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Local quartic interaction of scalars with higher spin gauge fields and commutator of linear gauge transformations

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Plan:

1. *Motivation*
2. Spin 2 test
3. Spin 4 case
3. *Commutator*
4. Outlook

Paper in preparation
arXive 2012.????

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MOTIVATION: CUBIC!!, ⟶ THEN QUARTIC ??!!??

Cubic : Full Success:

K. H. Bengtsson, I. Bengtsson and L. Brink, Nucl. Phys. B 227 (1983) 31
F. A. Berends, G. J. H. Burgers and H. Van Dam, “Z. Phys. C 24 (1984) 247
E. S. Fradkin and M. A. Vasiliev, Nucl. Phys. B 291 (1987) 141
R. R. Metsaev, Nucl. Phys. B 759 (2006)
R. M., K. Mkrtchyan and W. Ruehl, Nucl. Phys. B 836 (2010) 204; B 844 (2011) 348
K. Mkrtchyan, Phys. Rev. Lett. 120 (2018) no.22

Quartic: Non-Local ????:

P. Dempster and M. Tsulaia, Nucl. Phys. B 865 (2012) 353;
K. H. Bengtsson, JHEP 1612 (2016) 134;
M. Taronna, JHEP 1705 (2017) 026;
R. Roiban and A. A. Tseytlin, JHEP 1704 (2017) 139;

This work is based on

R. M. and W. R�uhl, Phys. Lett. B 593 (2004) 253 [arXiv:hep-th/0403241];
R. M. and K. Mkrtchyan, Mod. Phys. Lett. A 25(2010) 1333,arXiv:0903.0058;
R. M., K. Mkrtchyan, W. Ruhl and M. Tovmasyan, Phys. Lett. B 699 (2011), 187-191;
B. de Wit and D. Z. Freedman, Phys. Rev. D 21 (1980)

MOTIVATION: CUBIC!!, ⟶ THEN QUARTIC ??!!?? SPECIAL CASE

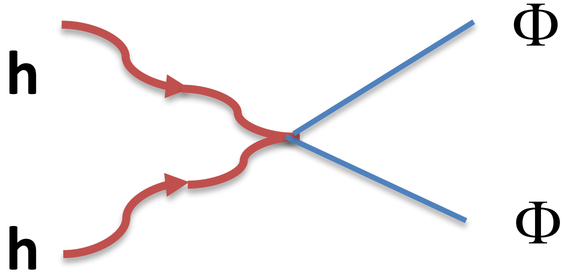
we construct quartic interaction of two scalar and two spin four fields using standard Noether's procedure

Motivations :??:

1) How far we can go with Noether's method after construction of cubic terms?
2) Is it possible to find some local Quartic interactions?

Here we explore this two possibility

Investigating the simplest case



SPIN TWO CASE: HOW IT WAS WITH CUBIC ?

Let us construct:

$$S^{\Phi\Phi h^{(2)}} = S_0(\Phi) + S_1(\Phi, h^{(2)})$$

Starting:

$$S_0(\Phi) = \frac{1}{2}\int d^d x \partial_\mu \Phi \partial^\mu \Phi,$$

Obtaining:

$$S_1(\Phi, h^{(2)}) = \frac{1}{2}\int d^d x h^{(2)\mu\nu}[-\partial_\mu \Phi \partial_\nu \Phi + \frac{\eta_{\mu\nu}}{2}\partial_\lambda \Phi \partial^\lambda \Phi]$$

From:

$$\delta_1 S_0(\Phi) + \delta_0 S_1(\Phi, h^{(2)}) = 0$$

Using:

$$\delta_1 \Phi = \varepsilon^\lambda \partial_\lambda \Phi \qquad \delta_0 h_{\mu\nu} = \partial_{(\mu}\varepsilon_{\nu)} = \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$$

$$\delta_0 \Phi = 0$$

Important point

Spin two case: Now quartic -- next order?

Equation: $$\delta_2 S_0(\Phi) + \delta_1 S_1(\Phi, h^{(2)}) + \delta_0 S_2(\Phi, h^{(2)}) = 0$$

Admitting that again $$\delta_2 \Phi = 0$$

Equation to solve

$$\delta_1 S_1(\Phi, h^{(2)}) + \delta_0 S_2(\Phi, h^{(2)}) = 0$$

assumption about the form of first order transformation of spin 2 gauge field:

$$\delta_1 h_{\mu\nu} = \varepsilon^{\lambda} \partial_{\lambda} h_{\mu\nu} + \bar{\delta}_1 h_{\mu\nu}$$

Variation

$$\delta_1 S_1(\Phi, h^{(2)}) = \int d^d x \{ -\frac{1}{2}\partial^\mu \Phi \partial^\nu \Phi [\bar{\delta}_1 h_{\mu\nu} - \partial_{(\mu}\varepsilon^\lambda h_{\nu)\lambda} + 2 h_\mu^\lambda \partial_{(\nu}\varepsilon_{\lambda)}]$$

$$+\frac{1}{2}\partial^\mu \Phi \partial^\nu \Phi [\partial_\lambda \varepsilon^\lambda h_{\mu\nu} + \frac{1}{2}\partial_{(\mu}\varepsilon_{\nu)} h_\alpha^\alpha] + \frac{1}{4}\partial^\lambda \Phi \partial_\lambda \Phi [\bar{\delta}_1 h_\alpha^\alpha - \partial_\beta \varepsilon^\beta h_\alpha^\alpha]\}$$

