Symmetry algebra of dynamical and discrete Calogero models

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Outline of talk

- Superintegrability of N-dimensional isotropic oscillator. $SO(N)$ angular momentum and Fradkin's tensors as providers of $U(N)$ symmetry.
- Superintegrability of N-particle (rational) Calogero model. Dunkl deformations of $SO(N)$ angular momentum and Fradkin's tensors as providers of superintegrability of Calgero model.
- Discrete Calogero system and it's relation to exactly solvable $SU(n)$ spin system with long-range interaction (Polychronakos-Frahm chain).
- The behavior of deformed $U(N)$ symmetry at the freezing limit and how it links the symmetries of isotropic oscillator with discrete Calogero systems.

Calogero model

The Calogero model describes $1d$ particles with $1/r^2$ interaction bound by harmonic potential [Calogero (1969,1971)],

$$\hat{H}_{\mathrm{C}} = \sum_{i=1}^{N} \frac{\hat{p}_i^2 + \omega^2 x_i^2}{2} + \sum_{i<j} \frac{g(g \mp \hbar)}{(x_i - x_j)^2}.$$

Where g is a positive *coupling constant* characterizing the strength of the interparticle two-body interaction and ω^2 is a nonnegative constant characterizing the strength of the interaction with an external *harmonic oscillator* potential.

Most its properties, like (super)integrability, spectrum, wave functions, and conservation laws, are conditioned by its *nonlocal* modification (called a *generalized* Calogero model) [Polychronakos (1992); Brink, Hansson, Vasiliev (1992)]:

$$\hat{H} = \frac{1}{2} \sum_{i=1}^{N} \left(\hat{p}_i^2 + \omega^2 x_i^2\right) + \sum_{i<j} \frac{g(g - \hbar M_{ij})}{(x_i - x_j)^2}.$$

The nonlocal operator M_{ij} permutes the coordinates x_i and x_j.

On *bosonic/fermionic* states $M_{ij} = \pm 1$ and $\hat{H}$ is reduced to $\hat{H}_{\mathrm{C}}$.

Dunkl operator representation

- Define a new momentum

$$\hat{\pi}_i = -\imath\hbar\nabla_i$$

$$\nabla_i = \partial_i - \frac{g}{\hbar}\sum_{j\neq i}\frac{1}{x_i - x_j}M_{ij}.$$

- ∇_i was constructed first by Dunkl [(1988)].

- Then the inverse-square Calogero interaction can be encapsulated into the Dunkl momentum operator as [Polychronakos (1992); Brink, Hansson, Vasiliev (1992)]:

$$\hat{H} = \frac{\hat{\pi}^2}{2} + \frac{\omega^2 x^2}{2}.$$

Properties of Dunkl operators

- Dunkl operators are nonlocal covariant derivatives. Their momenta mutually commute:

$$[\hat{\pi}_i, \hat{\pi}_j] = 0.$$

- But the standard commutations with coordinates are changed:

$$[\hat{\pi}_i, x_j] = -\imath \hat{S}_{ij},$$

where

$$\hat{S}_{ij} = (\delta_{ij} - 1) g M_{ij} + \delta_{ij} \left(\hbar + g \sum_{k \neq i} M_{ik} \right).$$

- In the $g = 0$ limit,

$$\hat{\pi}_i = \hat{p}_i, \qquad \hat{S}_{ij} = \hbar \delta_{ij}$$

recovering the Heisenberg algebra commutations.

