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Bogomolny equations from the Pseudo Analytic Functions Viewpoint

Tinatin Supatashvili (MS)

Supervisor: Dr. Prof. Gia Giorgadze

Ivane Javakhishvili Tbilisi State University

The Plan:

- Looking at interesting Physical problem through the differential geometry language.
- Writing equations for vortex solutions.
- Defining the Vortex Number and stating the theorem about the solutions written by Jaffe and Taubes in [1].
- Going to complex variables.
- Getting the same results but with a different approach to the Bogomolny equations, in particular, using the already existing theorems known in Pseudoanalytic Functions Theory.

Physical system:

- Lagrangian for the complex scalar field $\varphi(x)$ in (2+1)-dimensional space-time with $U(1)$ gauge symmetry:

$$\mathcal{L} = -\frac{1}{4} F_{\nu\mu} F^{\nu\mu} + \left(D_\mu \varphi\right)\overline{(D^\mu \varphi)} - V(\varphi) \qquad (1)$$

$\nu, \mu = 0,1,2;\ \eta_{\nu\mu} = diag(1,-1,-1)$

- $D_\mu = \partial_\mu - ieA_\mu; \quad F_{\nu\mu} = \partial_\mu A_\nu - \partial_\nu A_\mu; \quad V(\varphi) = \frac{\lambda}{2}(|\varphi|^2 - v^2)^2$

- $U(1)$ local transformation: $\varphi(x) \to e^{i\alpha(x)}\varphi(x)$ (3)

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x) \qquad (4)$$

- Static configuration: $\varphi = \varphi(\boldsymbol{x}),\ A_i = A_i(\boldsymbol{x}),\ A_0 = 0,\ \ i = 1,2.$

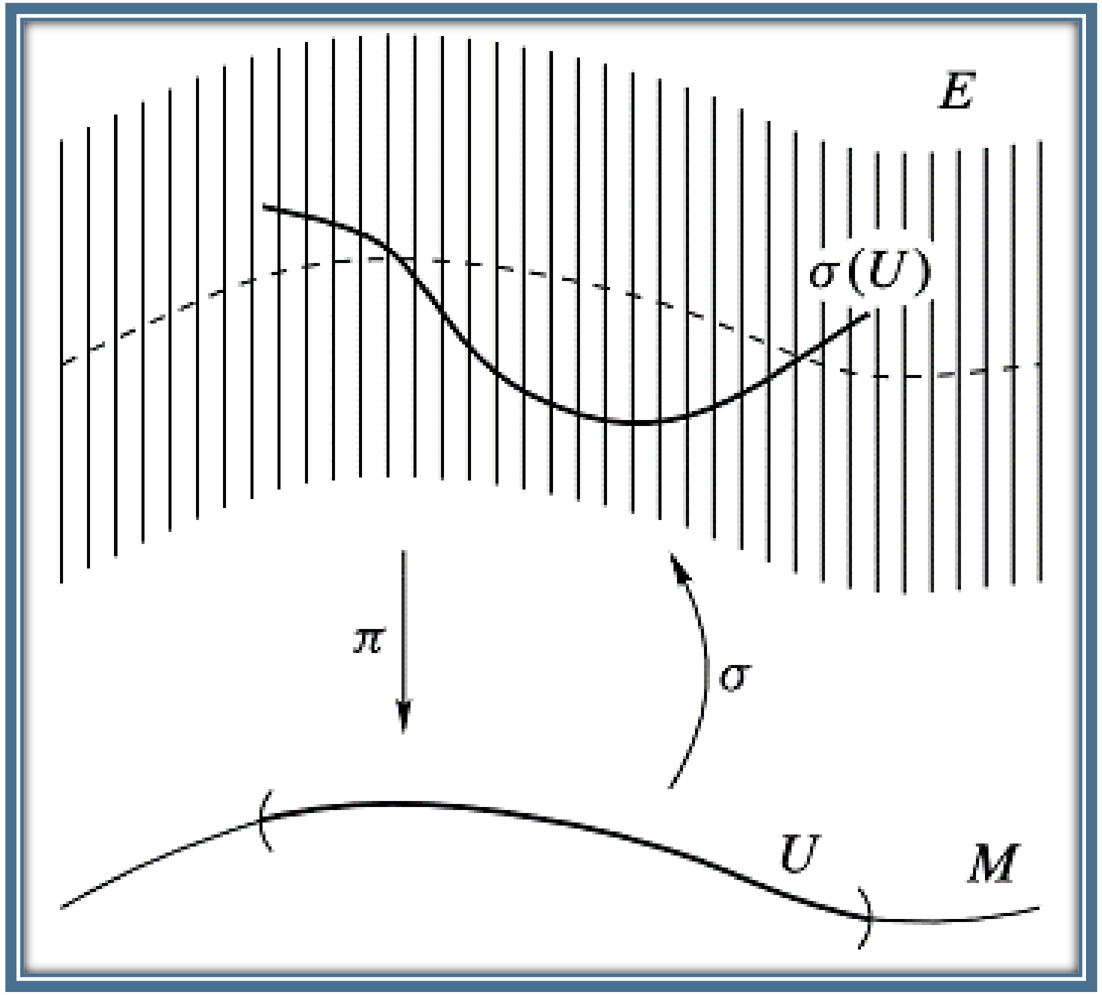
- $E[A_i, \varphi] = \int_{\mathbb{R}^2} d^2x \left[\frac{1}{4} F_{ij} F_{ij} + \overline{(D_i \varphi)}(D_i \varphi) + \frac{\lambda}{2}(|\varphi|^2 - v^2)^2\right]$ (5)