Non intagrable part leads to the definition of transformation

$$\bar{\delta}_1 h_{\mu\nu} = \partial_{(\mu}\varepsilon^\lambda h_{\nu)\lambda}$$

$$\bar{\delta}_1 h_\alpha^\alpha = 2\partial^\lambda \varepsilon^\alpha h_{\lambda\alpha}$$

$$\frac{1}{2}\delta_0 [h^{\lambda\alpha} h_{\lambda\alpha}]$$

We can Integrate using

$$\delta_0 h_{\mu\nu} = \partial_{(\mu}\varepsilon_{\nu)}$$

and

$$\bar{\delta}_0 h_\alpha^\alpha = 2\partial_\lambda \varepsilon^\lambda,$$

Cross-check of consistency

$$\delta_1 h_{\mu\nu} = \varepsilon^\lambda \partial_\lambda h_{\mu\nu} + \partial_\mu \varepsilon^\lambda h_{\nu\lambda} + \partial_\nu \varepsilon^\lambda h_{\mu\lambda} = \mathrm{L}_\varepsilon h_{\mu\nu}$$

SPIN TWO CASE: NOW QUARTIC -- NEXT ORDER?

SOLUTION!

$$S_2(\Phi, h^{(2)}) = \int d^d x \{ \frac{1}{2} \partial^\mu \Phi \partial^\nu \Phi h_\mu^\lambda h_{\nu\lambda} - \frac{1}{4} \partial^\mu \Phi \partial^\nu \Phi h_{\mu\nu} h_\alpha^\alpha$$

$$- \frac{1}{8} \partial^\lambda \Phi \partial_\lambda \Phi h^{\alpha\beta} h_{\alpha\beta} + \frac{1}{16} \partial^\lambda \Phi \partial_\lambda \Phi h_\alpha^\alpha h_\beta^\beta \}$$

SPIN FOUR CASE: HOW IT WAS WITH CUBIC ?

Starting point

$$S^{\Phi\Phi h^{(4)}}(\Phi, h^{(2)}, h^{(4)}) = S_0(\Phi) + S_1(\Phi, h^{(2)}) + S_1(\Phi, h^{(4)}),$$

$$S_1(\Phi, h^{(4)}) = \frac{1}{4}\int d^d x h^{\mu\nu\alpha\beta}[\partial_\mu\partial_\nu\Phi\partial_\alpha\partial_\beta\Phi - \eta_{\mu\nu}\partial_\alpha\partial^\gamma\Phi\partial_\beta\partial_\gamma\Phi]$$

Transformations

$$\delta_1\Phi(x) = \varepsilon^{\mu\nu\lambda}(x)\partial_\mu\partial_\nu\partial_\lambda\Phi(x),$$

$$\delta_0 h^{\mu\nu\lambda\rho} = \partial^{(\mu}\varepsilon^{\nu\lambda\rho)} = \partial^\mu\varepsilon^{\nu\lambda\rho} + \partial^\nu\varepsilon^{\mu\lambda\rho} + \partial^\lambda\varepsilon^{\mu\nu\rho} + \partial^\rho\varepsilon^{\mu\nu\lambda},$$

$$\delta_0 h^{\mu\nu} = \partial^{(\mu}\varepsilon^{\nu)}_{(2)}, \qquad \varepsilon^{\nu}_{(2)} = \partial_\alpha\partial_\beta\varepsilon^{\nu\alpha\beta}.$$

Noether's Equation

$$\delta_1 S_0(\Phi) + \delta_0[S_1(\Phi, h^{(2)}) + S_1(\Phi, h^{(4)})] = 0$$

SPIN FOUR CASE: HOW IT WAS WITH CUBIC ?

Important point: special field redefinition

$$\Phi \rightarrow \Phi + \frac{1}{2}\partial_\mu \left(h_\alpha^{\alpha\mu\nu} \partial_\nu \Phi \right)$$

Generalization:

- ***general spin S case***
- ***generalized Weyl invariance***

R. M. and W. Rühl (2004), R.M., K. Mkrtchyan (2009)

SPIN FOUR CASE: VARIATION OF CUBIC TERM

physical traceless and transfer gauge for our spin four field

$$\left.\begin{aligned} \partial_\mu h^{\mu\nu\lambda\rho} &= 0 \\ h^{\mu\lambda\rho}_{\mu} &= 0 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \partial_\mu \partial^\mu \varepsilon^{\alpha\beta\gamma} = \Box \varepsilon^{\alpha\beta\gamma} &= 0 \\ \partial_\alpha \varepsilon^{\alpha\beta\gamma} &= 0 \end{aligned}\right.$$

Simplified task: Starting from single cubic term

$$L_1 \sim h^{\mu\nu\lambda\rho} \partial_\mu \partial_\nu \Phi \partial_\lambda \partial_\rho \Phi$$

Prediction : **We will loose all divergences and traces and possible terms coming from variation of initial trace terms**

How it is in spin 2 case

$$\delta_1 S_1(\Phi, h^{(2)}) = \int d^d x\{-\frac{1}{2}\partial^\mu\Phi\partial^\nu\Phi[\bar{\delta}_1 h_{\mu\nu} - \partial_{(\mu}\varepsilon^\lambda h_{\nu)\lambda} + 2h^\lambda_\mu \partial_{(\nu}\varepsilon_{\lambda)}]$$

$$+\frac{1}{2}\partial^\mu\Phi\partial^\nu\Phi[\partial_\lambda\varepsilon^\lambda h_{\mu\nu} + \frac{1}{2}\partial_{(\mu}\varepsilon_{\nu)} h^\alpha_\alpha] + \frac{1}{4}\partial^\lambda\Phi\partial_\lambda\Phi[\bar{\delta}_1 h^\alpha_\alpha - \partial_\beta\varepsilon^\beta h^\alpha_\alpha]\}$$

Variation of the trace is not expressed through trace terms, but ….

