LATTICE METHODS FOR STRONGLY INTERACTING SYSTEMS

Lecture 2: Quantum Gauge Fields on a Lattice—A Primer

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JUST A RECAP OF THE LAST LECTURE

- Introduced Euclidean path integral and gave expression for expectation value of an operator O

$$\langle E_0|\hat{O}(\hat{x})|E_0\rangle = \frac{\int [d\boldsymbol{x}(t)]\, O(x) e^{-S[\boldsymbol{x}(t)]}}{\int [d\boldsymbol{x}(t)]\, e^{-S[\boldsymbol{x}(t)]}}$$

- Presented an algorithm to generate configurations with probability proportional to the Boltzmann factor

Metropolis-Hastings Algorithm

- Various properties of the system could be probed in the large time limit

WHAT YOU MIGHT HAVE NOTICED…

$$\begin{aligned}\langle E_0|\hat{O}(\hat{x})|E_0\rangle &= \frac{\int [d\boldsymbol{x}(t)]\, O(x) e^{-S[\boldsymbol{x}(t)]}}{\int [d\boldsymbol{x}(t)]\, e^{-S[\boldsymbol{x}(t)]}} \\ &= \lim_{t_f \gg t_i} \frac{\langle x_f, t_f|O(x)|x_i, t_i\rangle}{\langle x_f, t_f|x_i, t_i\rangle}\end{aligned}$$

Independent of $\boldsymbol{x_f}$ and $\boldsymbol{x_i}$?

$$\langle x_f|e^{-H(t_f-t_i)}|x_i\rangle = \langle x_f|U(t_f, t_i)|x_i\rangle \approx \left(\frac{m}{2\pi a}\right)^{N/2} \int_{-\infty}^{\infty} dx[1]\; dx[2]\; dx[3] \ldots dx[N-1] e^{-S_{lat}[x]}$$

IT'S REALLY STATISTICAL PHYSICS

set $x_f = x_i \equiv x[0]$ and integrate over all values of x[0]

PBCs

$$\int [dx] \to \left(\frac{m}{2\pi a}\right)^{(N+1)/2} \int_{-\infty}^{\infty} dx[0]dx[1]dx[2]dx[3]\ldots dx[N-1]$$

$$\begin{aligned}\langle E_0|\hat{O}(\hat{x})|E_0\rangle =& \frac{\int [d\boldsymbol{x}(t)]\, O(x) e^{-S[\boldsymbol{x}(t)]}}{\int [d\boldsymbol{x}(t)]\, e^{-S[\boldsymbol{x}(t)]}} \\ =& \frac{1}{Z}\,\mathrm{tr}\left[\hat{O} e^{-\beta H}\right]\end{aligned}$$

$$Z = \mathrm{tr}\left[e^{\beta H}\right]$$

HOW DO THINGS CHANGE WHEN DEALING WITH A QUANTUM FIELD THEORY?

- Recall that for field theories, degrees of freedom are represented as fields, e.g.

Quantum Mechanics $\boldsymbol{x}(t) \mapsto \phi(\boldsymbol{x}, t) = \phi(x)$ Quantum Field Theory

- Field theories are defined by their Lagrangian—no problem, this is what we've been using all along!

Quantum Mechanics $\mathcal{L}[\dot{\boldsymbol{x}}(t), \boldsymbol{x}(t)] \mapsto \mathcal{L}[\dot{\phi}(x), \phi(x)]$ Quantum Field Theory

LET'S TURN OUR 1-D HO INTO A FIELD THEORY!

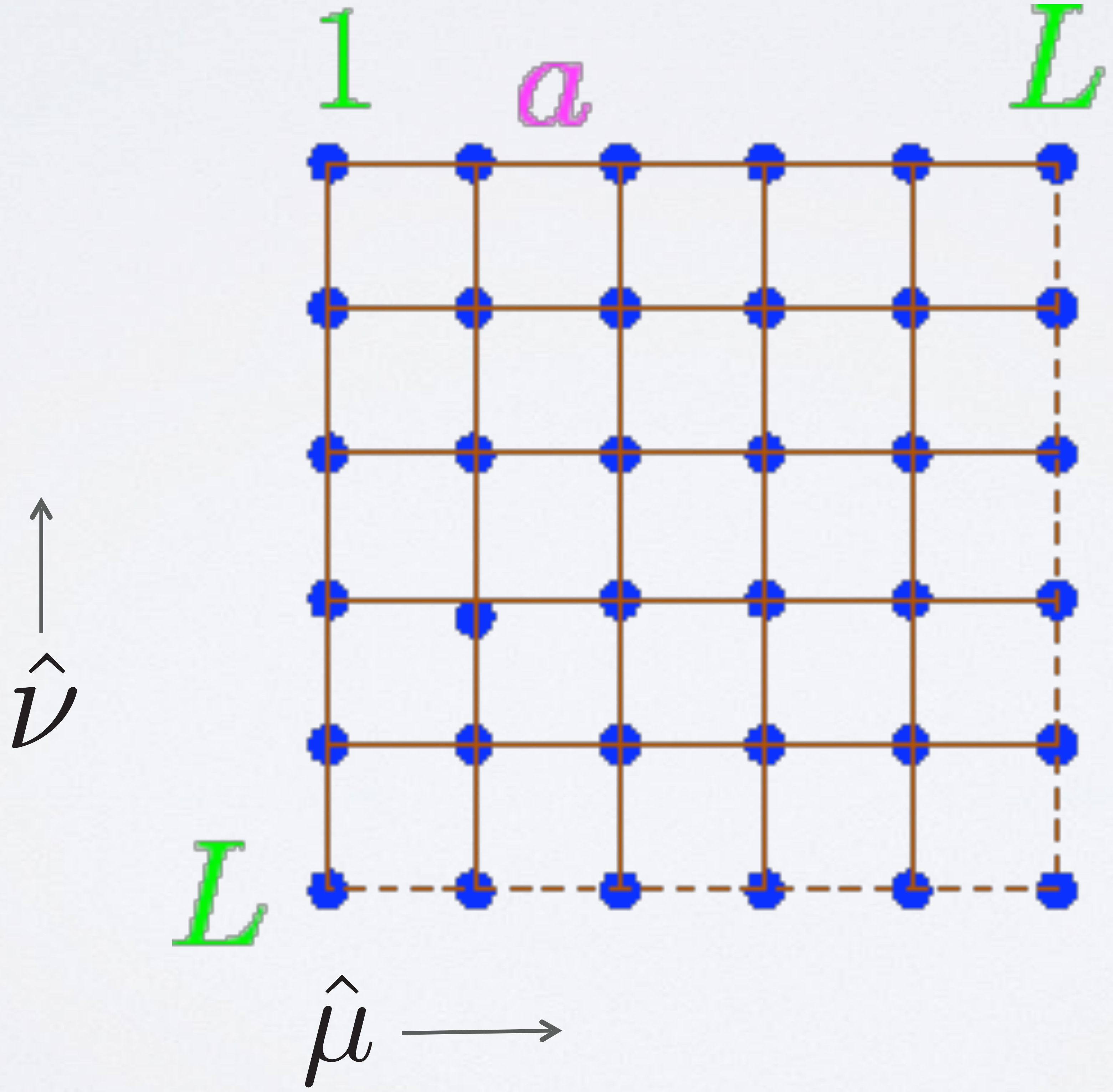
$$\frac{1}{2}\left(m\dot{x}^2 + m\omega^2 x^2\right) + \lambda m^2 \omega^3 x^4$$

$$\mapsto \frac{1}{2}\dot{\phi}(x)^2 + \frac{1}{2}(\nabla\phi(x))^2 + \frac{m^2}{2}\phi(x)^2 + \lambda\phi(x)^4$$

phi-4 theory (Euclidean space)