❖ For finite E: $|\varphi| \to v$ & $(D_i \varphi) \to 0\ for\ |\boldsymbol{x}| \to \infty$ (6)

Mathematical Problem:

- Taking two bundles with the Base Space $\mathbb{R}^2$:

1. Principal Bundle with the fiber $G = U(1)$;
2. Vector Bundle associated with this Principal Bundle. In our case, we have $\mathbb{C}$ as a fiber and G group acts on it by the simple multiplication of two complex numbers.



Mathematical Viewpoint:

- $\varphi(\boldsymbol{x}) = \varphi_1(\boldsymbol{x}) + i\varphi_2(\boldsymbol{x})$ as the section of Hermitian Line Bundle over the $\mathbb{R}^2$.

- $A_i(\boldsymbol{x})$ – Components of the connection on $\mathbb{R}^2$. Connection is:

$$A(\boldsymbol{x}) = A_1(\boldsymbol{x})dx_1 + A_2(\boldsymbol{x})dx_2 \qquad (7)$$

- F_{ij} – Components of the curvature :

$$F = dA + A \wedge A = \frac{1}{2}\sum_{i,j=1}^{2} F_{ij}dx^i \wedge dx^i \qquad (8)$$

Exterior derivative $d: \Omega^k(\mathbb{R}^2) \to \Omega^{k+1}(\mathbb{R}^2)$, where $\Omega^k(\mathbb{R}^2)$ is the space of smooth k-forms on the smooth manifold $\mathbb{R}^2$.
" $\wedge$ " – is a wedge product.

- In our case: $F = dA$ *and* $F_{ij} = \partial_i A_j - \partial_j A_i$ (9)

In the Language of Differential Forms and "Youg-Mills-Higgss Action":

- $\varphi -$ $\mathbb{C}$ - valued 0-form; $A -$ $\mathfrak{u}(1)$ algebra-valued 1-form; $F -$ $\mathfrak{u}(1)$ algebra-valued 2-form.
- Action: $\mathcal{A}[A_i, \varphi] = \frac{1}{2}\int_{\mathbb{R}^2}\left[F_A \wedge * F_A + (D_A\varphi) \wedge * \overline{(D_A\varphi)} + \frac{\lambda}{2} * (|\varphi|^2 - 1)^2\right]$ (10)
- Hodge Star Operator $* : \Omega^k(\mathbb{R}^2) \to \Omega^{dim(\mathbb{R}^2)-k}(\mathbb{R}^2)$.
- $D_A\varphi = (\nabla_A)_1\varphi dx_1 + (\nabla_A)_2\varphi dx_2$ (11)
- Covariant derivative of the φ : $(\nabla_A)_i\varphi = \nabla_i\varphi + \rho(A_i)\varphi$. (12)

$\nabla_i \equiv \partial_i$ and $\rho(A_j)$ is a representation of the Lie algebra corresponding to the considered group. The representation space is chosen to be a fiber of the associated Vector Bundle.

The Case When λ=1:

- Define Vortex Number:

$$N = \frac{1}{2\pi} \int_{\mathbb{R}^2} F_A \,, \qquad (13)$$

Which is an integer number $N \in \mathbb{Z}$.

