SU(2|1) supersymmetric spinning models of chiral superfields

Stepan Sidorov (BLTP JINR, Dubna)

RDP Online Workshop on Mathematical Physics

December 6, 2020

S. Sidorov, arXiv:2003.01023 [hep-th]

Introduction

- A new class of systems of $\mathcal{N}=4$, $d=1$ supersymmetric quantum mechanics called "Kähler oscillator" was introduced by S. Bellucci and A. Nersessian (Phys. Rev. D **67** (2003) 065013; arXiv:hep-th/0401232).

- They studied supersymmetric oscillator models on Kähler manifolds with the term of the first-order in time derivatives responsible for the presence of a constant magnetic field. The bosonic Lagrangian of such system can be written as

$$\mathcal{L}_{\text{bos.}} = g\dot{z}\dot{\bar{z}} + \frac{i}{2}\,\mathbf{B}\left(\dot{z}\,\partial_z K - \dot{\bar{z}}\,\partial_{\bar{z}} K\right) - \omega^2 g^{-1}\,\partial_z K\,\partial_{\bar{z}} K, \qquad g = \partial_z\partial_{\bar{z}} K\left(z, \bar{z}\right). \tag{1}$$

SU(2|1) supersymmetric mechanics

- It turned out, the presence of oscillator term and the interaction with a magnetic field deforms the standard $\mathcal{N}=4$, $d=1$ Poincaré supersymmetry to the so-called "Weak supersymmetry" (A. Smilga, Phys. Lett. B **585** (2004) 173).

- We showed that the deformed superalgebra of Weak supersymmetry corresponds to the worldline supersymmetry SU(2|1) (E. Ivanov, S. Sidorov, Class. Quant. Grav. **31** (2014) 0750; J. Phys. A **47** (2014) 292002).

- We initiated a study of deformed supersymmetric quantum mechanics by employing superfield approach based on the worldline supersymmetry SU(2|1) with a mass dimension deformation parameter m.

Superalgebra

The standard definition of the superalgebra $su(2|1)$ is

$$
\begin{aligned}
&\{Q^i, \bar{Q}_j\} = 2\delta^i_j \mathcal{H} - 2m I^i_j, \\
&[\mathcal{H}, Q^k] = -\frac{m}{2}\, Q^k, \qquad [\mathcal{H}, \bar{Q}_l] = \frac{m}{2}\, \bar{Q}_l\,, \\
&[I^i_j, \bar{Q}_l] = \frac{1}{2}\,\delta^i_j \bar{Q}_l - \delta^i_l \bar{Q}_j\,, \qquad [I^i_j, Q^k] = \delta^k_j Q^i - \frac{1}{2}\,\delta^i_j Q^k, \\
&[I^i_j, I^k_l] = \delta^k_j I^i_l - \delta^i_l I^k_j\,.
\end{aligned} \tag{2}
$$

The indices i, j ($i = 1, 2$) are SU(2) indices. The U(1) generator $\mathcal{H}$ is associated with the Hamiltonian. The generators I^i_j ($I^k_k = 0$) form SU(2) symmetry. In the limit $m = 0$, models of the standard $\mathcal{N} = 4$ supersymmetric mechanics are restored with $\mathcal{H}$ being a central charge generator.

Superalgebra

The standard definition of the superalgebra $su(2|1)$ is

$$
\begin{aligned}
&\{Q^i, \bar{Q}_j\} = 2\delta^i_j \mathcal{H}, \\
&\left[\mathcal{H}, Q^k\right] = 0, \qquad \left[\mathcal{H}, \bar{Q}_l\right] = 0\,, \\
&\left[I^i_j, \bar{Q}_l\right] = \frac{1}{2}\,\delta^i_j \bar{Q}_l - \delta^i_l \bar{Q}_j\,, \qquad \left[I^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{2}\,\delta^i_j Q^k, \\
&\left[I^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j\,.
\end{aligned} \tag{3}
$$

The indices i, j $(i=1,\,2)$ are $\mathrm{SU}(2)$ indices. The $\mathrm{U}(1)$ generator $\mathcal{H}$ is associated with the Hamiltonian. The generators I^i_j $(I^k_k=0)$ form $\mathrm{SU}(2)$ symmetry. In the limit $m=0$, models of the standard $\mathcal{N}=4$ supersymmetric mechanics are restored with $\mathcal{H}$ being a central charge generator.

$\mathcal{N} = 4$ multiplets

- Multiplets of $\mathcal{N}=4$, $d=1$ supersymmetry are denoted as $(\mathbf{k}, \mathbf{4}, \mathbf{4}-\mathbf{k})$ with $\mathbf{k}=\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$. These numbers correspond to the numbers of bosonic physical fields, fermionic physical fields and bosonic auxiliary fields.