Integrable: $$\bar{\delta}_1 h^\alpha_\alpha = 2\partial^\lambda\varepsilon^\alpha h_{\lambda\alpha} = \frac{1}{2}\delta_0[h^{\lambda\alpha}h_{\lambda\alpha}]$$

and is 'contracted' with trace of current:

$$S_2(\Phi, h^{(2)}) = \int d^d x\{\frac{1}{2}\partial^\mu\Phi\partial^\nu\Phi h^\lambda_\mu h_{\nu\lambda} - \frac{1}{4}\partial^\mu\Phi\partial^\nu\Phi h_{\mu\nu}h^\alpha_\alpha$$

$$-\frac{1}{8}\partial^\lambda\Phi\partial_\lambda\Phi h^{\alpha\beta}h_{\alpha\beta} + \frac{1}{16}\partial^\lambda\Phi\partial_\lambda\Phi h^\alpha_\alpha h^\beta_\beta\}$$

SPIN FOUR CASE: VARIATION OF CUBIC TERM

$$\delta_1(h^{\mu\nu\lambda\rho}\partial_\mu\partial_\nu\Phi\partial_\lambda\partial_\rho\Phi) = \frac{1}{3}\delta_1(h^{\mu\nu\lambda\rho}J^{(4)}_{\mu\nu\lambda\rho}) = \delta_1 h^{\mu\nu\lambda\rho}\partial_\mu\partial_\nu\Phi\partial_\lambda\partial_\rho\Phi$$

$$+\frac{1}{50}\left[\varepsilon^{\mu(\alpha\beta}\partial_\mu h^{\gamma\nu\lambda\rho)} - \partial_\mu\varepsilon^{(\alpha\beta\gamma}h^{\nu\lambda\rho)\mu}\right]J^{(6)}_{\nu\lambda\rho\alpha\beta\gamma}$$

$$+\frac{1}{5}\left[\partial_\alpha\varepsilon^{\mu\nu(\beta}\partial_\mu\partial_\nu h^{\gamma\lambda\rho)\alpha} - \partial_\mu\partial_\nu\varepsilon^{\alpha(\beta\gamma}\partial_\alpha h^{\lambda\rho)\mu\nu}\right]J^{(4)}_{\lambda\rho\beta\gamma}$$

$$+\frac{2}{15}\left[\partial_\alpha\partial_\beta\partial_\gamma\varepsilon^{(\mu\nu\lambda}h^{\rho)\alpha\beta\gamma} - \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho}\right]J^{(4)}_{\mu\nu\lambda\rho}$$

$$+\frac{1}{5}[\partial_\mu\partial_\nu\varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho} - \partial_\mu\partial_\nu\partial_\gamma\varepsilon^{\alpha\beta(\lambda}\partial_\alpha\partial_\beta h^{\rho)\mu\nu\gamma}]J^{(2)}_{\lambda\rho}$$

$$+\frac{1}{5}\partial_\mu\partial_\nu\partial_\lambda\partial_\rho\varepsilon^{\alpha\beta\gamma}\partial_\alpha h^{\mu\nu\lambda\rho}J^{(2)}_{\beta\gamma}$$

SPIN FOUR CASE: VARIATION OF CUBIC TERM

$$J^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} = \partial_{(\nu}\partial_{\lambda}\partial_{\rho}\Phi\partial_{\alpha}\partial_{\beta}\partial_{\gamma)}\Phi = \partial_{\nu}\partial_{(\lambda}\partial_{\rho}\Phi\partial_{\alpha}\partial_{\beta}\partial_{\gamma)}\Phi,$$

$$J^{(4)}_{\mu\nu\lambda\rho} = \partial_{(\mu}\partial_{\nu}\Phi\partial_{\lambda}\partial_{\rho)}\Phi = \partial_{\mu}\partial_{(\nu}\Phi\partial_{\lambda}\partial_{\rho)}\Phi,$$

$$J^{(2)}_{\mu\nu} = \partial_{\mu}\Phi\partial_{\nu}\Phi,$$

1) we cannot integrate Noether' s equation without introduction of the cubic interaction for gauge field of spin 6 **coupled to the spin 6 current** :

$$h^{\nu\lambda\rho\alpha\beta\gamma} J^{(6)}_{\nu\lambda\rho\alpha\beta\gamma}$$

2) $J^{(4)}_{\mu\nu\lambda\rho}$ ***terms arose with different weight 1/5 and 2/15. But we will see below that they should come with same weight to complete integration for interaction terms.***

3) we have three unwanted $J^{(2)}_{\mu\nu}$ ***terms . We should discover way to get rid of them***

SPIN FOUR CASE: TUNING OF CUBIC INTERACTION

We can modify our initial interaction with higher spin currents adding gradients of lower spin currents with some coefficients:

$$J^{(6)}_{\alpha\beta\mu\nu\lambda\rho} \Rightarrow J^{(6)}_{\alpha\beta\mu\nu\lambda\rho} + A\partial_{(\alpha}\partial_{\beta}J^{(4)}_{\mu\nu\lambda\rho)} + B\partial_{(\alpha}\partial_{\beta}\partial_{\mu}\partial_{\nu}J^{(2)}_{\lambda\rho)}$$

$$J^{(4)}_{\mu\nu\lambda\rho} \Rightarrow J^{(4)}_{\mu\nu\lambda\rho} + C\partial_{(\mu}\partial_{\nu}J^{(2)}_{\lambda\rho)}$$

And it works!