- $$\mathcal{A}[A_i, \varphi] = \int_{\mathbb{R}^2} d^2x \left\{ \frac{1}{2} [(\partial_1 \varphi_1 + A_1 \varphi_2) \mp (\partial_2 \varphi_2 - A_2 \varphi_1)]^2 + \right. \\ + \frac{1}{2} [(\partial_2 \varphi_1 + A_2 \varphi_2) \pm (\partial_1 \varphi_2 - A_1 \varphi_1)]^2 + \\ \left. + \frac{1}{2} \left[F_{12} \pm \frac{1}{2} (\varphi_1^2 + \varphi_2^2 - 1) \right]^2 \right\} \pm \frac{1}{2} \int_{\mathbb{R}^2} d^2x \, F_{12} \qquad (14)$$

- $\mathcal{A} \geq \pi |N|$, where N is a vortex number.

Bogomolny Equations:

- If $N > 0$, the minimum is achieved when:
- $(\partial_1\varphi_1 + A_1\varphi_2) - (\partial_2\varphi_2 - A_2\varphi_1) = 0$ (15)
- $(\partial_2\varphi_1 + A_2\varphi_2) + (\partial_1\varphi_2 - A_1\varphi_1) = 0$ (16)
- $F_{12} + \frac{1}{2}(\varphi_1^2 + \varphi_2^2 - 1) = 0$ (17)
- Solutions of (15),(16),(17) also satisfy the variational equations:

$$d * F_A = \frac{i}{2} * (\varphi\overline{D_A\varphi} - \bar{\varphi}D_A\varphi) \quad (18)$$

$$D_A * D_A\varphi = \frac{\lambda}{2} * (|\varphi|^2 - 1)\varphi \quad (19)$$

- In components: $\partial_i F_{ij} = Im\left(\varphi\overline{(\nabla_A)_j\varphi}\right)$ (20)
- $\nabla_A^2\varphi = \frac{\lambda}{2}\varphi(|\varphi|^2 - 1)$ (21)

Complex Vairables:

- Connection: $A = \alpha dz + \bar{\alpha} d\bar{z}$, where $\alpha = \frac{1}{2}(A_1 - iA_2)$; $\bar{\alpha} = \frac{1}{2}(A_1 + iA_2)$ (22)
- $D_A\varphi = (\partial_z - i\alpha)\varphi dz + (\partial_{\bar{z}} - i\bar{\alpha})\varphi d\bar{z}$
  - (15) → $(\partial_z + i\alpha)\bar{\varphi} + (\partial_{\bar{z}} - i\bar{\alpha})\varphi = 0$ (23)
  - (16) → $(\partial_z + i\alpha)\bar{\varphi} - (\partial_{\bar{z}} - i\bar{\alpha})\varphi = 0$ (24)
- (17) → $Im(\partial_{\bar{z}}\alpha) = \frac{1}{8}(1 - \varphi\bar{\varphi})$ (25)
- (23) and (24) are *Real* and *Imaginary* parts of:

$$D_A\varphi - i * D_A\varphi = 2(\partial_{\bar{z}} - i\bar{\alpha})\varphi d\bar{z} = 0$$

❖ $(\partial_{\bar{z}} - i\bar{\alpha})\varphi = 0$ (26)

- This is the main equation that we are going to study.

Main Theorem:

- Given an integer $N \geq 0$ and a set $\{z_i\}$, $i = 1, \ldots, N$, of N points in $\mathbb{C}$, there exists a finite action solution to equations (15),(16),(17) unique up to gauge equivalence, with the following properties:

1. The solution is globally C^∞.

2. The zeros of φ are the set of points $\{z_i\}$, and as $z \to z_i$:

$$\varphi(z, \bar{z}) \sim c_i (z - z_i)^{n_i}, \qquad c_i \neq 0$$

3.

$$N = \frac{1}{2\pi} \int_{\mathbb{R}^2} F_A = \sum_{\substack{distinct \\ z_i}} n_i = \frac{1}{\pi} \mathcal{A}$$

Solutions for which $N \neq 0$ are called “N –vortex” solutions. In case of $N = 0$, we have a *classical vacuum* solution for φ and A_i.