- Wess-Zumino (WZ) type Lagrangians (sometimes referred as Chern-Simons) for $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ and $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ were presented in the framework of the $\mathcal{N}=4$, $d=1$ harmonic superspace (E. Ivanov, O. Lechtenfeld, JHEP **0309** (2003) 073). For example, the simplest WZ Lagrangian for $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ reads

$$\mathcal{L}_{\rm WZ} = \frac{i}{2}\left(z^i\dot{\bar z}_i - \dot z^i \bar z_i\right) + \psi^a\bar\psi_a\,, \qquad i=1,2, \qquad a=1,2. \tag{4}$$

- Without kinetic Lagrangian containing bosonic terms second-order in time derivatives, this Lagrangian describes a semi-dynamical multiplet. Fermionic fields become auxiliary, while bosonic fields satisfy the following Dirac brackets:

$$\left\{z^i, \bar z_j\right\} = i\,\delta^i_j\,. \tag{5}$$

Coupling

- Coupling of dynamical and semi-dynamical multiplets was proposed by S. Fedoruk, E. Ivanov, O. Lechtenfeld, Phys. Rev. D **79** (2009) 105015. This idea provided harmonic superfield construction of $\mathcal{N}=4$ extension of Calogero system with the additional spin (isospin) degrees of freedom z^i, z_j .

- This work was followed by a further study of "spinning" models considering couplings of dynamical and semi-dynamical multiplets of various types (S. Bellucci, S. Krivonos, A. Sutulin, Phys. Rev. D **81** (2010) 105026, E. Ivanov, M. Konyushikhin, A. Smilga, JHEP **1005** (2010) 033, etc).

Spinning models of chiral superfields

- The main goal of the present talk is to employ $SU(2|1)$ superfield approach to spinning models of chiral superfields instead of harmonic ones.
- In this paper we consider the coupling of the chiral multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ and the semi-dynamical multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ of the mirror type.
- We construct $SU(2|1)$ supersymmetric models based on this coupling, undeformed versions of which were studied by S. Bellucci, N. Kozyrev, S. Krivonos, A. Sutulin, Phys. Rev. D **85** (2012) 065024.

Superspace

The SU(2|1) superspace is defined as a supercoset (E. Ivanov, S. Sidorov, J. Phys. A **47** (2014) 292002):

$$\frac{\mathrm{SU}(2|1)}{\mathrm{SU}(2)} \sim \frac{\{\mathcal{H}, Q^i, \bar{Q}_j\,, I^i_j\}}{\{I^i_j\}} = \{t, \theta_i, \bar{\theta}^j\}\,. \tag{6}$$

It has a chiral subspace identified with the coset

$$\frac{\{\mathcal{H}, Q^i, \bar{Q}_j\,, I^i_j\}}{\{\bar{Q}_j\,, I^i_j\}} = \{t_{\mathrm{L}}, \theta_i\}\,. \tag{7}$$

The chiral condition reads

$$\bar{\mathcal{D}}_j \Phi\left(t_{\mathrm{L}}, \theta_i\right) = 0, \tag{8}$$

where $\bar{\mathcal{D}}_j$ and $\mathcal{D}_j$ are SU(2|1) covariant derivatives.

Generalized chiral superspace

A generalization of the chiral subspace is identified with the coset

$$\frac{\left\{\mathcal{H}, \hat{Q}^i, \hat{\bar{Q}}_j\,, I^i_j\right\}}{\left\{\hat{\bar{Q}}_j\,, I^i_j\right\}} = \left\{\hat{t}_{\rm L}, \hat{\theta}_i\right\}, \tag{9}$$

where

$$\hat{Q}^i = \cos\lambda\, Q^i - \sin\lambda\, \bar{Q}^i, \qquad \hat{\bar{Q}}_j = \cos\lambda\, \bar{Q}_j + \sin\lambda\, Q_j\,. \tag{10}$$

The generalized chiral condition reads

$$\left(\cos\lambda\, \bar{\mathcal{D}}_j - \sin\lambda\, \mathcal{D}_j\right)\Phi\left(\hat{t}_{\rm L}, \hat{\theta}_i\right) = 0. \tag{11}$$

Generalized chiral multiplet

- The generalized chiral condition describes a new type of the chiral multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ defined on the generalized chiral superspace and depending on two deformation parameters: m, λ.

- Exactly this multiplet is a basis for the construction of supersymmetric Kähler oscillator models with the frequency of oscillator $\omega = m\sin 2\lambda\,/2$ and the strength of an external magnetic field $\mathbf{B} = m\cos 2\lambda$.

- Both parameters disappear in the limit $m = 0$ and the rotation parameter λ becomes just an external automorphism parameter of the standard $\mathcal{N} = 4$ Poincaré supersymmetry.