AS BEFORE, WE DISCRETIZE THE COORDINATES

- Not only do we discretize the time coordinate, we discretize the spatial coordinates as well since the fields also depend on them



- Under this discretization, for example, we can approximate the Laplacian in the following manner:

$$\sum_{\mu=1}^{4} (\partial_\mu)^2 \phi(x) \approx \frac{1}{a^2} \sum_{\mu=1}^{4} (\phi(x + \hat{\mu}a) + \phi(x - \hat{\mu}a) - 2\phi(x))$$

- The action, defined as

$$S = \int d^4x \ \mathcal{L}[\dot{\phi}(x), \phi(x)]$$

can be approximated by discrete sums and differences over the field $\boldsymbol{\Phi}(\boldsymbol{x})$ at the discrete lattice points

THERE ARE SOME FUNDAMENTAL DIFFERENCES, HOWEVER

- Generally, we are interested in calculating expectation values of various operators with respect to the ground state, but here the ground state of a field theory is simply the vacuum,

Quantum Mechanics

$$\underset{\text{“vacuum state”}}{|E_0\rangle \mapsto |0\rangle}$$

Quantum Field Theory

- What are the quantized excitations of a field theory?

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Quantum Mechanics $|E_0\rangle \mapsto |0\rangle$ Quantum Field Theory

"vacuum state"

- What are the quantized excitations of a field theory?

Answer—The excitations of a field correspond to particles

THE PATH INTEGRAL FORMALISM IS AN INDISPENSIBLE TOOL FOR QF THEORISTS

- Expectation values are just as you would expect:

$$\langle 0|\hat{O}[\phi]|0\rangle \equiv \langle \hat{O}[\phi]\rangle = \frac{\int [d\phi]\, O[\phi]e^{-S[\phi]}}{\int [d\phi]\, e^{-S[\phi]}}$$

- We can extract information about the spectrum of particles (i.e. excitations) associated with a field by using various operators that act on the vacuum state

$$\langle \hat{\phi}(t')\hat{\phi}(t)\rangle = \frac{\int [d\phi]\, \phi(t')\phi(t)e^{-S[\phi]}}{\int [d\phi]\, e^{-S[\phi]}}$$

$$= \sum_n \mathbb{C}_n e^{-E_n(t'-t)} \xrightarrow{t'\gg t} \mathbb{C}_0 e^{-E_0(t'-t)}$$

"Correlator"

Problem #1 (moderate):

We can also extract information about excited states in our quantum mechanical 1-D oscillator problem from the previous lecture.

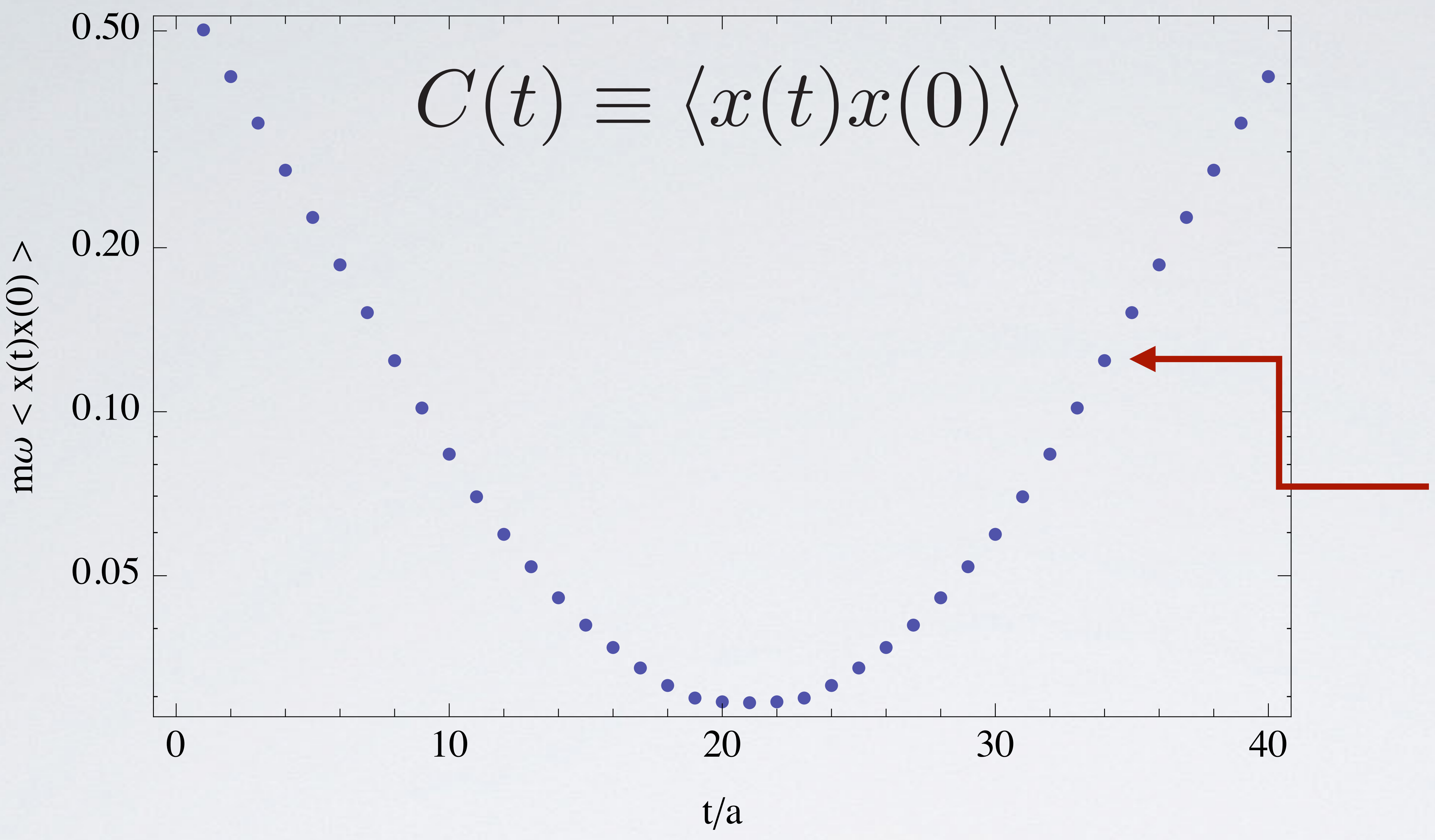
First show that the following relation holds:

$$\langle x(t')x(t)\rangle \xrightarrow{t'\gg t} \mathbb{C}_0 e^{-(E_1-E_0)(t'-t)}$$

What is $\boldsymbol{C_0}$?