Spectrum generating operators ($\omega = 1$)

- Dunkl-operator analog of lowering-rising operators:

$$\hat{a}_i^{\pm} = \frac{x_i \mp \imath\hat{\pi}_i}{\sqrt{2}}.$$

- The commutators:

$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^+, \hat{a}_j^+] = 0,$$

$$[\hat{a}_i, \hat{a}_j^+] = \hat{S}_{ij}.$$

- The generalized Calogero Hamiltonian can be expressed in terms of them:

$$\hat{H} = \frac{1}{2}\sum_i (\hat{a}_i^+ \hat{a}_i + \hat{a}_i \hat{a}_i^+)$$

- Operators $\hat{a}_i^{\pm}$ obey a standard spectrum generating relations [Brink, Hansson, Vasiliev (1992); Minhatan, Polychronakos (1992)]:

$$[\hat{H}, \hat{a}_i^{\pm}] = \pm\hbar\hat{a}_i^{\pm}.$$

Spectrum of Calogero system ($\hbar = 1$)

Bosonic ground state wavefuntion must obey [Brink, Hansson, Vasiliev (1992)]

$$a_i\psi_0 = 0 \qquad \text{or} \qquad \frac{\partial_i\psi_0}{\psi_0} = -x_i + \sum_{j\neq i}\frac{g}{x_i - x_j}.$$

The solution is:

$$\psi_0 = \prod_{i<j}|x_i - x_j|^g e^{-\frac{1}{2}\sum_i x_i^2}$$

with ground state energy

$$E_0 = \frac{N}{2} + g\frac{N(N-1)}{2}.$$

Excitations are generated by creation operators:

$$|k_1, \ldots, k_N\rangle = (A_1^+)^{k_1}(A_2^+)^{k_2}\ldots(A_N^+)^{k_N}\psi_0, \qquad A_l^+ = \sum_{i=1}^{N}(a_i^+)^l$$

with $k_i = 0, 1, 2\ldots$ and energy

$$E_{k_1\ldots k_N} = E_0 + k_1 + 2k_2 + \cdots + Nk_N$$

Wavefunction for bosonic system:

$$\psi_{k_1\ldots k_N}(x) = \text{const}\prod_{i<j}|x_i - x_j|^g \text{Sym}\left\{\prod_{i=1}^{N}H_{k_i}(x_i)\right\}e^{-\frac{1}{2}x^2},$$

Dunkl analog of $U(N)$ symmetry

- As a consequence of spectrum-generating relations, the elements

$$\hat{E}_{ij} = \hat{a}_i^+ \hat{a}_j$$

satisfies the conservation law.

- $\hat{E}_{ij}$ are Dunkl-analogs of $U(N)$ generators.

- Together with permutations S_{ij}, they generate the symmetries of the nonlocal Calogero model:

$$[\hat{H}, \hat{E}_{ij}] = 0, \qquad [\hat{H}, \hat{S}_{ij}] = 0.$$

- They imply following deformation of $u(N)$ commutations[Feigin, T.H. (2015); Correa, T.H., Lechtenfeld, Nersessian (2016)]:

$$[\hat{E}_{ij}, \hat{E}_{kl} + \hat{S}_{kl}] = \hat{E}_{il}\hat{S}_{kj} - \hat{S}_{il}\hat{E}_{kj}.$$

Liouville integrals

As a consequence, the diagonal elements are closed under commutation. They are not Abelian but obey a simple commutation,

$$[\hat{E}_{ii}, \hat{E}_{kk}] = (\hat{E}_{ii} - \hat{E}_{kk})\hat{S}_{ik}.$$

The above algebra ensures that the power sums form a system of Liouville integrals of the Calogero system [Polychronakos (1992)],

$$\hat{\mathcal{E}}_k = \sum_i \hat{E}_{ii}^k, \qquad [\hat{\mathcal{E}}_k, \hat{\mathcal{E}}_l] = 0.$$

The generalized Hamiltonian itself is expressed in terms of the first member in this family,

$$\hat{H} = \hat{\mathcal{E}}_1 - S + \frac{N\hbar}{2},$$

where the permutation invariant element

$$S = \sum_{i<j} M_{ij}.$$

Moreover, it is a unique Casimir element (up to a nonessential constant term) of the Dunkl-deformed $u(N)$ algebra [Feigin, Hakobyan (2015)].