Superfield action

The chiral superfield is solved by

$$\Phi\left(\hat{t}_{\rm L},\hat{\theta}_j\right) = z + \sqrt{2}\,\hat{\theta}_k\xi^k + \hat{\theta}_k\hat{\theta}^k B, \tag{12}$$

where

$$\hat{t}_{\rm L} = t + i\,\bar{\hat{\theta}}^k\hat{\theta}_k\,, \qquad \hat{\theta}_i = \left(\cos\lambda\,\theta_i\,e^{\frac{i}{2}mt} + \sin\lambda\,\bar{\theta}_i\,e^{-\frac{i}{2}mt}\right)\left(1 - \frac{m}{2}\,\bar{\theta}^k\theta_k\right). \tag{13}$$

Superfield invariant action for the chiral superfield Φ is given by

$$S_{\rm kin.} = \frac{1}{4}\int dt\,d^2\theta\,d^2\bar{\theta}\,\left(1 + 2m\,\bar{\theta}^k\theta_k\right)K\left(\Phi,\bar{\Phi}\right). \tag{14}$$

where $K\left(\Phi,\bar{\Phi}\right)$ is a Kähler potential. Superpotential is given by the standard superfield action

$$S_{\rm pot.} = \int d\hat{t}_{\rm L}\,d^2\hat{\theta}\,f\left(\Phi\right) + \int d\hat{t}_{\rm R}\,d^2\bar{\hat{\theta}}\,\bar{f}\left(\bar{\Phi}\right). \tag{15}$$

Mirror multiplets

- The standard $\mathcal{N}=4$ multiplets have their mirror counterparts characterized by the interchange of two $\mathrm{SU}(2)$ groups which form $\mathrm{SU}(2)\times\mathrm{SU}'(2)\to\mathrm{SO}(4)$ automorphism group of the standard $\mathcal{N}=4$ Poincaré supersymmetry [E. Ivanov, J. Niederle, Phys. Rev. D **80** (2009) 065027].

- Since this interchange $(i,j\longleftrightarrow i',j')$ has no essential impact on Poincaré supersymmetry, $\mathcal{N}=4$ multiplets and their mirror counterparts are mutually equivalent when dealing with only one multiplet from such a pair.

- Deformation to $\mathrm{SU}(2|1)$ supersymmetry breaks the equivalence, because the first $\mathrm{SU}(2)$ group becomes subgroup of $\mathrm{SU}(2|1)$ and the second group $\mathrm{SU}'(2)$ is broken. It means that $\mathrm{SU}(2|1)$ multiplets differ from their mirror counterparts.

Multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$

First, let us consider the ordinary multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ satisfying the constraints

$$\mathcal{D}^{(i}q^{j)a} = 0, \qquad \bar{\mathcal{D}}^{(i}q^{j)a} = 0, \qquad \overline{(q^{ia})} = q_{ia}\,, \qquad a = 1,2. \tag{16}$$

One can rewrite these constraints as

$$\mathcal{D}^{(i}_{i'}q^{j)a} = 0, \tag{17}$$

where the covariant derivatives are written via $\mathrm{SU}(2)\times\mathrm{SU}'(2)$ indices as

$$\mathcal{D}^{i} := \mathcal{D}^{i\,1'} = -\,\mathcal{D}^{i}_{2'}\,, \qquad \bar{\mathcal{D}}^{i} := \mathcal{D}^{i\,2'} = \mathcal{D}^{i}_{1'} \quad \Rightarrow \quad \mathcal{D}^{i\,i'} \tag{18}$$

Mirror multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$

Hence, its mirror counterpart is written as

$$\mathcal{D}^{i\,(i'}Y^{j')A} = 0, \qquad \overline{(Y^{i'A})} = Y_{i'A}\,, \qquad A = 1, 2. \tag{19}$$

The superfield $Y^{i'A}$ can be denoted as $Y^A = Y^{2'A}$, $\bar{Y}^A = -\,Y^{1'A}$ that leads to the constraints

$$\bar{\mathcal{D}}_i Y^A = 0, \qquad \mathcal{D}^i \bar{Y}^A = 0, \qquad \mathcal{D}_i Y^A = \bar{\mathcal{D}}_i \bar{Y}^A, \qquad \overline{(Y^A)} = \bar{Y}_A\,. \tag{20}$$

One can see that first two constraints are (anti)chiral conditions describing the pair of chiral multiplets $(\mathbf{2}, \mathbf{4}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{4}, \mathbf{2})$. The third constraint kills the half of component fields leaving the field content to be $(\mathbf{4}, \mathbf{4}, \mathbf{0})$.

Generalized mirror multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$

In order to couple the mirror $(\mathbf{4},\mathbf{4},\mathbf{0})$ and generalized chiral multiplets, we consider the generalized constraints

$$\tilde{\bar{\mathcal{D}}}_i Y^A = 0, \qquad \tilde{\mathcal{D}}^i \bar{Y}^A = 0, \qquad \tilde{\mathcal{D}}_i Y^A = \tilde{\bar{\mathcal{D}}}_i \bar{Y}^A, \qquad \overline{(Y^A)} = \bar{Y}_A\,, \tag{21}$$

where

$$\tilde{\bar{\mathcal{D}}}_i = \cos\lambda\,\bar{\mathcal{D}}_i - \sin\lambda\,\mathcal{D}_i\,, \qquad \tilde{\mathcal{D}}^i = \cos\lambda\,\mathcal{D}^i + \sin\lambda\,\bar{\mathcal{D}}^i. \tag{22}$$

