\, \sqrt{\frac{1}{4} - 8\pi G\,\frac{\gamma}{3}}\,\cosh 2T$



cosmological scale factor $a(T)$ for same data as above

Regular solution requires $\quad 8\pi G\,\frac{4}{3}\gamma \,\leq\, 1 \quad\Leftrightarrow\quad 8\pi G \,\leq\, g^2/(2\,C(j)\,V_0)$

Action for reduced coupled system?

$a(T)$ analytic in de Sitter frame, $\Psi(\tau)$ analytic in conformal frame, but $T \leftrightarrow \tau$ unknown

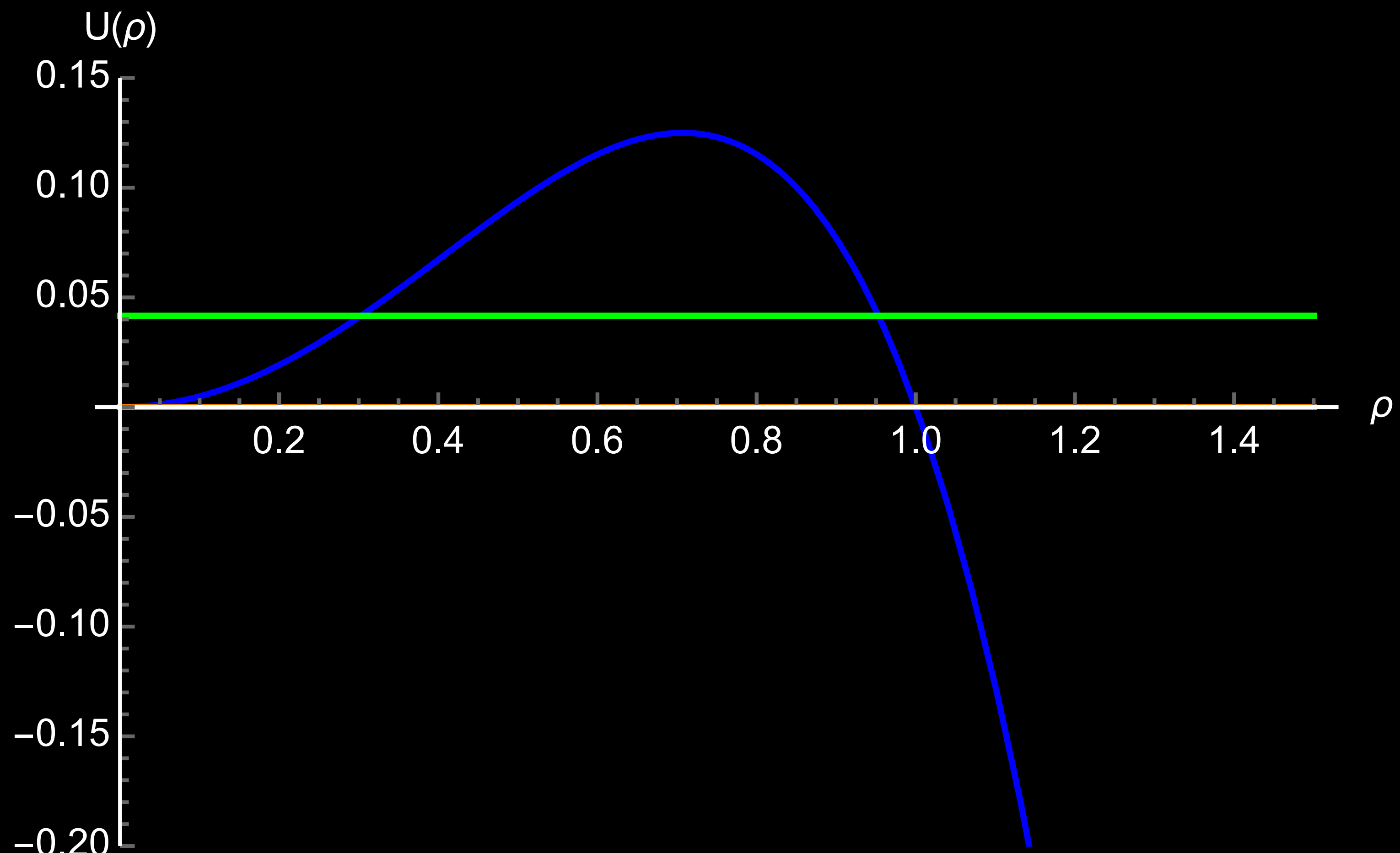
one-way decoupling only in conformal frame $\Rightarrow$ go to cylinder: $a(T(\tau)) =: \rho(\tau)$

$$\ddot{\rho} + \partial_\rho U(\rho) = 0 \quad \text{and} \quad \tfrac{1}{2}\dot{\rho}^2 + U(\rho) = 8\pi G\,\tfrac{\gamma}{6} \quad \text{for} \quad U(\rho) = \tfrac{1}{2}\rho^2 - \tfrac{\Lambda}{6}\rho^4$$

together with

$$\ddot{\Psi} + \partial_\Psi V(\Psi) = 0 \quad \text{and} \quad \tfrac{1}{2}\dot{\Psi}^2 + V(\Psi) = \frac{2g^2\gamma}{3\,C(j)} \quad \text{for} \quad V(\Psi) = 8\Psi^2(\Psi-1)^2$$

Coupling only via zero-sum of conserved energies $\frac{C(j)}{4g^2}\,\mathcal{E}_\Psi - \frac{1}{8\pi G}\,\mathcal{E}_\rho = \frac{\gamma}{6} - \frac{\gamma}{6} = 0$



cosmological potential $U(\rho)$ and $U(\rho) = \frac{\gamma}{6}$ levels for same data as above

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RIEMANN CENTER for Geometry and Physics

THANK YOU !

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Quantum Structure of Spacetime

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