Dunkl angular momentum. Dunkl-Fradkin tensor

- The antisymmetric combinations of $\hat{E}_{ij}$ yield the *Dunkl angular momentum* components [Feigin (2003); Kuznetsov (1996)],

$$\hat{L}_{ij} = \hat{E}_{ij} - \hat{E}_{ji} = x_i \hat{\pi}_j - x_j \hat{\pi}_i.$$

- Together with permutations, they produce a deformation of $so(N)$ algebra.
- The Casimir element is a deformation of the usual angular momentum square [Feigin, T.H. (2015)],

$$\hat{\mathcal{L}}_2 = \hat{L}^2 + S^2 - \hbar(N-2)S, \qquad [\hat{L}_{ij}, \hat{\mathcal{L}}_2] = 0.$$

- $\hat{\mathcal{L}}_2$ is an angular Hamiltonian describing the spherical part of the Calogero model [Feigin, Lechtenfeld, Polychronakos (2013)].
- The symmetric combinations of the deformed $u(N)$ produce a Dunkl-operator deformation of the *Fradkin tensor* [Correa, T.H., Lechtenfeld, Nersessian (2016)],

$$\hat{I}_{ij} = \hat{E}_{ij} + \hat{E}_{ji} - \hat{S}_{ij} = \hat{\pi}_i \hat{\pi}_j + x_i x_j.$$

Complete set of integrals for Calogero model

- More general invariants may include the nondiagonal generators E_{ij}.
- The most general integrals of the Calogero Hamiltonian are *symmetric polynomials* $P_{\rm sym}$ on the generators $\hat{E}_{ij}$, (or $\hat{L}_{ij}$ and $\hat{I}_{ij}$) and permutation S_{ij},

$$P_{\rm sym}(\hat{E}_{ij}, S_{ij}) = P'_{\rm sym}(\hat{L}_{ij}, \hat{I}_{ij}, S_{ij})$$

$$[P_{\rm sym}, \hat{H}_{\rm C}] = 0.$$

- Some simplest integrals:

$$\sum_{i,j} \hat{I}_{ij}^k, \qquad \sum_{i,j} \hat{L}_{ij}^{2k},$$

$$\sum_{i<j} \hat{I}_{ij}^k M_{ij}, \qquad \sum_{i,j} \hat{I}_{ii}^k \hat{L}_{ij}^{2l}.$$

Reduction to discrete system

- Set constant $g = 1$ in the nonlocal Calogero system:

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{N}\left(\hat{p}_i^2 + \hat{x}_i^2\right) + \sum_{i<j}\frac{1-\hbar M_{ij}}{\hat{x}_{ij}^2}.$$

- Consider the *dynamical* system at the *equilibrium* point $\hat{x} = x$, where the (classical) confining Calogero potential is minimal:

$$\frac{\partial V}{\partial x_i} = 0, \qquad V(x) = \sum_{i=1}^{N}\frac{x_i^2}{2} + \sum_{i<j}\frac{1}{x_{ij}^2}.$$

- These equilibrium coordinates are roots of Hermite polynomial [Frahm (1993)],

$$H_N(x_i) = 0$$

- All roots differ, so there are $N!$ equivalent minima related by permutations M_{ij}.
- Permutations are the only allowed evolutions in *frozen* system, so the system becomes discrete.

Generalized Polychronakos-Frahm model

Apply $\hbar$ expansion to dynamical system [Mathieu, Xudous (2001)]

$$\hat{H} = V + \hbar H^{(1)} - \frac{\hbar^2 \partial^2}{2}.$$

Generalized Polychronakos-Frahm Hamiltonian [Polychronakos (1993)]:

$$H^{(1)} = \sum_{i<j} \frac{1}{x_{ij}^2} M_{ij}.$$

The $SU(n)$ symmetric Polychronakos-Frahm spin chain is recovered after the replacement of the *coordinate permutations* with *spin exchange* operators,

$$H_{\mathrm{PF}} = \sum_{i<j} \frac{P_{ij}}{x_{ij}^2}.$$

P_{ij} permutes $SU(n)$ spins $s = 1, \ldots, n$:

$$P_{ij} | s_1 \ldots s_i \ldots s_j \ldots s_N \rangle = | s_1 \ldots s_j \ldots s_i \ldots s_N \rangle.$$