Their solution reads

$$Y^A\left(\hat{t}_{\rm L}, \hat{\theta}_i\right) = y^A + \sqrt{2}\,\hat{\theta}_i \psi^{iA} + i\,\hat{\theta}_k \hat{\theta}^k\,\dot{y}^A, \qquad \overline{(y^A)} = \bar{y}_A\,, \qquad \overline{(\psi^{iA})} = \psi_{iA}\,. \tag{23}$$

Superpotential

The superpotential action

$$S_{\text{pot.}} = \frac{\mu}{2}\int d\hat{t}_{\text{L}}\, d^2\hat{\theta}\, h\left(Y^A\right) + \frac{\mu}{2}\int d\hat{t}_{\text{R}}\, d^2\bar{\hat{\theta}}\, \bar{h}\left(\bar{Y}_A\right), \tag{24}$$

yields Wess-Zumino type Lagrangian:

$$S_{\text{pot.}} = \int dt\, \mathcal{L}_{\text{WZ}}\,, \qquad \mathcal{L}_{\text{WZ}} = \mu\left[i\,\dot{\bar{y}}^A\,\partial_A h\left(y^A\right) + \frac{1}{2}\,\psi^{iA}\psi_i^B\,\partial_A\partial_B h\left(y^A\right) + \text{c.c.}\right]. \tag{25}$$

Gauged mirror multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$

We can also assume that the chiral superfields Y^A are subjected to the local $\mathrm{U}(1)$ transformations

$$(Y^1)' = e^{\frac{1}{2}(\Lambda - \bar{\Lambda})}\, Y^1, \qquad (Y^2)' = e^{-\frac{1}{2}(\Lambda - \bar{\Lambda})}\, Y^2, \tag{26}$$

where $\Lambda := \Lambda\left(\hat{t}_{\rm L}, \hat{\theta}_i\right)$, $\bar{\Lambda} := \bar{\Lambda}\left(\hat{t}_{\rm R}, \bar{\hat{\theta}}^j\right)$. The superfields then satisfy the new gauge invariant constraints

$$\begin{aligned}
&\left(\bar{\tilde{\mathcal{D}}}^i + \frac{1}{2}\left[\bar{\tilde{\mathcal{D}}}^i, X\right]\right) Y^1 = 0, \quad \left(\bar{\tilde{\mathcal{D}}}^i - \frac{1}{2}\left[\bar{\tilde{\mathcal{D}}}^i, X\right]\right) Y^2 = 0, \qquad \text{c.c.},\\
&\left(\tilde{\mathcal{D}}^i - \frac{1}{2}\left[\tilde{\mathcal{D}}^i, X\right]\right) Y^1 = \left(\bar{\tilde{\mathcal{D}}}^i + \frac{1}{2}\left[\bar{\tilde{\mathcal{D}}}^i, X\right]\right) \bar{Y}^1,\\
&\left(\tilde{\mathcal{D}}^i + \frac{1}{2}\left[\tilde{\mathcal{D}}^i, X\right]\right) Y^2 = \left(\bar{\tilde{\mathcal{D}}}^i - \frac{1}{2}\left[\bar{\tilde{\mathcal{D}}}^i, X\right]\right) \bar{Y}^2.
\end{aligned} \tag{27}$$

where the real superfield X is a gauge superfield transforming as

$$X' = X + \Lambda + \bar{\Lambda}\,. \tag{28}$$

Gauge superfield

The superfield X satisfies the additional gauge invariant constraint

$$\tilde{\mathcal{D}}_{(i}\bar{\tilde{\mathcal{D}}}_{j)}X = 0. \tag{29}$$

It is solved by

$$\begin{aligned} X\left(t,\hat{\theta}_i,\bar{\hat{\theta}}^j\right) \;=\;& x + \sqrt{2}\left(\hat{\theta}_k\bar{\chi}^k + \bar{\hat{\theta}}^k\chi_k\right) + 2\,\bar{\hat{\theta}}_k\hat{\theta}^k\mathcal{A} + \hat{\theta}_k\hat{\theta}^k\,D + \bar{\hat{\theta}}^k\bar{\hat{\theta}}_k\,\bar{D} \\ & + \sqrt{2}\,i\,\bar{\hat{\theta}}^k\hat{\theta}_k\left(\hat{\theta}_i\dot{\bar{\chi}}^i - \bar{\hat{\theta}}^i\dot{\chi}_i\right) - \frac{1}{4}\,\hat{\theta}_i\hat{\theta}^i\,\bar{\hat{\theta}}^j\bar{\hat{\theta}}_j\,\ddot{x}, \\ & \overline{(x)} = x, \qquad \overline{(\mathcal{A})} = \mathcal{A}\,, \qquad \overline{(D)} = \bar{D}, \qquad \overline{(\chi^i)} = \bar{\chi}_i\,. \end{aligned} \tag{30}$$

This superfield describes the mirror multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ that differs from the ordinary one because of the deformation. Using the $\mathrm{U}(1)$ gauge freedom $X' = X + \Lambda + \bar{\Lambda}$, we can choose the WZ gauge:

$$X_{\mathrm{WZ}} = 2\,\bar{\hat{\theta}}^k\hat{\theta}_k\,\mathcal{A}\,, \qquad \mathcal{A}'\,(t) = \mathcal{A}\,(t) - \dot{\alpha}\,(t)\,. \tag{31}$$