Use the ensemble of configurations that you generated from the previous lecture to extract this energy shift. Use periodic boundary conditions in the time direction.

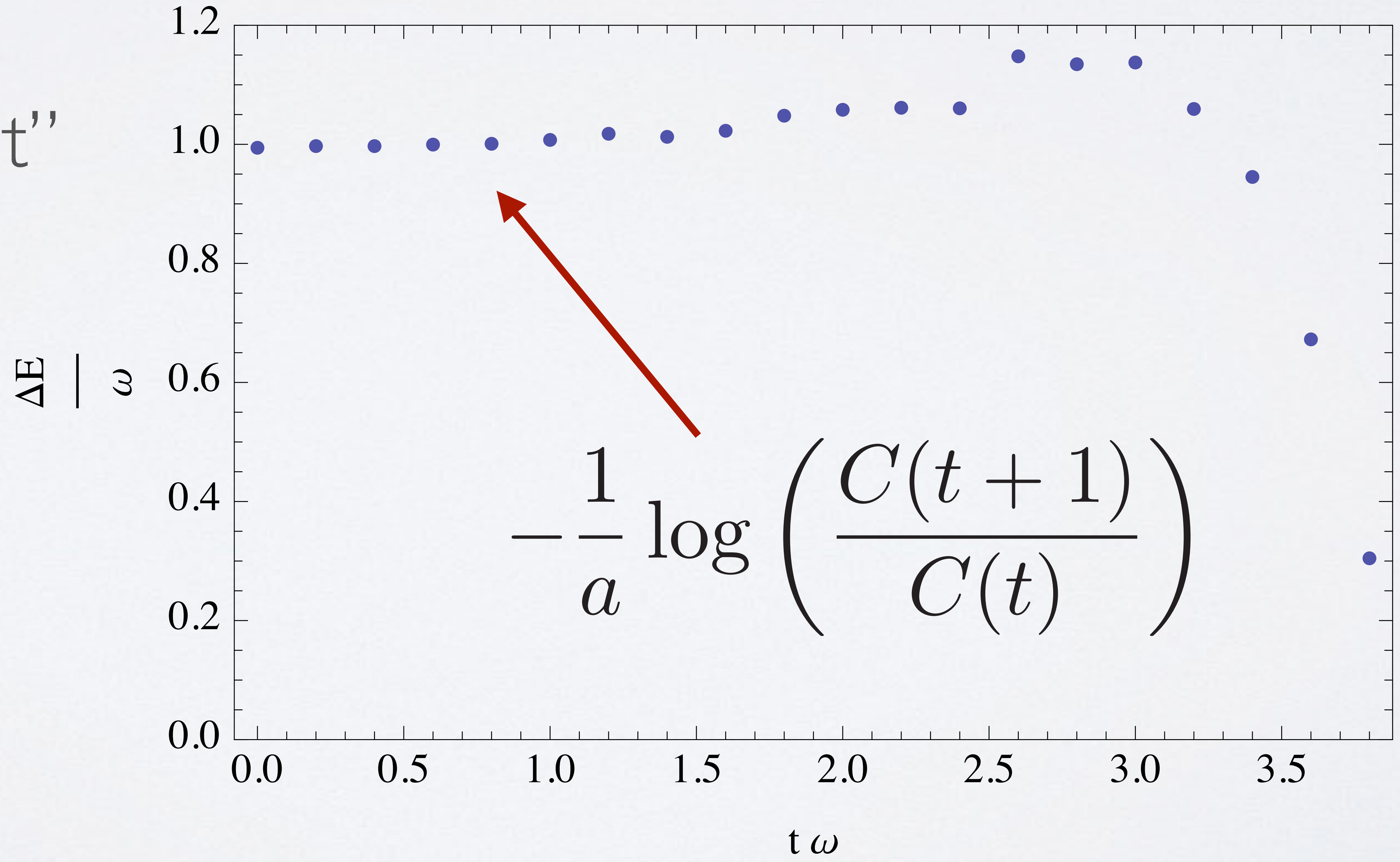
Correlator



$$C(t) \equiv \langle x(t)x(0) \rangle$$

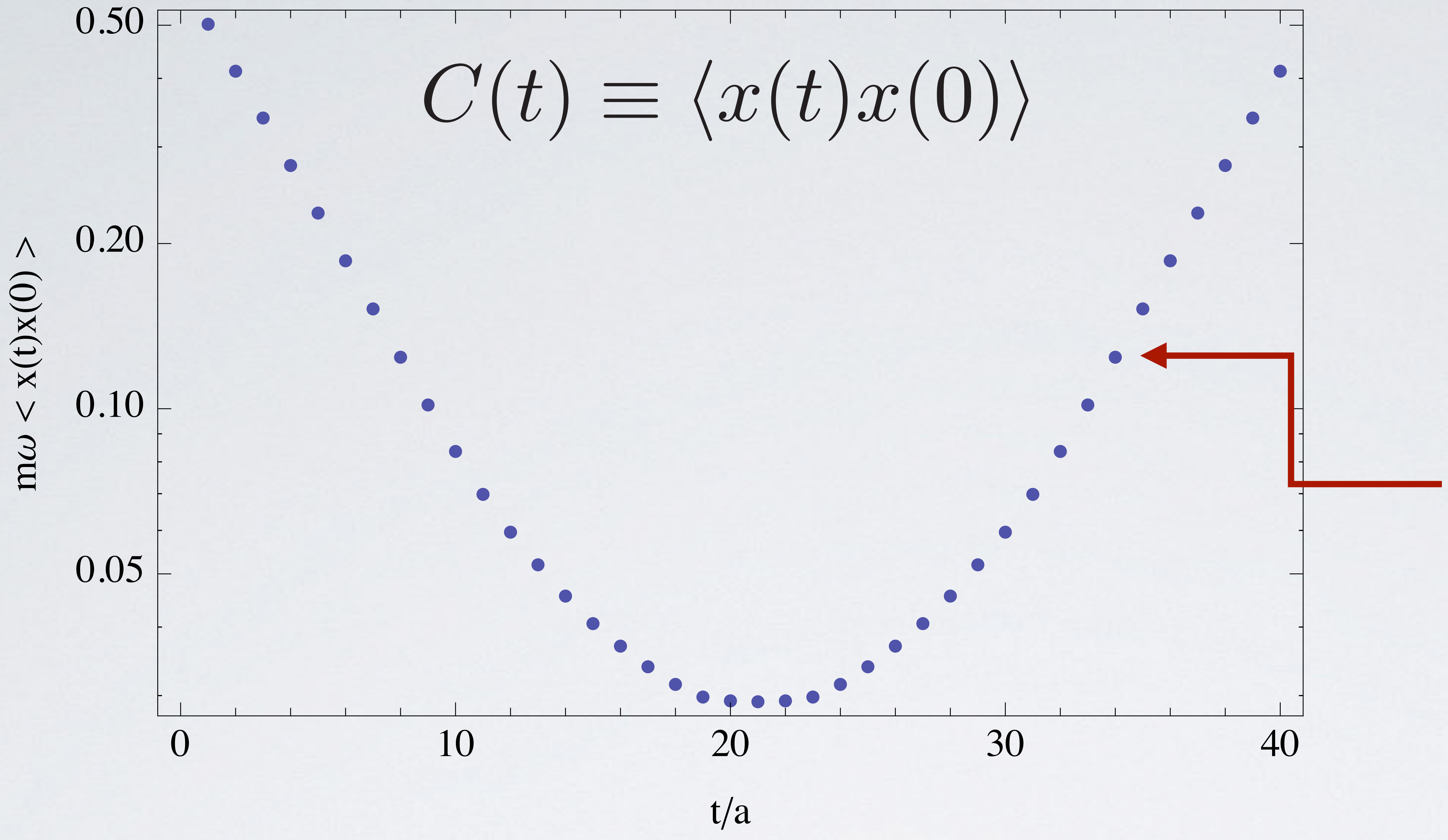
Periodic boundary conditions

"Effective Mass Plot"



$$-\frac{1}{a}\log\left(\frac{C(t+1)}{C(t)}\right)$$

Correlator

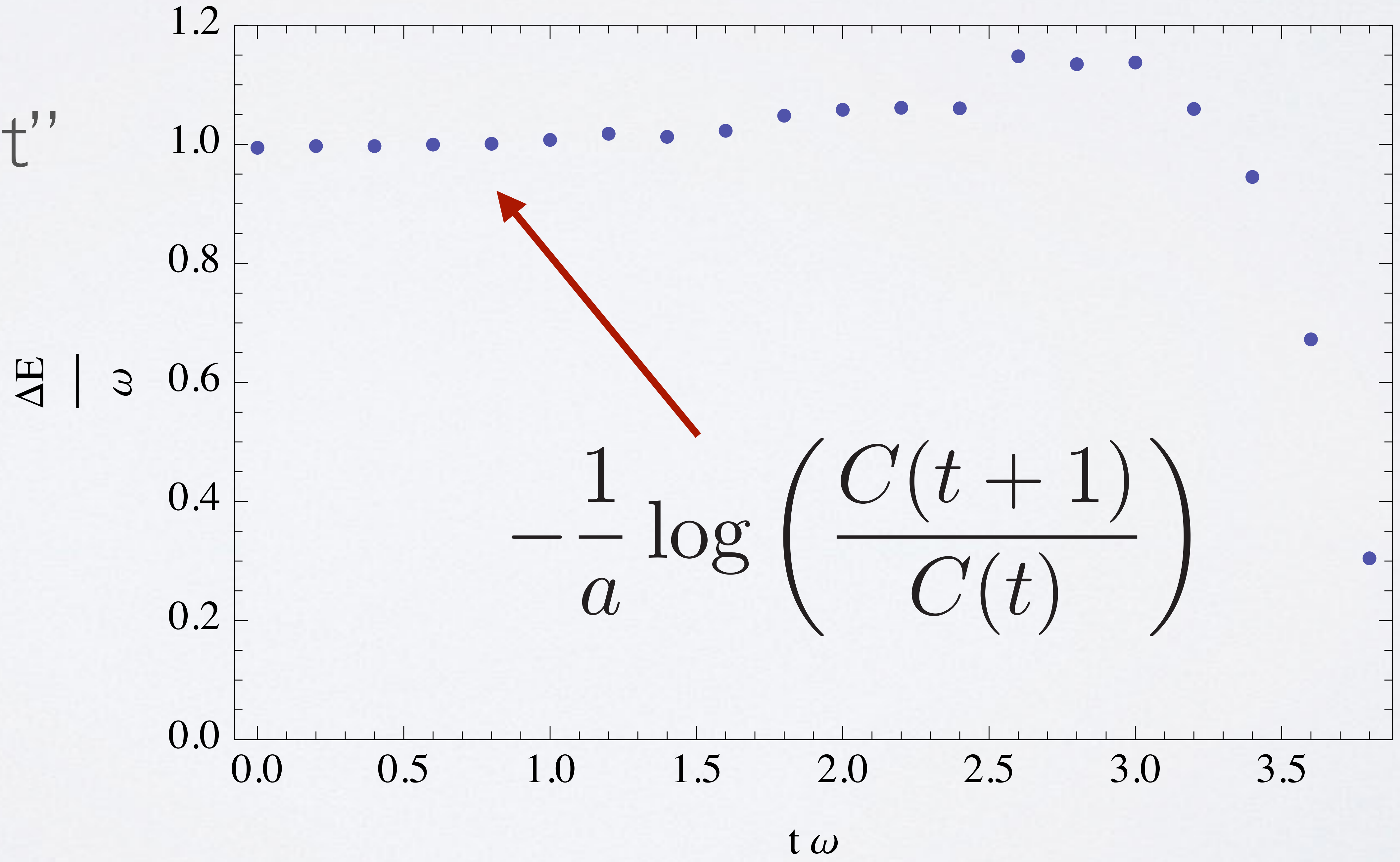


$$C(t) \equiv \langle x(t)x(0) \rangle$$

Periodic boundary
conditions

Hey Tom! Where are your uncertainties?

“Effective Mass Plot”



$$-\frac{1}{a}\log\left(\frac{C(t+1)}{C(t)}\right)$$

SOME DISCUSSION POINTS REGARDING QUANTUM FIELD THEORIES ON A LATTICE

- Quantum field theories, from a mathematical point of view, are ill-behaved in the ultra-violet limit—i.e. they suffer from infinities
  - Many people have contributed to "removing" these infinities (e.g. Feynman, Dyson, etc.)
  - Introduce cutoff; express observables in terms of physical parameters, instead of bare parameters; remove cutoff dependence with relevant counter-terms, i.e. renormalization

- By putting QFTs on a discretized lattice, we are already introducing a cutoff: the lattice spacing α

- Removing lattice discretization effects, as well as taking the continuum limit ("renormalization"), is a tricky business!