Equivalence between spin and coordinate chains for identical particles

Endow the particles x_i with additional spin degrees of freedom, s_i:

$$\xi_i = (x_i, s_i)$$

- Then in bosonic and fermionic sectors, the coordinate Hamiltonian becomes identical to the spin system
$$H^{(1)} = \pm H_{PF}$$
- For bosonic states:
$$M_{ij} P_{ij} = 1 \quad \text{so that} \quad M_{ij} = P_{ij}$$
- For fermionic states:
$$M_{ij} P_{ij} = -1 \quad \text{so that} \quad M_{ij} = -P_{ij}$$
- The projection inverts the order of permutations so that the operator $M_{ij} M_{kl}$ must be substituted by $P_{kl} P_{ij}$.

Discrete Dunkl momentum

Planck's expansion of Dunkl momentum:

$$\hat{\pi}_i = \pi_i - \imath\hbar\partial_i.$$

The zero-term is a *discrete* analog of *Dunkl momentum* operator [Polychronakos (1993)]:

$$\pi_i = \sum_{j\neq i} \frac{\imath}{x_{ij}} M_{ij}.$$

Planck's expansion of permutation matrix:

$$\hat{S}_{ij} = S_{ij} + \hbar\delta_{ij} \qquad \text{with} \qquad S_{ij} = \begin{cases} -M_{ij}, & \text{if } i \neq j, \\ \sum_{k\neq i} M_{ik}, & \text{otherwise.} \end{cases}$$

Canonical commutations resemble their original form:

$$[\pi_i, \pi_j] = 0, \qquad [x_i, \pi_j] = \imath S_{ij}.$$

Discrete spectrum-generating operators

- The *discrete lowering-rising* operators:

$$a_i^{\pm} = \frac{x_i \mp \imath\pi_i}{\sqrt{2}}, \qquad \text{where} \qquad \hat{a}_i^{\pm} = a_i^{\pm} \pm \frac{\hbar}{\sqrt{2}}\partial_i$$

- They obey similar commutations as $\hat{a}_i^{\pm}$ under replacement $\hat{S}_{ij} \to S_{ij}$ [Polychronakos (1993)]:

$$[a_i, a_j] = [a_i^{+}, a_j^{+}] = 0, \qquad [a_i, a_j^{+}] = S_{ij}.$$

- The spectrum generating relation remains valid for the discrete system too [Polychronakos (1993)]:

$$[H^{(1)}, a_i^{\pm}] = \pm a_i^{\pm}.$$

- However, unlike the dynamical case (or oscillator case), the discrete Hamiltonian *is not expressed* via lowering-rising operators,

$$H^{(1)} \neq \frac{1}{2}\sum_i (a_i^{+} a_i + a_i a_i^{+})$$

Discrete analog of Dunkl $U(N)$ and $SO(N)$ symmetries

The constants of motion of the dynamical system have Planck's expansion:

$$\hat{E}_{ij} = E_{ij} + \hbar E_{ij}^{(1)} - \frac{\hbar^2}{2}\partial_i\partial_j.$$