Triplet

One can introduce accompanying chiral superfields

$$\mathcal{V}_{\rm WZ}\left(\hat{t}_{\rm L},\hat{\theta}_i\right) = \hat{\theta}_k\hat{\theta}^k\mathcal{A}\,,\qquad \bar{\mathcal{V}}_{\rm WZ}\left(\hat{t}_{\rm R},\bar{\hat{\theta}}^j\right) = \bar{\hat{\theta}}^k\bar{\hat{\theta}}_k\,\mathcal{A}\,, \tag{32}$$

satisfying

$$\tilde{\mathcal{D}}_i\,X_{\rm WZ} = \bar{\tilde{\mathcal{D}}}_i\bar{\mathcal{V}}_{\rm WZ}\,,\qquad \bar{\tilde{\mathcal{D}}}_i\,X_{\rm WZ} = -\,\tilde{\mathcal{D}}_i\mathcal{V}_{\rm WZ}\,. \tag{33}$$

These superfields can be combined in the form a triplet superfield $\mathcal{V}^{i'j'} \equiv \mathcal{V}^{(i'j')}$ as

$$\mathcal{V}^{i'j'} = \hat{\theta}^{k\,(i'}\hat{\theta}^{j')}_k\mathcal{A}\,,\qquad \tilde{\mathcal{D}}^{i\,(i'}\mathcal{V}^{j'k')} = 0. \tag{34}$$

Thus, it can be interpreted as a mirror counterpart of the "topological" gauge multiplet described by the harmonic superfield V^{++} in the WZ gauge [F. Delduc, E. Ivanov, Nucl. Phys. B **753** (2006) 211-241; Nucl. Phys. B **770** (2007) 179-205].

Solution

According to the (anti)chiral conditions, the superfield solution is modified as

$$
\begin{aligned}
Y^1\left(t,\hat{\theta}_i,\bar{\hat{\theta}}^j\right) &= e^{-\frac{X}{2}}\, Y^1_{\rm L}\left(\hat{t}_{\rm L},\hat{\theta}_i\right), \qquad \left(Y^1_{\rm L}\right)' = e^{\Lambda}\, Y^1_{\rm L}\,,\\
Y^2\left(t,\hat{\theta}_i,\bar{\hat{\theta}}^j\right) &= e^{\frac{X}{2}}\, Y^2_{\rm L}\left(\hat{t}_{\rm L},\hat{\theta}_i\right), \qquad \left(Y^2_{\rm L}\right)' = e^{-\Lambda}\, Y^2_{\rm L}\,.
\end{aligned} \tag{35}
$$

Solving the additional constraint the left chiral superfield $Y^A_{\rm L}$ has the following θ-expansion:

$$
Y^A_{\rm L}\left(\hat{t}_{\rm L},\hat{\theta}_i\right) = y^A + \sqrt{2}\,\hat{\theta}_i\psi^{iA} + i\,\hat{\theta}_k\hat{\theta}^k\,\nabla_t\bar{y}^A, \qquad \nabla_t\bar{y}^1 = \left(\partial_t + i\mathcal{A}\right)\bar{y}^1, \quad \nabla_t\bar{y}^2 = \left(\partial_t - i\mathcal{A}\right)\bar{y}^2. \tag{36}
$$

Finally, the superpotential action must be written as a function of the only possible invariant $Y^1_{\rm L}Y^2_{\rm L}$, since it is the only gauge invariant object defined on the left chiral subspace.

Coupling of dynamical and semi-dynamical multiplets

The simplest superpotential term of interacting multiplets reads

$$S_{\text{int.}} = \frac{\mu}{2}\int d\hat{t}_{\rm L}\, d^2\hat{\theta}\, Y^1_{\rm L} Y^2_{\rm L}\, f\,(\Phi) - \frac{\mu}{2}\int d\hat{t}_{\rm R}\, d^2\bar{\hat{\theta}}\, \bar{Y}^1_{\rm R} \bar{Y}^2_{\rm R}\, \bar{f}\,(\bar{\Phi})\,, \tag{37}$$

where f is an arbitrary holomorphic function of Φ. The Lagrangian is then given by

$$\begin{aligned} \mathcal{L}_{\text{int.}} \;=\; & \mu\Big[i\left(y^1\,\nabla_t\bar{y}_1 - y^2\,\nabla_t\bar{y}_2\right) f + \psi^{i1}\psi^2_i\, f + B\, y^1 y^2\,\partial_z f \\ & + \xi^i\left(\psi_{i1}\, y^1 - \psi_{i2}\, y^2\right)\partial_z f - \frac{\xi_i\xi^i}{2}\, y^1 y^2\,\partial_z\partial_z f + \text{c.c.}\Big]. \end{aligned} \tag{38}$$