Pseudo Analytic Functions Viewpoint:

- **Definition**: A pair of complex functions F and G in the domain - Ω, which have Höldercontinuous partial derivatives with respect to the real variables is called a generating pair if the next inequality holds: $\mathrm{Im}(\bar{F}G) > 0$ in Ω

- Generalized Cauchy-Riemann equations:

$$\omega_{\bar{z}} = a\omega + b\bar{\omega}, \omega_{\bar{z}} \equiv \partial_{\bar{z}}\omega \qquad (27)$$

ω is (F, G)pseudoanalytic function of the first kind. a, b are defined by the generating pair (F, G):

$$a_{(F,G)} = -\frac{\bar{F}G_{\bar{z}} - F_{\bar{z}}\bar{G}}{F\bar{G} - \bar{F}G} \qquad b_{(F,G)} = \frac{FG_{\bar{z}} - F_{\bar{z}}G}{F\bar{G} - \bar{F}G} \qquad (28)$$

- $\omega(z)$ is (F, G)pseudoanalytic function of the first kind if and only if it satisfies Generalized Cauchy-Riemann equation.

Pseudo Analytic Functions Viewpoint:

- We consider such $a_{(F,G)}$ and $b_{(F,G)}$ that satisfy the regularity condition:

$$a_{(F,G)}, b_{(F,G)} \in L_{p,2}(\mathbb{C}) \quad p > 2$$

- $L_{p,2}(\mathbb{C}),\ p > 2$ is a set of all functions $f(z)$ defined on the complex plane and satisfying the conditions:
- $f(z) \in L_p(E_1) \quad E_1 = \{z|\ |z| \le 1\}$
- $f_2(z) = \frac{1}{|z|^2} f\left(\frac{1}{z}\right) \in L_p(E_1)$
- Our equation is:

$$\varphi_{\bar{z}} = i\bar{\alpha}\varphi$$

- So here: $b = 0$ and $a = i\bar{\alpha}$.

Similarity Principle:

- $q(z) = T_{\mathbb{C}}[g(z)] = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{g(\lambda)}{\lambda - z} d\xi d\eta \qquad \lambda = \xi + i\eta$ (29)

  $g(z) \in L_{p,2}(\mathbb{C}),\ p > 2$. $q(z) \in D_{\bar{z}}(\mathbb{C})$.

- $D_{\bar{z}}(\mathbb{C})$ − is a linear space of functions that have generalized derivative in the Sobolev sense with respect to $\bar{z}$.

- "*Similarity Principle*": Let ω(z) satisfy the generalized Cauchy-Riemann equation, then the following representation is valid:

$$\omega = \Phi e^{s} \tag{30}$$

- Where Φ(z) is analytic and $s(z) = \begin{cases} T_{\mathbb{C}}\left[a + b\frac{\bar{\omega}}{\omega}\right], if\ \omega(z) \neq 0\ \ z \in \Omega \\ T_{\mathbb{C}}[a + b], if\ \omega(z) = 0\ \ z \in \Omega \end{cases}$ (31)

- **Corollary1** (Carleman's theorem): A pseudoanalytic function, which does not vanish identically, has only isolated zeros.

Similarity Principle:

- From Similarity Principle follows that we can rewrite φ as:

$$\varphi(z) = \Phi e^{T_{\mathbb{C}}[a]} = \Phi(z) e^{-\frac{1}{\pi}\int_{\mathbb{C}} \frac{a(\lambda)}{\lambda - z} d\xi d\eta} = \Phi(z) e^{-\frac{i}{\pi}\int_{\mathbb{C}} \frac{\overline{\alpha}(\lambda)}{\lambda - z} d\xi d\eta} = \\ = \Phi(z) e^{-\frac{i}{2\pi}\int_{\mathbb{C}} \frac{(A_1 + iA_2)(\lambda)}{\lambda - z} d\xi d\eta} \\ = \Phi(z) e^{\frac{1}{4\pi}\int_{\mathbb{C}} \frac{(A_1 + iA_2)(\lambda)}{\lambda - z} d\lambda d\bar{\lambda}} \qquad (32)$$

❖ The same result is obtained using $\bar{\partial}$-Poincaré lemma.

- For it to be a solution of our problem, it needs to satisfy the following condition:

$$|\varphi(z)| = \left| \Phi(z) e^{\frac{1}{4\pi}\int_{\Omega} \frac{(A_1 + iA_2)(\lambda)}{\lambda - z} d\lambda d\bar{\lambda}} \right| \to 1 \quad as \ \ |z| \to \infty \qquad (33)$$

❖ Corollary1- if φ has zeros, they are isolated.