Expansion for the antisymmetric (Dunkl angular momentum) is simpler:

$$\hat{L}_{ij} = L_{ij} - \imath\hbar(x_i\partial_j - x_j\partial_i).$$

It produces a *discrete Dunkl angular momentum* operator,

$$L_{ij} = x_i\pi_j - x_j\pi_i.$$

A similar expansion for the symmetrised components of $\hat{E}_{ij}$ is more complex:

$$\hat{I}_{ij} = I_{ij} + \hbar I_{ij}^{(1)} - \hbar^2\partial_i\partial_j.$$

It defines a *discrete* analog of the *Fradkin's tensor*:

$$I_{ij} = x_i x_j + \pi_i\pi_j.$$

Algebra of discrete Dunkl $U(N)$ generators

The mapping from the *dynamical* to *discrete Dunkl* $U(N)$ generators

$$\hat{E}_{ij} \to E_{ij}, \qquad \hat{S}_{ij} \to S_{ij}$$

provides the following relation for the discrete symmetries,

$$[E_{ij}, E_{kl} + S_{kl}] = E_{il}S_{kj} - S_{il}E_{kj},$$

$$[E_{ii}, E_{kk}] = (E_{ii} - E_{kk})S_{ik}.$$

The power sums of diagonal elements

$$\mathcal{E}_k = \sum_i E_{ii}^k$$

yield Liouville integrals of the Polychronakos-Frahm chain [Polychronakos (1993)]:

$$[\mathcal{E}_k, \mathcal{E}_l] = 0, \qquad [H^{(1)}, \mathcal{E}_k] = 0.$$

Discrete Dunkl $U(N)$ generators: additional relations

- For the dynamical system following quadratic relation is *the only constraint* on $\hat{E}_{ij}$ [Feigin, T.H. (2015)].

$$\hat{E}_{ij}(\hat{E}_{kl} + \hat{S}_{kl}) = \hat{E}_{il}(\hat{E}_{kj} + \hat{S}_{kj}).$$

But for the discrete system, there are *a lot of other restrictions* on them. Such as:

1. Linear relations (consequence of the zero centre-of-mass):

$$\sum_i E_{ik} = \sum_i E_{ki} = \sum_i L_{ik} = \sum_i I_{ik} = 0.$$

2. Triviality of the first Liouville integral (for $k \geq 2$ they are more complicate):

$$\mathcal{E}_1 = S + \tfrac{1}{2}N(N-1).$$

3. In the equilibrium limit angular Calogero Hamiltonian is just a constant:

$$\mathcal{L}_2 = r^4, \qquad r^2 = \tfrac{1}{2}N(N-1)$$

Two-dimensional Calogero system with Dihedral symmetry

1. The Dihedral group D_n is a finite group that describes the symmetries of a regular polygon with rotations and reflections.

2. In the complex plane,

$$z = x_1 + \imath x_2, \qquad \bar{z} = x_1 - \imath x_2,$$

3. It consists of the n discrete rotations and n reflections with respect to the symmetry axes

$$r_k(z) = w^{2k} z, \qquad w = e^{\frac{\imath\pi}{n}}, \qquad s_k(z) = w^{-2k}$$

with $k = 0, 1, \ldots, n-1$

4. The Dunkl operators are defined as [C.F.Dunkl(1989)].

$$\nabla_z = \partial_z - g \sum_{k=0}^{n-1} \frac{w^{-k}}{f_k} s_k \qquad \text{with} \qquad f_k = z w^{-k} - \bar{z} w^{k}$$

5. The two-dimensional analogs of both dynamical and descrete Calogero models, remain invariant with respect to the Dihedral group D_n with odd n.

Summary

1. The symmetries of the Calogero model have been described by means of the Dunkl-operators deformation of $U(N)$ generators (symmetry algebra of $d = N$ isotropic oscillator).

2. The symmetries of the discrete Calogero model are described by means of a discrete version Dunkl-deformed $U(N)$ algebra. Polychronakos-Frahm spin chain has been described as discrete Calogero model for identical particals.

3. The complete set of independent integrals of motion for the discrete Calogero Hamiltonian are retrieved using dynamical-to-discrete transition relations.

4. Some of the integrals of motion become mere constants for discrete Calogero model (H, $\mathcal{L}_2$).

5. The dynamical-to-discrete relations of Calogero models can be extended with general finite reflection groups (the dihedral group D_n provides the simplest example in $d = 2$).