Fayet-Iliopoulos term can be constructed as

$$S_{\rm FI} = -\frac{c}{4}\left[\int d\hat{t}_{\rm L}\, d^2\hat{\theta}\,\mathcal{V}_{\rm WZ} + \int d\hat{t}_{\rm R}\, d^2\bar{\hat{\theta}}\,\bar{\mathcal{V}}_{\rm WZ}\right] \quad\Rightarrow\quad \mathcal{L}_{\rm FI} = -\,c\,\mathcal{A}\,, \quad c = \text{const.} \tag{39}$$

Total Lagrangian

- We consider the model of the dynamical multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ interacting with the semi-dynamical multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$. The total Lagrangian is written as

$$\mathcal{L} = \mathcal{L}_{\text{kin.}} + \mathcal{L}_{\text{int.}} + \mathcal{L}_{\text{FI}}\,. \tag{40}$$

- Eliminating the auxiliary fields B and ψ^{iA} by their equations of motion and performing the following redefinition

$$\begin{aligned} &y^1 = v\left(f+\bar f\right)^{-\frac12}, \qquad y^2 = \bar w\left(f+\bar f\right)^{-\frac12}, \qquad \xi^i = g^{-\frac12}\,\eta^i, \\ &\bar y_1 = \bar v\left(f+\bar f\right)^{-\frac12}, \qquad \bar y_2 = w\left(f+\bar f\right)^{-\frac12}, \qquad \bar\xi_j = g^{-\frac12}\,\bar\eta_j\,, \end{aligned} \tag{41}$$

we obtain the total on-shell Lagrangian $\mathcal{L}$.

Component Lagrangian

Component Lagrangian reads

$$
\begin{aligned}
\mathcal{L} \; = \;& g\,\dot{\bar z}\dot z + \frac{i}{2}\left(\eta^i\dot{\bar\eta}_i - \dot\eta^i\bar\eta_i\right) + \frac{i}{2}\,\mu\,(v\dot{\bar v} + w\dot{\bar w} - \dot v\bar v - \dot w\bar w) + \frac{i\mu\left(\dot{\bar z}\,\partial_{\bar z}\bar f - \dot z\,\partial_z f\right)}{2(f+\bar f\,)}\,(v\bar v - w\bar w) \\
& + \frac{i}{2}\,(\dot{\bar z}\,\partial_{\bar z} g - \dot z\,\partial_z g)\,g^{-1}\,\eta^k\bar\eta_k - \frac{\partial_z f\,\partial_{\bar z}\bar f}{g\,(f+\bar f\,)^2}\left[\mu\,(v\bar v - w\bar w)\,\eta^k\bar\eta_k + \mu^2\,w\bar w\,v\bar v\right] \\
& - \frac{\mu\,v\bar w}{g\,(f+\bar f\,)}\left[\frac{\partial_z\partial_z f}{2} - \frac{\partial_z f\,\partial_z f}{(f+\bar f\,)} - \frac{\partial_z f}{2}\,g^{-1}\,\partial_z g\right]\eta_i\eta^i \\
& - \frac{\mu\,w\bar v}{g\,(f+\bar f\,)}\left[\frac{\partial_{\bar z}\partial_{\bar z}\bar f}{2} - \frac{\partial_{\bar z}\bar f\,\partial_{\bar z}\bar f}{(f+\bar f\,)} - \frac{\partial_{\bar z}\bar f}{2}\,g^{-1}\,\partial_{\bar z} g\right]\bar\eta^j\bar\eta_j \\
& + \frac{1}{4g^2}\left(\partial_z\partial_{\bar z} g - g^{-1}\,\partial_z g\,\partial_{\bar z} g\right)\eta_i\eta^i\,\bar\eta^j\bar\eta_j + \left[\mu\,(v\bar v + w\bar w) - c\right]\mathcal{A} + \mathcal{L}_m\,, \qquad (42)
\end{aligned}
$$

Component Lagrangian

where $\mathcal{L}_m$ is a deformed part:

$$\begin{aligned}\mathcal{L}_m \;=\;& \frac{i}{2}\, m\cos 2\lambda\,(\dot z\,\partial_z K - \dot{\bar z}\,\partial_{\bar z} K) - \frac{(m\sin 2\lambda)^2}{4g}\,\partial_z K\,\partial_{\bar z} K \\ &+ \frac{m\sin 2\lambda}{4g}\left[\left(\partial_z\partial_z K - g^{-1}\,\partial_z K\,\partial_z g\right)\eta_i\eta^i + \left(\partial_{\bar z}\partial_{\bar z} K - g^{-1}\,\partial_{\bar z} K\,\partial_{\bar z} g\right)\bar\eta^j\bar\eta_j\right] \\ &+ \frac{m\sin 2\lambda}{2g\,(f+\bar f\,)}\left(\mu\, w\bar v\,\partial_z f\,\partial_{\bar z} K + \mu\, v\bar w\,\partial_{\bar z}\bar f\,\partial_z K\right) - \frac{m\cos 2\lambda}{2}\,\eta^k\bar\eta_k\,. \end{aligned} \tag{43}$$

Indeed, it depends on two deformation parameters:

$$\mathbf{B} = m\cos 2\lambda\,, \qquad \omega = \frac{m\sin 2\lambda}{2}\,. \tag{44}$$