We can prove for our constrained field and parameter several relations

First important relation

$$\frac{1}{15}\left[\partial_{\mu}\varepsilon^{(\alpha\beta\gamma}h^{\nu\lambda\rho)\mu} - \varepsilon^{\mu(\alpha\beta}\partial_{\mu}h^{\gamma\nu\lambda\rho)}\right]\partial_{(\nu}\partial_{\lambda}J^{(4)}_{\rho\alpha\beta\gamma)} =$$

$$-\left[\partial_{\alpha}\varepsilon^{\mu\nu(\beta}\partial_{\mu}\partial_{\nu}h^{\gamma\lambda\rho)\alpha} - \partial_{\mu}\partial_{\nu}\varepsilon^{\alpha(\beta\gamma}\partial_{\alpha}h^{\lambda\rho)\mu\nu}\right]J^{(4)}_{\lambda\rho\beta\gamma}$$

$$+\left[\partial_{\alpha}\partial_{\beta}\partial_{\gamma}\varepsilon^{(\mu\nu\lambda}h^{\rho)\alpha\beta\gamma} - \varepsilon^{\alpha\beta\gamma}\partial_{\alpha}\partial_{\beta}\partial_{\gamma}h^{\mu\nu\lambda\rho}\right]J^{(4)}_{\mu\nu\lambda\rho}$$

Second important relation

$$\frac{1}{30}\left[\partial_{\mu}\varepsilon^{(\alpha\beta\gamma}h^{\nu\lambda\rho)\mu}-\varepsilon^{\mu(\alpha\beta}\partial_{\mu}h^{\gamma\nu\lambda\rho)}\right]\partial_{(\nu}\partial_{\lambda}\partial_{\rho}\partial_{\alpha}J^{(2)}_{\beta\gamma)}=$$

$$+[\partial_{\mu}\partial_{\nu}\partial_{\gamma}\varepsilon^{\alpha\beta(\lambda}\partial_{\alpha}\partial_{\beta}h^{\rho)\mu\nu\gamma}-\partial_{\mu}\partial_{\nu}\varepsilon^{\alpha\beta\gamma}\partial_{\alpha}\partial_{\beta}\partial_{\gamma}h^{\mu\nu\lambda\rho}]J^{(2)}_{\lambda\rho}$$

$$+\frac{3}{2}\partial_{\mu}\partial_{\nu}\partial_{\lambda}\partial_{\rho}\varepsilon^{\alpha\beta\gamma}\partial_{\alpha}h^{\mu\nu\lambda\rho}J^{(2)}_{\beta\gamma}$$

Using these we can

- ***improve discrepancy in numbers***
- ***Cancel the first line with spin two current***.

Finally we obtain the following expression for variation:

$$\delta_1(h^{\mu\nu\lambda\rho}\partial_\mu\partial_\nu\Phi\partial_\lambda\partial_\rho\Phi) = \frac{1}{3}\delta_1(h^{\mu\nu\lambda\rho}J^{(4)}_{\mu\nu\lambda\rho}) = \delta_1 h^{\mu\nu\lambda\rho}\partial_\mu\partial_\nu\Phi\partial_\lambda\partial_\rho\Phi$$

$$+\frac{1}{50}\left[\varepsilon^{\mu(\alpha\beta}\partial_\mu h^{\gamma\nu\lambda\rho)} - \partial_\mu\varepsilon^{(\alpha\beta\gamma}h^{\nu\lambda\rho)\mu}\right]\widetilde{J}^{(6)}_{\nu\lambda\rho\alpha\beta\gamma}$$

$$+\frac{1}{6}\left[\partial_\alpha\varepsilon^{\mu\nu(\beta}\partial_\mu\partial_\nu h^{\gamma\lambda\rho)\alpha} - \partial_\mu\partial_\nu\varepsilon^{\alpha(\beta\gamma}\partial_\alpha h^{\lambda\rho)\mu\nu}\right]J^{(4)}_{\lambda\rho\beta\gamma}$$

$$+\frac{1}{6}\left[\partial_\alpha\partial_\beta\partial_\gamma\varepsilon^{(\mu\nu\lambda}h^{\rho)\alpha\beta\gamma} - \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho}\right]J^{(4)}_{\mu\nu\lambda\rho}$$

$$+\frac{1}{2}\partial_\mu\partial_\nu\partial_\lambda\partial_\rho\varepsilon^{\alpha\beta\gamma}\partial_\alpha h^{\mu\nu\lambda\rho}J^{(2)}_{\beta\gamma}$$

Where

$$\tilde{J}^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} = J^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} + \frac{1}{9}\partial_{(\alpha}\partial_\beta J^{(4)}_{\gamma\nu\lambda\rho)} + \frac{1}{3}\partial_{(\nu}\partial_\lambda\partial_\rho\partial_\alpha J^{(2)}_{\beta\gamma)}$$