- QFTs also respect certain symmetries
  - Lorentz invariance
  - Gauge symmetries

- These symmetries are important since conservation laws are derived from them

- How/Can these symmetries be preserved on a lattice?
  - Lorentz invariance? 🚫 But we understand its effects
  - Gauge symmetries? Yes, but does require some extra work—come back to this later

- Fermions obey particular statistics—How is this captured in our numerical path integral?
  - Good question! I will answer this question by avoiding it (for the time being)!

A FIRST LOOK AT QCD—BUT WITHOUT FERMIONS

- The QCD Lagrangian w/o fermions is sometimes referred to as Yang-Mills theory under SU(3):

$$\mathcal{L}_{ym} = \frac{1}{4}\,\mathrm{Tr}\left(F_{\mu\nu}(x)F_{\mu\nu}(x)\right)$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig\,\underbrace{[A_\mu, A_\nu]}$$

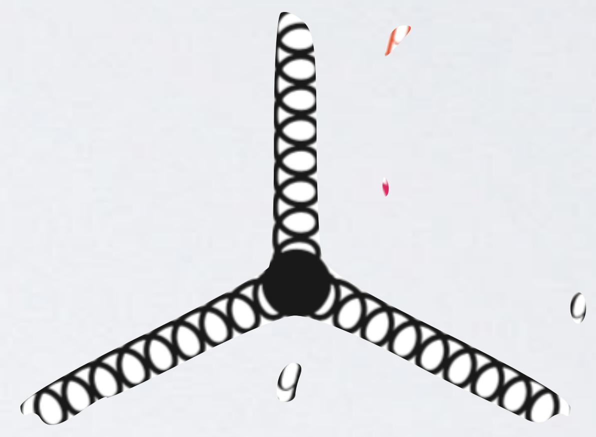


Fig 2

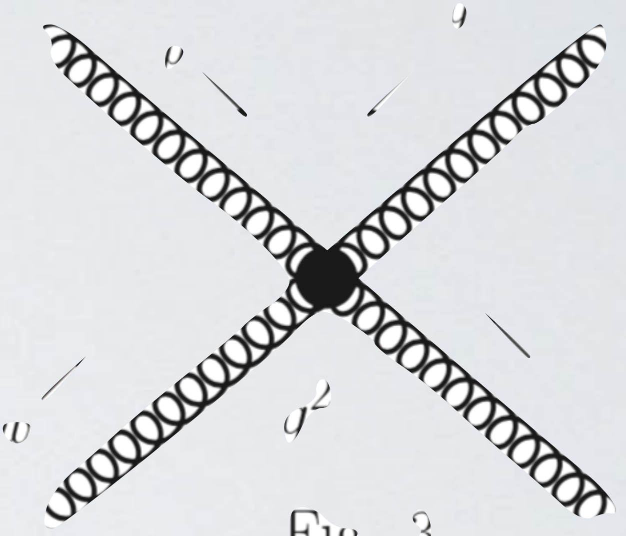


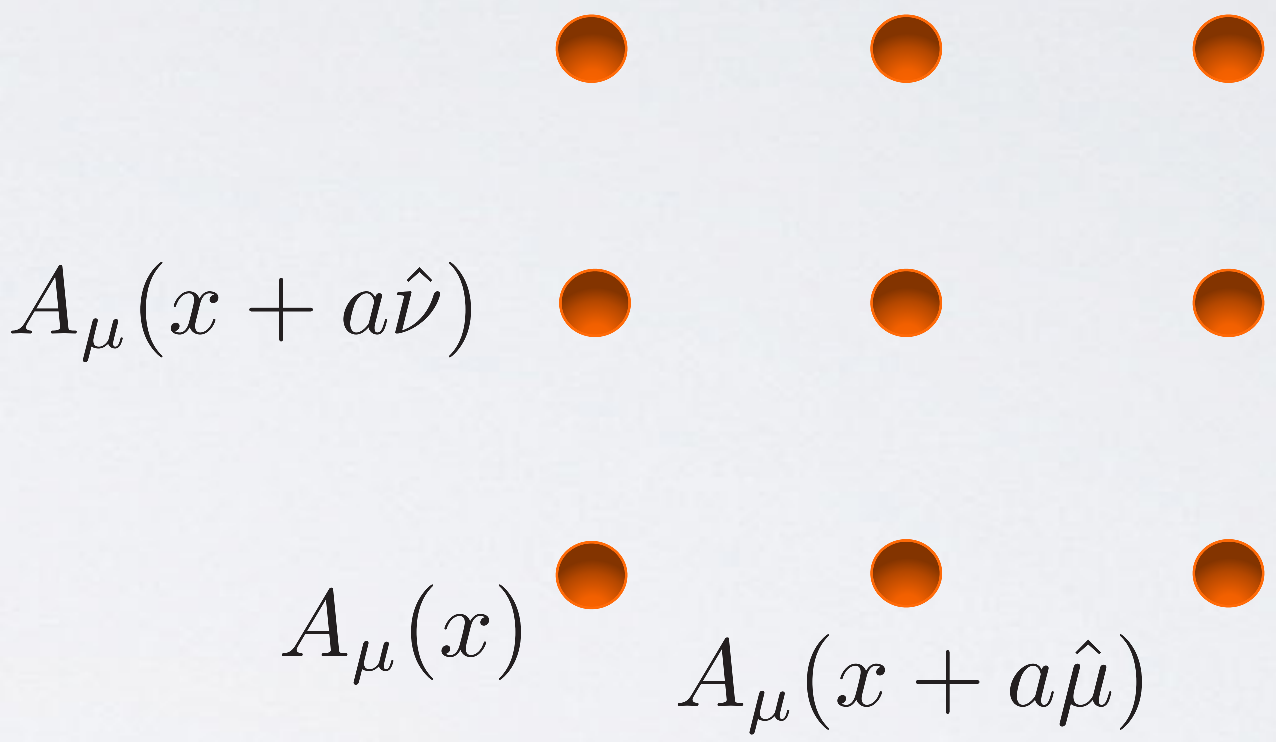
Fig 3

The gauge fields are actually 3x3 Hermitian matrices that are part of the Lie algebra of SU(3)

This commutator is not zero ➔ The gauge fields interact with each other! A consequence of the ***non-Abelian*** nature of SU(3).

SO LET'S DISCRETIZE THIS THEORY

- A naïve first step would be to just define values of the vector gauge field $A_\mu(x)$ at each lattice point



But this is really bad for the following reasons…

- A defining feature of QCD is that it is that the theory is invariant under SU(3) rotations—In other words, the Lagrangian doesn't change when

$$F_{\mu v}(x) \to \Omega(x) F_{\mu v}(x) \Omega^\dagger(x)$$

position-dependent SU(3) matrix

- This symmetry keeps the number of input parameters to QCD to a minimum and conserves color charge
    - From a numerical standpoint, we don't want many parameters in our simulations since they potentially all have to be tuned

- Unfortunately, finite differences and sums of the vector gauge fields **DO NOT** preserve this symmetry under the Lagrangian!