The U(1) gauge field $\mathcal{A}$ plays the role of a Lagrange multiplier enforcing the constraint

$$\mu\,(v\bar v + w\bar w) - c = 0. \tag{45}$$

Classical mechanics

Classical Hamiltonian reads

$$
\begin{aligned}
\mathcal{H} \;=\;& g^{-1} P_z P_{\bar z} - \frac{1}{4}\left(\partial_z\partial_{\bar z} g - g^{-1}\,\partial_z g\,\partial_{\bar z} g\right) g^{-2}\,\eta_i\eta^i\,\bar\eta^j\bar\eta_j + \frac{2\,\partial_z f\,\partial_{\bar z}\bar f\,S_3}{g\left(f+\bar f\,\right)^2}\,\eta^k\bar\eta_k + \frac{m\cos 2\lambda}{2}\,\eta^k\bar\eta_k \\
&+ \frac{S_+}{g\left(f+\bar f\,\right)}\left[\frac{\partial_z\partial_z f}{2} - \frac{\partial_z f\,\partial_z f}{\left(f+\bar f\,\right)} - \frac{\partial_z f}{2}\,g^{-1}\,\partial_z g\right]\eta_k\eta^k \\
&+ \frac{S_-}{g\left(f+\bar f\,\right)}\left[\frac{\partial_{\bar z}\partial_{\bar z}\bar f}{2} - \frac{\partial_{\bar z}\bar f\,\partial_{\bar z}\bar f}{\left(f+\bar f\,\right)} - \frac{\partial_{\bar z}\bar f}{2}\,g^{-1}\,\partial_{\bar z} g\right]\bar\eta^k\bar\eta_k \\
&- \frac{m\sin 2\lambda}{4}\left[\left(\partial_z\partial_z K - g^{-1}\,\partial_z K\,\partial_z g\right) g^{-1}\,\eta_i\eta^i + \left(\partial_{\bar z}\partial_{\bar z} K - g^{-1}\,\partial_{\bar z} K\,\partial_{\bar z} g\right) g^{-1}\,\bar\eta^j\bar\eta_j\right] \\
&+ g^{-1}\left[\frac{\partial_z f\,S_+}{\left(f+\bar f\,\right)} - \frac{m\sin 2\lambda}{2}\,\partial_z K\right]\left[\frac{\partial_{\bar z}\bar f\,S_-}{\left(f+\bar f\,\right)} - \frac{m\sin 2\lambda}{2}\,\partial_{\bar z} K\right], \qquad (46)
\end{aligned}
$$

Classical mechanics

where

$$S_3 = \frac{\mu}{2}\,(v\bar{v} - w\bar{w})\,, \qquad S_+ = \mu\, v\bar{w}, \qquad S_- = \mu\, w\bar{v},$$
$$P_z = p_z - \frac{i}{2}\, m\cos 2\lambda\, \partial_z K + \frac{i\,\partial_z f\, S_3}{(f+\bar{f}\,)} + \frac{i}{2}\, g^{-1}\,\partial_z g\, \eta^i \bar{\eta}_i\,,$$
$$P_{\bar{z}} = p_{\bar{z}} + \frac{i}{2}\, m\cos 2\lambda\, \partial_{\bar{z}} K - \frac{i\,\partial_{\bar{z}} \bar{f}\, S_3}{(f+\bar{f}\,)} - \frac{i}{2}\, g^{-1}\,\partial_{\bar{z}} g\, \eta^j \bar{\eta}_j\,. \qquad (47)$$

Poisson (Dirac) brackets are imposed as

$$\{p_z\,, z\} = -1, \qquad \{p_{\bar{z}}\,, \bar{z}\} = -1, \qquad \{\eta^i, \bar{\eta}_j\} = -i\delta^i_j\,,$$
$$\{v, \bar{v}\} = i\mu^{-1}, \qquad \{w, \bar{w}\} = i\mu^{-1}. \qquad (48)$$

SU(2|1) generators

SU(2|1) supercharges are

$$Q^i = \sqrt{2}\, e^{\frac{i}{2}mt}\, g^{-\frac{1}{2}} \Big\{ \cos\lambda\, \eta^i \Big[P_z - \frac{i}{2}(1-\cos 2\lambda)\, m\, \partial_z K \Big] - \sin\lambda\, \bar\eta^i \Big[P_{\bar z} - \frac{i}{2}(1+\cos 2\lambda)\, m\, \partial_{\bar z} K \Big] - \frac{i}{(f+\bar f)} \big(\cos\lambda\, \partial_{\bar z} \bar f\, S_-\, \bar\eta^i - \sin\lambda\, \partial_z f\, S_+\, \eta^i \big) \Big\}, \quad (49)$$

$$\bar Q_j = \sqrt{2}\, e^{-\frac{i}{2}mt}\, g^{-\frac{1}{2}} \Big\{ \cos\lambda\, \bar\eta_j \Big[P_{\bar z} + \frac{i}{2}(1-\cos 2\lambda)\, m\, \partial_{\bar z} K \Big] + \sin\lambda\, \eta_j \Big[P_z + \frac{i}{2}(1+\cos 2\lambda)\, m\, \partial_z K \Big] - \frac{i}{(f+\bar f)} \big(\cos\lambda\, \partial_z f\, S_+\, \eta_j + \sin\lambda\, \partial_{\bar z} \bar f\, S_-\, \bar\eta_j \big) \Big\}. \quad (50)$$