Third important relation

$$6\partial_\mu\partial_\nu\partial_\lambda\partial_\rho\varepsilon^{\alpha\beta\gamma}\partial_\alpha h^{\mu\nu\lambda\rho}J^{(2)}_{\beta\gamma} =$$

$$\left[\partial_\alpha\varepsilon^{\mu\nu(\beta}\partial_\mu\partial_\nu h^{\gamma\lambda\rho)\alpha} - \partial_\mu\partial_\nu\varepsilon^{\alpha(\beta\gamma}\partial_\alpha h^{\lambda\rho)\mu\nu}\right]\partial_\lambda\partial_\rho J^{(2)}_{\beta\gamma}$$

$$+\left[\partial_\alpha\partial_\beta\partial_\gamma\varepsilon^{(\mu\nu\lambda}h^{\rho)\alpha\beta\gamma} - \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho}\right]\partial_\mu\partial_\nu J^{(2)}_{\lambda\rho}$$

We will not do that

$$\widetilde{J}^{(4)}_{\nu\lambda\rho\gamma} = J^{(4)}_{\nu\lambda\rho\gamma} + \frac{1}{2}\partial_{(\nu}\partial_\lambda J^{(2)}_{\rho\gamma)},$$

We prefer to keep initial spin 4 current unchanged and cancel last unnecessary term by traceless Stueckelberg like transformation of the spin two gauge field from linear coupling with spin two current

$$\delta_1 h^{\beta\gamma} \sim \partial_\mu\partial_\nu\partial_\lambda\partial_\rho\varepsilon^{\alpha\beta\gamma}\partial_\alpha h^{\mu\nu\lambda\rho}$$

SPIN FOUR CASE: INTEGRATION AND INTERACTION

Now we start to integrate expression:

$$\frac{1}{50}\left[\varepsilon^{\mu(\alpha\beta}\partial_{\mu}h^{\gamma\nu\lambda\rho)} - \partial_{\mu}\varepsilon^{(\alpha\beta\gamma}h^{\nu\lambda\rho)\mu}\right]\tilde{J}^{(6)}_{\nu\lambda\rho\alpha\beta\gamma}$$

$$+\frac{1}{6}\left[\partial_{\alpha}\varepsilon^{\mu\nu(\beta}\partial_{\mu}\partial_{\nu}h^{\gamma\lambda\rho)\alpha} - \partial_{\mu}\partial_{\nu}\varepsilon^{\alpha(\beta\gamma}\partial_{\alpha}h^{\lambda\rho)\mu\nu}\right]J^{(4)}_{\lambda\rho\beta\gamma}$$

$$\delta_0 h\partial\partial h + \partial\partial\delta_0 hh = \delta_0(h\partial\partial h)$$

$$+\frac{1}{6}\left[\partial_{\alpha}\partial_{\beta}\partial_{\gamma}\varepsilon^{(\mu\nu\lambda}h^{\rho)\alpha\beta\gamma} - \varepsilon^{\alpha\beta\gamma}\partial_{\alpha}\partial_{\beta}\partial_{\gamma}h^{\mu\nu\lambda\rho}\right]J^{(4)}_{\mu\nu\lambda\rho}$$

To extract interactions and linear on gauge field transformations we can use the following important relations

$$\partial_{\mu}\varepsilon^{\alpha\beta\gamma} = \delta_0 h^{\alpha\beta\gamma}_{\mu} - \partial^{(\alpha}\varepsilon^{\beta\gamma)}_{\mu}$$

$$\partial_{\mu}\partial_{\nu}\varepsilon^{\alpha\beta\gamma} = \frac{1}{2}\partial_{(\nu}\delta_0 h^{\alpha\beta\gamma}_{\mu)} - \frac{1}{2}\partial^{(\alpha}\delta_0 h^{\beta\gamma)}_{\mu\nu} + \partial^{(\alpha}\partial^{\beta}\varepsilon^{\gamma)}_{\mu\nu}$$

$$\partial_{\mu}\partial_{\nu}\partial_{\lambda}\varepsilon^{\alpha\beta\gamma} = \frac{1}{3}\partial_{(\nu}\partial_{\lambda}\delta_0 h^{\alpha\beta\gamma}_{\mu)} - \frac{1}{6}\partial^{(\alpha}\partial_{(\lambda}\delta_0 h^{\beta\gamma)}_{\mu\nu)} + \frac{1}{3}\partial^{(\alpha}\partial^{\beta}\delta_0 h^{\gamma)}_{\mu\nu\lambda} - \partial^{\alpha}\partial^{\beta}\partial^{\gamma}\varepsilon_{\mu\nu\lambda}$$

SPIN FOUR CASE: INTEGRATION AND INTERACTION

Spin 6 part:

$$L_2^1 = \frac{1}{10} h_{\mu}^{\alpha\beta\gamma} h^{\nu\lambda\rho\mu} \widetilde{J}^{(6)}_{\nu\lambda\rho\alpha\beta\gamma}$$

$$\delta_1 h^{\mu\nu\lambda\alpha\beta\gamma} = \varepsilon^{\rho(\alpha\beta} \partial_\rho h^{\gamma\mu\nu\lambda)} + \partial^{(\alpha} \varepsilon_{\rho}^{\beta\gamma} h^{\mu\nu\lambda)\rho}$$