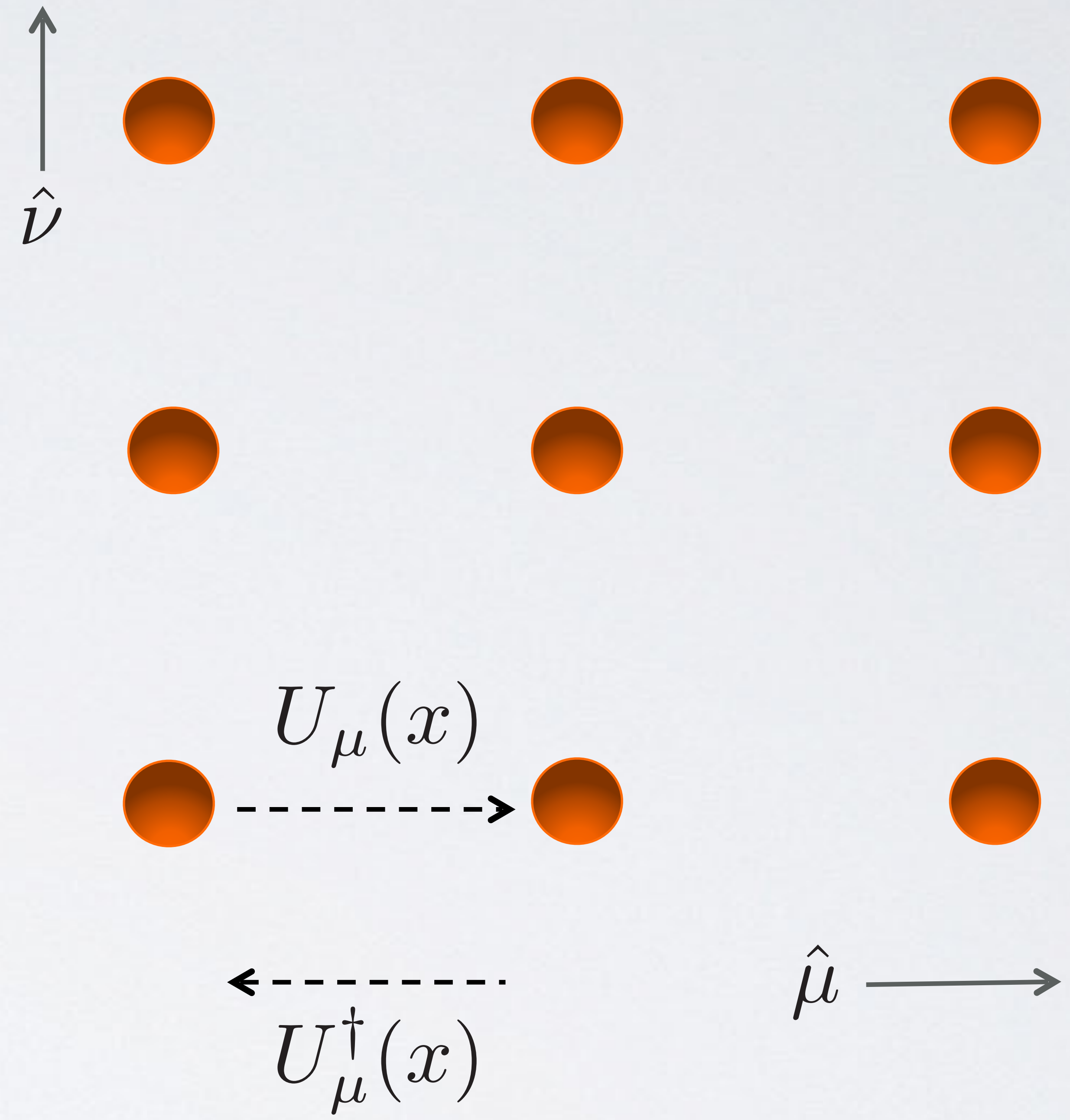
- The upshot: for gauge fields, we need an alternative discretization scheme that preserves gauge symmetry

PRESERVING GAUGE INVARIANCE ON A LATTICE

Kenneth Wilson

- Let us define the following objects:

$$U_\mu(x) = e^{-i\int_x^{x+a\hat{\mu}} gA_\mu(x')dx'} \approx e^{-igaA_\mu(x)}$$



- What are these objects?
  - They are simply 3x3 unitary matrices with det = 1, i.e. SU(3) matrices
  - They depend on position

- They do not reside at lattice points, but between them
- "link variables"

CONSTRUCTING GAUGE INVARIANT OBJECTS USING LINK VARIABLES

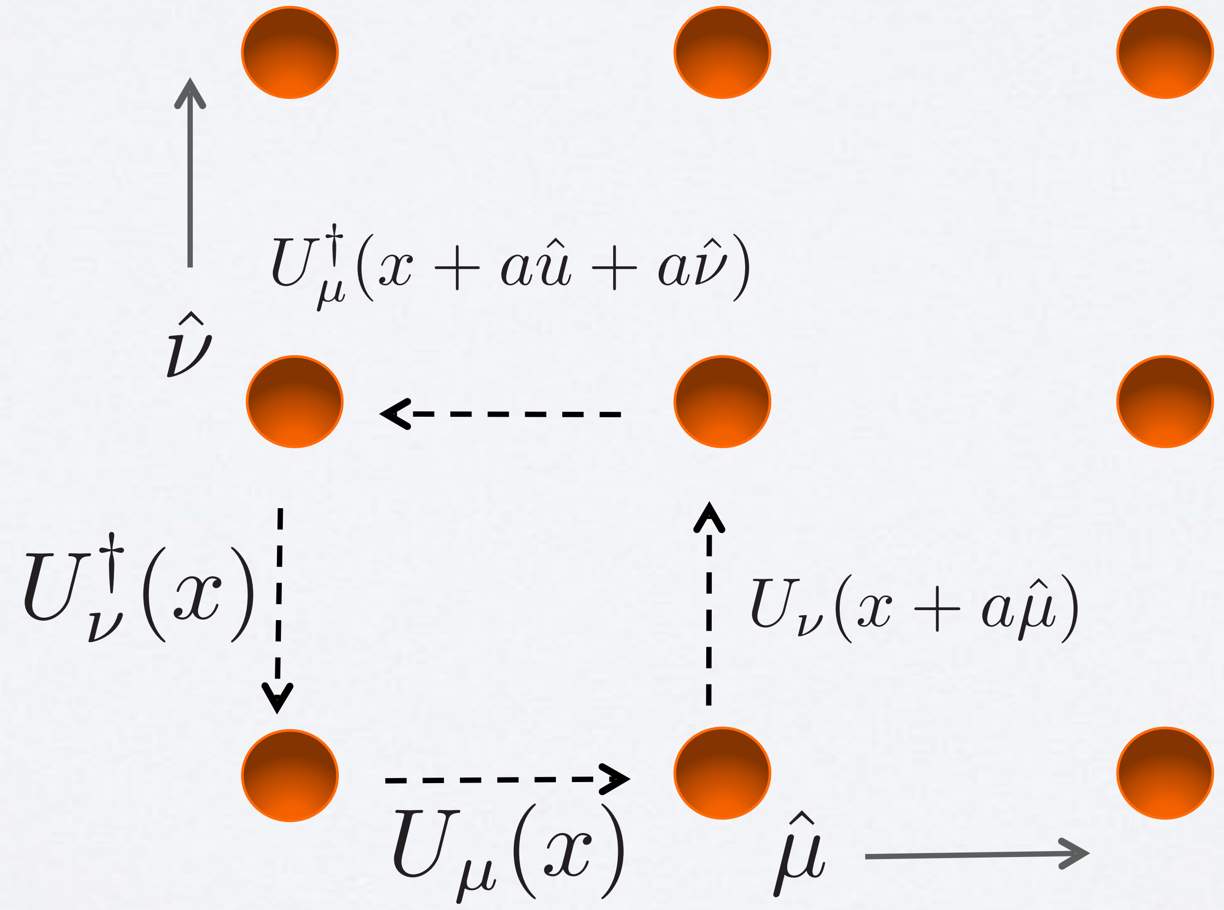
- How do these link variables transform under SU(3) rotations?