SU(2) subgroup generators are written as

$$I^i_j = \eta^i \bar\eta_j - \frac{\delta^i_j}{2}\, \eta^k \bar\eta_k\,. \quad (51)$$

SU(2) group of spin variables

The generators S_3 and $S_\pm$, written trough spin variables as

$$S_3 = \frac{\mu}{2}\,(v\bar{v} - w\bar{w})\,, \qquad S_+ = \mu\, v\bar{w}, \qquad S_- = \mu\, w\bar{v}, \tag{52}$$

form the $su(2)$ algebra:

$$\{S_3, S_\pm\} = \mp\, iS_\pm\,, \qquad \{S_+, S_-\} = -\,2iS_3\,. \tag{53}$$

The Hamiltonian commutes with the Casimir operator

$$C_{\rm SU(2)} = S_+S_- + (S_3)^2\,. \tag{54}$$

SU(2) group of spin variables

According to the constraint

$$\mu\left(v\bar{v}+w\bar{w}\right)-c=0, \tag{55}$$

the Casimir operator is determined by the constant

$$C_{\mathrm{SU}(2)}=\frac{\mu^{2}\left(v\bar{v}+w\bar{w}\right)^{2}}{4}=\frac{c^{2}}{4}\,. \tag{56}$$

Its quantum counterpart (up to the ordering ambiguity) is given by $(c\approx 2s)$

$$C_{\mathrm{SU}(2)}=s\left(s+1\right), \tag{57}$$

where s is a spin of the quantum system.

Conclusions

- We proposed new models of SU(2|1) supersymmetric mechanics with the use of dynamical and gauged semi-dynamical multiplets. As an alternative of harmonic superspace, we exploited the generalized chiral superspace.

Conclusions

- We proposed new models of $\mathrm{SU}(2|1)$ supersymmetric mechanics with the use of dynamical and gauged semi-dynamical multiplets. As an alternative of harmonic superspace, we exploited the generalized chiral superspace.
- We considered in details the deformed model on the pseudo-sphere $\mathrm{SU}(1,1)/\mathrm{U}(1)$ (Lobachevsky space) with the corresponding metric $g = (1 - \gamma\, z\bar{z})^{-2}$ and the special choice of $f\,(z)$.

Conclusions

- We proposed new models of $\mathrm{SU}(2|1)$ supersymmetric mechanics with the use of dynamical and gauged semi-dynamical multiplets. As an alternative of harmonic superspace, we exploited the generalized chiral superspace.
- We considered in details the deformed model on the pseudo-sphere $\mathrm{SU}(1,1)/\mathrm{U}(1)$ (Lobachevsky space) with the corresponding metric $g = (1 - \gamma\, z\bar{z})^{-2}$ and the special choice of $f\,(z)$.
- In distinct from [S. Bellucci, N. Kozyrev, S. Krivonos, A. Sutulin, Phys. Rev. D **85** (2012) 065024], the symmetry $\mathrm{SU}(1,1) \times \mathrm{U}(1)$ of the model is broken. We can define only the remaining $\mathrm{U}(1) \subset \mathrm{SU}(1,1)$ generator J_3 that commutes with all $\mathrm{SU}(2|1)$ generators. However, one can restore this symmetry in the limit $\lambda = 0$.

Conclusions

- We tried to obtain superconformal symmetry for considered models, but it is impossible at least for a single-particle model. Superconformal models can possibly be obtained for two or more chiral dynamical multiplets.
- It would be interesting to study mirror counterparts of the multiplets $(\mathbf{1},\mathbf{4},\mathbf{3})$ and $(\mathbf{3},\mathbf{4},\mathbf{1})$. The latter one is described by a triplet consisting of real and chiral superfields, *i.e.* X, $\mathcal{V}$ and $\bar{\mathcal{V}}$.
- One can try to couple the mirror multiplets $(\mathbf{3},\mathbf{4},\mathbf{1})$ and $(\mathbf{4},\mathbf{4},\mathbf{0})$ in $\mathrm{SU}(2|1)$ chiral superspace.

Conclusions

- We tried to obtain superconformal symmetry for considered models, but it is impossible at least for a single-particle model. Superconformal models can possibly be obtained for two or more chiral dynamical multiplets.
- It would be interesting to study mirror counterparts of the multiplets $(\mathbf{1},\mathbf{4},\mathbf{3})$ and $(\mathbf{3},\mathbf{4},\mathbf{1})$. The latter one is described by a triplet consisting of real and chiral superfields, *i.e.* X, Φ and $\bar{\Phi}$.
- One can try to couple the mirror multiplets $(\mathbf{3},\mathbf{4},\mathbf{1})$ and $(\mathbf{4},\mathbf{4},\mathbf{0})$ in $\mathrm{SU}(2|1)$ chiral superspace.

Thank you for your attention!