Behavior Near Zeros:

- Let $\varphi(z_k \in \Omega) = 0$, in the neighborhood of z_k

$$\Phi(z) = g(z)(z - z_k)^{n_k},$$

n_k- multiplicity of z_k. $g(z)$ is analytic function and $g(z_k) \neq 0$.
Furthermore, for φ we can write:

$$\varphi = g(z)(z - z_k)^{n_k} e^s = h(z)(z - z_k)^{n_k} = |h(z)| e^{iarg(h(z))} |z - z_k|^{n_k} e^{in_k arg(z - z_k)}$$

$h(z) = g(z)e^s$, $h(z_k) \neq 0$and we come to the result,
that all zeros of $\varphi(z)$ are at the same time singular points for the “phase” :

$$\theta(z) \equiv arg\big(h(z)\big) + n_k arg(z - z_k) \qquad (35)$$

Winding Number:

- Constructing a map $f\colon\ S^1 \rightarrow S'^1$, where S^1 is in $\mathbb{R}^2$, or equivalently in $\mathbb{C}$ and S'^1 is a set of values of $\varphi(z)$ when $|z| \rightarrow \infty$.
- $deg(f) = winding\ number = \frac{1}{2\pi i}\oint_{S^1} \frac{\partial_z f}{f}\, dz$ (36)
- Since : $f = \lim_{|z|\rightarrow\infty} \varphi(z)$, we can represent f as :
- $f = exp\{2i \sum_k arg(z - z_k)\}$ (37)
- $deg(f) = \frac{1}{2\pi i}\oint_{S^1} \frac{\partial_z f}{f}\, dz = \frac{1}{2\pi i}\oint_{S^1} \frac{1}{z - z_k}\, dz = \sum_{\substack{distinct \\ z_i}} n_i$ (38)
- $deg(f) = Winding\ number = \sum_{\substack{distinct \\ z_i}} n_i$

Vortex Number:

- when $|z| \to \infty : |\varphi| \to 1$; $D_A\varphi = d\varphi - iA\varphi \ \to 0 \ \Rightarrow \ d\varphi \to iA\varphi$.
  On and beyond a sufficiently large circle $|\varphi| = 1, d\varphi = iA\varphi$:
- $\partial_z\varphi dz + \partial_{\bar{z}}\varphi d\bar{z} = i\alpha\varphi dz + i\bar{\alpha}\varphi d\bar{z} \quad \to \begin{cases}(\partial_{\bar{z}} - i\bar{\alpha})\varphi = 0 \\ (\partial_z - i\alpha)\varphi = 0\end{cases}$
- Using (37) $:(\partial_z - i\alpha)\varphi = (\partial_z - i\alpha)exp\{2i\sum_k arg(z - z_k)\} = 0$

$$\alpha(z) = \partial_z 2\sum_k \arg(z - z_k) = -i\sum_k \frac{1}{z - z_k} \tag{39}$$

- For any $z \in \mathbb{C}$, $\alpha(z) = -i\mathcal{X}(z,\bar{z})\sum_k \frac{1}{z-z_k}$; $\quad \lim_{|z|\to\infty} \mathcal{X}(z,\bar{z}) = 1$
- $N = \frac{1}{2\pi}\int_{\mathbb{R}^2} F_A = \frac{1}{2\pi}\oint_{S^1(|z|\to\infty)} A = \frac{1}{2\pi}\left(-i\oint_{S^1(|z|\to\infty)} \sum_k \frac{1}{z-z_k}dz + i\oint_{S^1(|z|\to\infty)} \sum_k \frac{1}{\bar{z}-\overline{z_k}}d\bar{z}\right) =$

$$= \sum_{\substack{distinct \\ z_i}} n_i \tag{40}$$

Summary:

- After rewriting and minimizing the Yung-Mills-Higgss action, we get Bogomolny equations. Going to the complex variables, two of them together take a form of the Generalized Cauchy-Riemann equations for the pseudoanalytic functions of the first kind.
- Using The Similarity Principle we easily arrive to the useful representation for φ.
- From that, we immediately conclude that if φ has zeros, they must be isolated.
- Using a representation of φ we also arrive to its behavior near the zeros.
- Constructing a map $f\colon \quad S^1 \rightarrow S'^1$ we can compute its degree and see that the number is characterized by the sum of multiplicities of zeros for φ.
- Using the asymptotic behavior of the field, its covariant derivative and previous results we finally write formula the Vortex Number.

References:

[1] A.Jaffe, C.Taubes -Vortices And Monopoles. Birkhauser,1980

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[3] G.Akhalaia,G.Giorgadze, G.Gulagashvili – "The Analysis of Vortex Equations Using Methods of Generalized Analytic Functions" –Bull.TICMI, Vol.22, No.2, 2018, 135-141.

Thank you for your attention!