Spin 4 part:

$$L_2^2 = -\frac{2}{3} h_{\mu}^{\alpha\beta\gamma} \partial_\alpha \partial_\beta h^{\mu\nu\lambda\rho} J^{(4)}_{\nu\lambda\rho\gamma} + \frac{1}{2} \partial_\nu h_{\mu}^{\alpha\beta\gamma} \partial_\alpha h^{\mu\nu\lambda\rho} J^{(4)}_{\lambda\rho\beta\gamma} - \frac{1}{4} \partial^\alpha h_{\mu\nu}^{\beta\gamma} \partial_\alpha h^{\mu\nu\lambda\rho} J^{(4)}_{\lambda\rho\beta\gamma}$$

$$-\partial^\beta h_{\mu\nu}^{\alpha\gamma} \partial_\alpha h^{\mu\nu\lambda\rho} J^{(4)}_{\lambda\rho\beta\gamma} + \frac{1}{3} \partial^\beta h_{\mu\nu\lambda}^{\gamma} \partial^\alpha h^{\mu\nu\lambda\rho} J^{(4)}_{\rho\alpha\beta\gamma}$$

$$\delta_1 h^{\mu\nu\lambda\rho} \sim$$

$$\varepsilon^{\alpha\beta\gamma} \partial_\alpha \partial_\beta \partial_\gamma h^{\mu\nu\lambda\rho} + \partial^{(\mu} \varepsilon_{\gamma}^{|\alpha\beta|} \partial_\alpha \partial_\beta h^{\nu\lambda\rho)\gamma} + \partial^{(\mu} \partial^\nu \varepsilon_{\beta\gamma}^{|\alpha|} \partial_\alpha h^{\lambda\rho)\beta\gamma} + \partial^{(\mu} \partial^\nu \partial^\lambda \varepsilon_{\alpha\beta\gamma} h^{\rho)\alpha\beta\gamma}$$

SPIN FOUR CASE: REMOVING REMINDER

After all this manipulation we still have four remaining terms of two types:
First two remaining terms contain divergences of spin 4 current

$$\partial_\lambda \partial^{(\nu} \varepsilon^{\beta\gamma\mu)} h^{\rho\lambda}_{\mu\nu} \partial^\alpha J^{(4)}_{\rho\alpha\beta\gamma} - \frac{2}{3} \partial^\beta \partial^{(\gamma} \varepsilon^{\mu\nu\lambda)} h^\rho_{\mu\nu\lambda} \partial^\alpha J^{(4)}_{\rho\alpha\beta\gamma} =$$

$$-\frac{1}{6} \partial^{(\mu} (\partial_\beta \partial_\gamma \varepsilon^{\nu\lambda}_\alpha h^{\rho)\alpha\beta\gamma}) J^{(4)}_{\mu\nu\lambda\rho} + \frac{1}{18} \partial^{(\mu} (\partial^\nu \partial^\lambda \varepsilon_{\alpha\beta\gamma} h^{\rho)\alpha\beta\gamma}) J^{(4)}_{\mu\nu\lambda\rho}$$

We can cancel them introducing first order redefinition of

$$\delta_0 h^{\mu\nu\lambda\rho} \Rightarrow \delta_0 h^{\mu\nu\lambda\rho} + \bar{\delta}_0 h^{\mu\nu\lambda\rho}$$

$$\bar{\delta}_0 h^{\mu\nu\lambda\rho} \sim \partial^{(\mu} (\partial_\beta \partial_\gamma \varepsilon^{\nu\lambda}_\alpha h^{\rho)\alpha\beta\gamma}) - \frac{1}{3} \partial^{(\mu} (\partial^\nu \partial^\lambda \varepsilon_{\alpha\beta\gamma} h^{\rho)\alpha\beta\gamma})$$

Second two remainders contain contractions between derivatives of gauge parameter and gauge fields

$$-\frac{4}{3}\partial^{\alpha}\varepsilon_{\mu}^{\beta\gamma}\partial_{\alpha}\partial_{\beta}h^{\nu\lambda\rho\mu}J^{(4)}_{\nu\lambda\rho\gamma}-2\partial^{\alpha}\partial^{\beta}\varepsilon^{\gamma}_{\mu\nu}\partial_{\alpha}h^{\lambda\rho\mu\nu}J^{(4)}_{\lambda\rho\beta\gamma}$$

Using the following (up to total derivatives) identity:

$$\partial_{\mu}A\partial^{\mu}BC=\frac{1}{2}(AB\Box C-\Box ABC-A\Box BC)$$

for on-shell spin 4 gauge field

$$\Box h^{\mu\nu\lambda\rho}=0$$

we can transform

$$\frac{1}{12}\{-8\varepsilon_{\mu}^{\beta\gamma}\partial_{\beta}h^{\nu\lambda\rho\mu}-24\partial^{\beta}\varepsilon^{\gamma}_{\mu\nu}h^{\lambda\rho\mu\nu}-12\partial^{\mu}\varepsilon^{\beta\gamma}_{\nu}h^{\lambda\rho\mu\nu}\}\Box J^{(4)}_{\nu\lambda\rho\gamma}$$

One more interaction term

$$+\frac{1}{4}\delta_{0}\{h^{\beta\gamma}_{\mu\nu}h^{\lambda\rho\mu\nu}\}\Box J^{(4)}_{\lambda\rho\beta\gamma}$$

$$L_{2}^{3}=-\frac{1}{4}h^{\beta\gamma}_{\mu\nu}h^{\lambda\rho\mu\nu}\Box J^{(4)}_{\lambda\rho\beta\gamma}$$

exactly the trace of our spin 6 gauge field transformation

$$\delta_{1}h^{\mu\nu\lambda\rho\alpha}_{\alpha}\Box J^{(4)}_{\mu\nu\lambda\rho}=\{8\varepsilon^{\alpha\beta\rho}\partial_{\alpha}h^{\mu\nu\lambda}_{\beta}+12\partial^{\beta}\varepsilon^{\lambda\rho}_{\alpha}h^{\mu\nu\alpha}_{\beta}+24\partial^{\mu}\varepsilon^{\nu\beta}_{\alpha}h^{\lambda\rho\alpha}_{\beta}\}\Box J^{(4)}_{\mu\nu\lambda\rho}$$