$$U_\mu(x) \rightarrow \Omega(x) U_\mu(x) \Omega^\dagger(x + a\hat{u})$$

- The simplest gauge invariant quantity that one can construct is the "plaquette"

$$P_{\mu\nu}(x) = \frac{1}{3} \mathrm{Re}\,\mathrm{Tr}\left[U_\mu(x) U_\nu(x + a\hat{\mu}) U_\mu^\dagger(x + a\hat{u} + a\hat{\nu}) U_\nu^\dagger(x)\right]$$

$$U_\mu(x) = e^{-igaA_\mu(x)}$$



AND FINALLY, THE ACTION…

- It turns out that the original action can be expressed in terms of the plaquettes

$$\begin{aligned} S_{ym} &= \int d^4x \ \frac{1}{4} \, \mathrm{Tr}\left[F_{\mu\nu}(x) F_{\mu\nu}(x)\right] \\ &= \frac{6}{g^2} \sum_{x,\mu>\nu} \left(1 - P_{\mu\nu}(x)\right) + O\left(a^2\right) \end{aligned}$$

“Wilson action”

Problem 1.5: prove this!

- Since the plaquettes are gauge invariant, our discretized action is gauge invariant!
  - It turns out that correction terms are gauge invariant also!

SO HOW DO WE GENERATE A CONFIGURATION OF THESE GAUGE FIELDS?

Procedure to generate $\{U\}_{n+1}$ from $\{U\}_n$:

```
def updateGaugeFields( U, beta, mu, nT, nX ):
    global num_of_updates, num_of_accepts

    for t in xrange( nT ):  # loop through all gauge links
      for x in xrange( nX ):
         for y in xrange( nX ):
            for z in xrange( nX ):
               for u in xrange( 4 ):
```

Loop through all links j

At link j, change $U(x)$ to $U(x)\ M$, where M is a random SU(3) matrix

```
                  Gamma_u = actions.calc_Gamma_u( [t,x,y,z], U , u , nT, nX )

                  for j in xrange( 10 ): # at each site update 10 times
                     num_of_updates += 1
                     old_U = U[t][x][y][z][u]
                     U[t][x][y][z][u] = dot( generateSU3_matrix(mu), U[t][x][y][z][u] )

                     dS = actions.deltaS_Wilson( U[t][x][y][z][u], old_U , beta, Gamma_u )
```

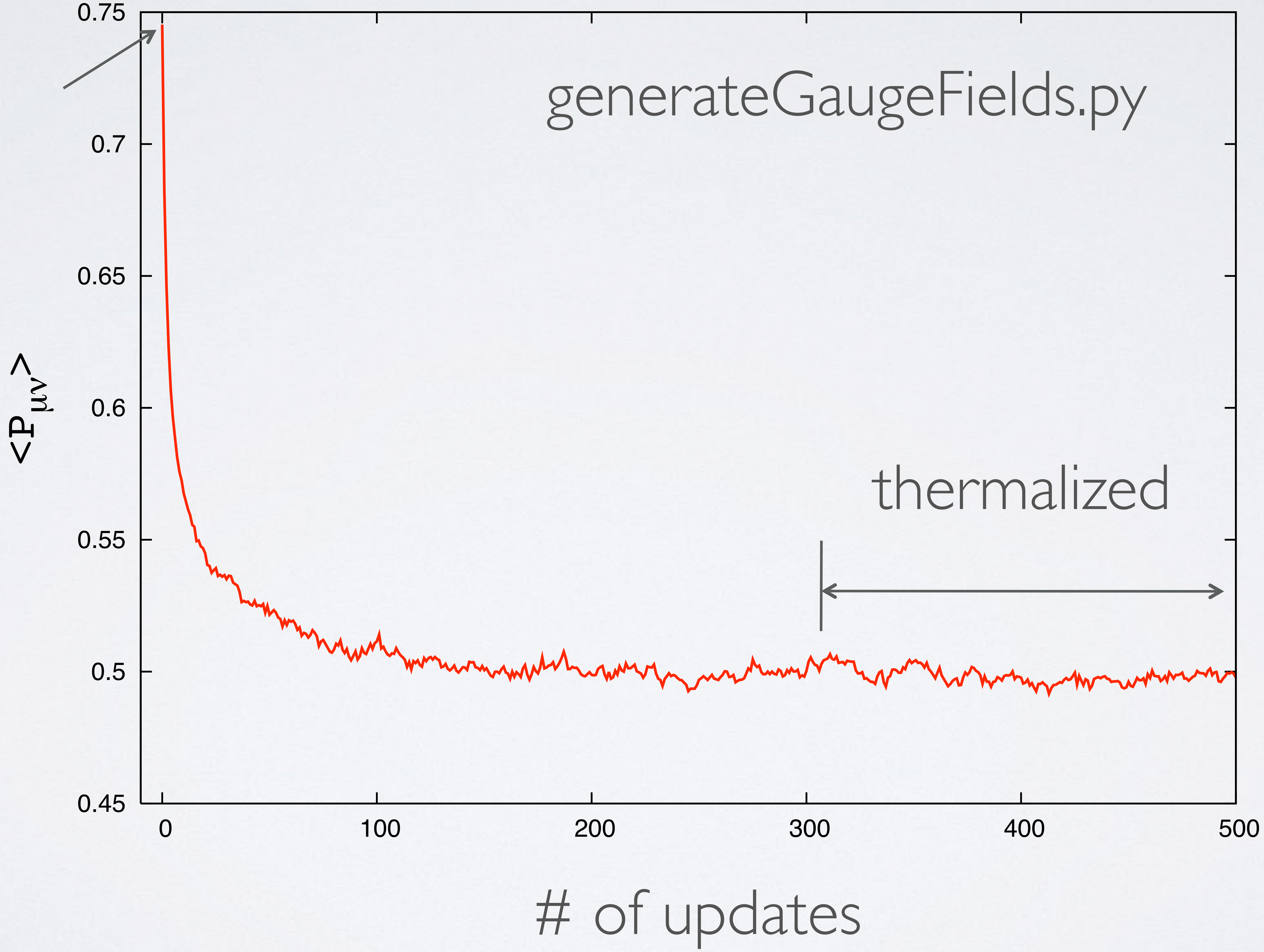
Replace $U(x) \rightarrow U(x)\ M$ and compute the change in action ΔS

```
                     if dS > 0 and exp(-dS) < uniform(0,1):
                        U[t][x][y][z][u]  = old_U            # don't accept change
                     else:
                        num_of_accepts += 1     # tally acceptance
                        actions.actionS += dS    # update action
```