SPIN FOUR CASE: INTERACTION

$$L_2 = \frac{1}{10} h_{\mu}^{\alpha\beta\gamma} h^{\nu\lambda\rho\mu} \tilde{J}^{(6)}_{\nu\lambda\rho\alpha\beta\gamma}$$

$$-\frac{2}{3} h_{\mu}^{\alpha\beta\gamma} \partial_\alpha \partial_\beta h^{\mu\nu\lambda\rho} J^{(4)}_{\nu\lambda\rho\gamma} + \frac{1}{2} \partial_\nu h_{\mu}^{\alpha\beta\gamma} \partial_\alpha h^{\mu\nu\lambda\rho} J^{(4)}_{\lambda\rho\beta\gamma} - \frac{1}{4} \partial^\alpha h^{\beta\gamma}_{\mu\nu} \partial_\alpha h^{\mu\nu\lambda\rho} J^{(4)}_{\lambda\rho\beta\gamma}$$

$$-\partial^\beta h^{\alpha\gamma}_{\mu\nu} \partial_\alpha h^{\mu\nu\lambda\rho} J^{(4)}_{\lambda\rho\beta\gamma} + \frac{1}{3} \partial^\beta h^{\gamma}_{\mu\nu\lambda} \partial^\alpha h^{\mu\nu\lambda\rho} J^{(4)}_{\rho\alpha\beta\gamma} - \frac{1}{4} h^{\beta\gamma}_{\mu\nu} h^{\lambda\rho\mu\nu} \Box J^{(4)}_{\lambda\rho\beta\gamma}$$

Transformation $\delta_1 h^{\mu\nu\lambda\rho} \sim$

$$\varepsilon^{\alpha\beta\gamma} \partial_\alpha \partial_\beta \partial_\gamma h^{\mu\nu\lambda\rho} + \partial^{(\mu} \varepsilon_{\gamma}^{|\alpha\beta|} \partial_\alpha \partial_\beta h^{\nu\lambda\rho)\gamma} + \partial^{(\mu} \partial^{\nu} \varepsilon_{\beta\gamma}^{|\alpha|} \partial_\alpha h^{\lambda\rho)\beta\gamma} + \partial^{(\mu} \partial^{\nu} \partial^{\lambda} \varepsilon_{\alpha\beta\gamma} h^{\rho)\alpha\beta\gamma}$$

$$\bar{\delta}_0 h^{\mu\nu\lambda\rho} \sim \partial^{(\mu} (\partial_\beta \partial_\gamma \varepsilon_{\alpha}^{\nu\lambda} h^{\rho)\alpha\beta\gamma}) - \frac{1}{3} \partial^{(\mu} (\partial^\nu \partial^\lambda \varepsilon_{\alpha\beta\gamma} h^{\rho)\alpha\beta\gamma})$$

$$\delta_1 h^{\mu\nu\lambda\alpha\beta\gamma} = \varepsilon^{\rho(\alpha\beta} \partial_\rho h^{\gamma\mu\nu\lambda)} + \partial^{(\alpha} \varepsilon_{\rho}^{\beta\gamma} h^{\mu\nu\lambda)\rho}$$

SPIN FOUR CASE: COMMUTATOR

Gauge Transformation

$$\delta_1^{(\varepsilon)} h_{\mu\nu\lambda\rho} = \varepsilon^{\alpha\beta\gamma} \partial_\alpha \partial_\beta \partial_\gamma h_{\mu\nu\lambda\rho} + \partial_{(\mu} \varepsilon^{\alpha\beta\gamma} \partial_{|} \alpha \partial_\beta h_{\gamma|\nu\lambda\rho)} + \partial_{(\mu} \partial_\nu \varepsilon^{\alpha\beta\gamma} \partial_{|\alpha} h_{\beta\gamma|\lambda\rho)}$$

$$+ \partial_{(\mu} \partial_\nu \partial_\lambda \varepsilon^{\alpha\beta\gamma} h_{\rho)\alpha\beta\gamma}$$

$$\delta_1^{(\varepsilon)} h_{\mu\nu\lambda\rho} = \varepsilon^{\alpha\beta\gamma} \Gamma_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) + \partial_{(\mu} \Lambda_{\nu\lambda\rho)}(\varepsilon, h)$$

$$\Lambda_{\nu\lambda\rho}(\varepsilon, h) = \varepsilon^{\alpha\beta\gamma} \partial_\alpha \partial_\beta h_{\gamma\nu\lambda\rho} + \frac{1}{2}\left[\partial_{(\nu} \varepsilon^{\alpha\beta\gamma} \partial_{|\alpha} h_{\beta\gamma|\lambda\rho)} - \varepsilon^{\alpha\beta\gamma} \partial_{(\nu} \partial_{|\alpha} h_{\beta\gamma|\lambda\rho)} \right]$$

$$+ \frac{1}{3}\left[\partial_{(\nu} \partial_\lambda \varepsilon^{\alpha\beta\gamma} h_{\rho)\alpha\beta\gamma} + \varepsilon^{\alpha\beta\gamma} \partial_{(\nu} \partial_\lambda h_{\rho)\alpha\beta\gamma} - \frac{1}{2} \partial_{(\nu} \varepsilon^{\alpha\beta\gamma} \partial_\lambda h_{\rho)\alpha\beta\gamma} \right]$$

SPIN FOUR CASE: COMMUTATOR

Generalized Christoffel Tensor

$$\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) = \partial_\alpha \partial_\beta \partial_\gamma h_{\mu\nu\lambda\rho} - \frac{1}{3}\partial_{<\alpha}\partial_\beta \partial_{(\mu} h_{\nu\lambda\rho)\gamma>}$$

$$+\frac{1}{3}\partial_{<\alpha}\partial_{(\mu}\partial_\nu h_{\lambda\rho)\beta\gamma>} - \partial_{(\mu}\partial_\nu \partial_\lambda h_{\rho)\alpha\beta\gamma}$$

$$\delta_0^{(\varepsilon)}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) = -4\partial_\mu \partial_\nu \partial_\lambda \partial_\rho \varepsilon_{\alpha\beta\gamma}$$

$$\delta_1^{(\varepsilon)} h_{\mu\nu} = \mathfrak{L}_{\varepsilon^\lambda} h_{\mu\nu} = \varepsilon^\alpha \Gamma^{(1)}_{\alpha;\mu\nu} + \partial_{(\mu}\left(\varepsilon^\alpha h_{\nu)\alpha}\right)$$

$$\Gamma^{(1)}_{\alpha;\mu\nu} = \partial_\alpha h_{\mu\nu} - \partial_{(\mu} h_{\nu)\alpha}$$

SPIN FOUR CASE: COMMUTATOR

$$[\delta_1^{(\omega)},\delta_1^{(\varepsilon)}]h_{\mu\nu\lambda\rho}=\varepsilon^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(\delta_1^{(\omega)}h)-4\varepsilon^{\alpha\beta\gamma}\partial_\mu\partial_\nu\partial_\lambda\partial_\rho\Lambda_{\alpha\beta\gamma}(\omega,h)$$
$$+\partial_{(\mu}\Lambda_{\nu\lambda\rho)}(\varepsilon,\delta_1^{(\omega)}h)-(\varepsilon\leftrightarrow\omega)$$

$$[\delta_1^{(\omega)},\delta_1^{(\varepsilon)}]h_{\mu\nu\lambda\rho}\sim\varepsilon^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(\delta_1^{(\omega)}h)+\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h)\delta_0^{(\omega)}\Lambda^{\alpha\beta\gamma}(\varepsilon,h)-(\varepsilon\leftrightarrow\omega)$$

Factorize on

$$\partial_{(\mu}\Lambda_{\nu\lambda\rho)}(\varepsilon,\delta_1^{(\omega)}h)-\partial_{(\mu}\Lambda_{\nu\lambda\rho)}(\omega,\delta_1^{(\varepsilon)}h)$$

and

$$\delta_0^{(\omega)}\delta_2^{(\varepsilon)}h_{\mu\nu\lambda\rho}-\delta_0^{(\varepsilon)}\delta_2^{(\omega)}h_{\mu\nu\lambda\rho}$$

SPIN FOUR CASE: COMMUTATOR RESULT

$$[\delta_1^{(\omega)},\delta_1^{(\varepsilon)}]h_{\mu\nu\lambda\rho} = <<\omega,\varepsilon>>^{\alpha\beta\gamma}\,\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h)$$

$$+3\varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\omega^{\alpha\beta\gamma}R^{(4)}_{\eta\alpha\beta\gamma;\mu\nu\lambda\rho}(h)+3\varepsilon_\delta^{\sigma\eta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_\sigma R^{(4)}_{\eta\alpha\beta\gamma;\mu\nu\lambda\rho}(h)-(\varepsilon\leftrightarrow\omega)$$

+ Mixed Symmetry Field Gauge Transformation Contribution

$$<<\omega,\varepsilon>>^{\alpha\beta\gamma} = \varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\partial_\eta\omega^{\alpha\beta\gamma}+\partial^{(\alpha}\varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\omega_\eta^{\beta\gamma)}+\partial^{(\alpha}\partial^\beta\varepsilon^{\delta\sigma\eta}\partial_\delta\omega_{\sigma\eta}^{\gamma)}+\frac{1}{3}\partial^{(\alpha}\partial^\beta\varepsilon^{\delta\sigma\eta}\partial^{\gamma)}\omega_{\delta\sigma\eta}$$

$$-\frac{1}{3}\partial^{(\alpha}\varepsilon^{\delta\sigma\eta}\partial^\beta\partial^{\gamma)}\omega_{\delta\sigma\eta}-\frac{1}{12}\partial^{(\alpha}\partial^\beta\varepsilon^{\delta\sigma\eta}\delta_0^{(\omega)}h_{\delta\sigma\eta}^{\gamma)}-\frac{3}{16}\partial^{[\alpha}\varepsilon^{\delta]\sigma\eta}\delta_0^{(\omega)}\partial_{[\delta}h_{\sigma\eta}^{\beta]\gamma}$$

$$+symetrizationin(\alpha\beta\gamma)-(\varepsilon\leftrightarrow\omega)$$

SPIN FOUR CASE: OUTLOOK

What can be done

- “Degauging “ and “Off-Shelling”
- Generalization for spin 2, 4 ….S, S+2, .. and scalars
- Weyl Invariance?
- What's cooking in the case of different spins?
- More special cases?

Thank You for your attention!