If $\Delta S < 0$, accept the new value of $U(x)$and continue to link j+1

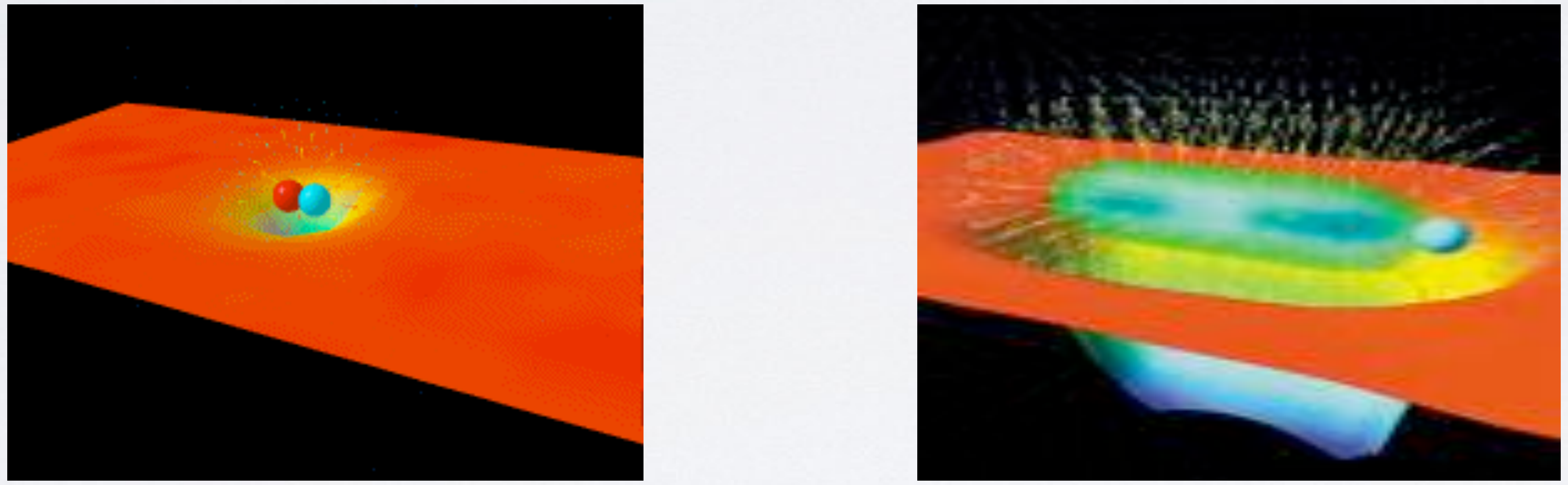
If $\Delta S > 0$, sample a number ρ uniformly distributed from 0 to 1. If exp(-ΔS) >ρ accept the new value of $U(x)$, otherwise reject change. Continue to link j+1

EXAMPLE OF THERMALIZING GAUGE FIELD CONFIGURATIONS



CONFINEMENT WITHIN YANG-MILLS THEORY

- One of the great hallmarks of QCD is that it exhibits confinement: there is never a lonely quark in the universe



Indicative of a linear potential at large separations

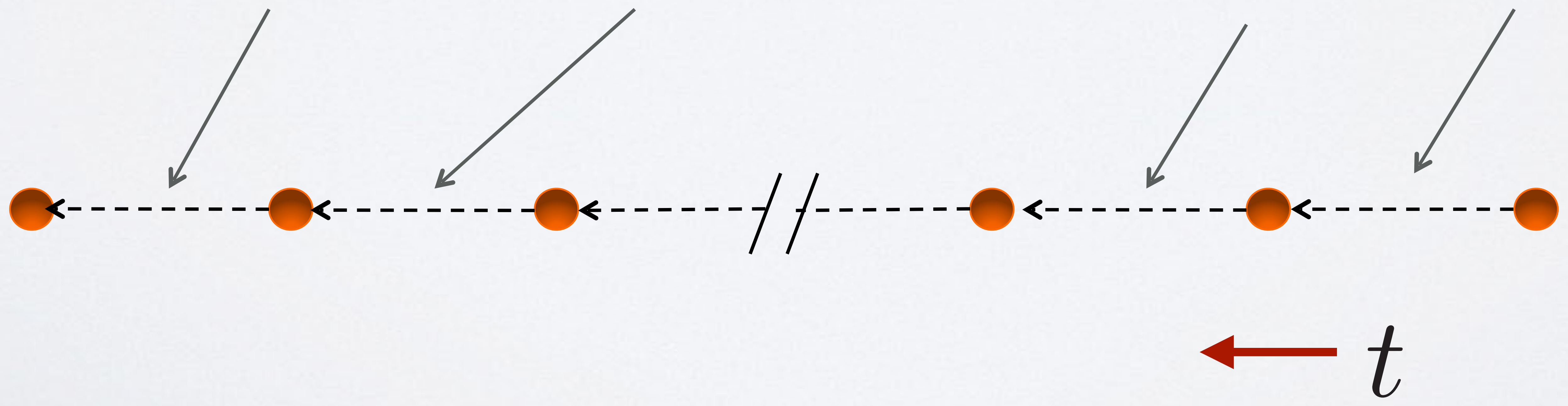
- Our theory of QCD w/o fermions already shows this feature!

HEAVY (INFINITE) QUARK POTENTIALS

- We can insert an infinitely massive quark and anti-quark within our soup of gauge fields and compute the potential between them as a function of separation distance

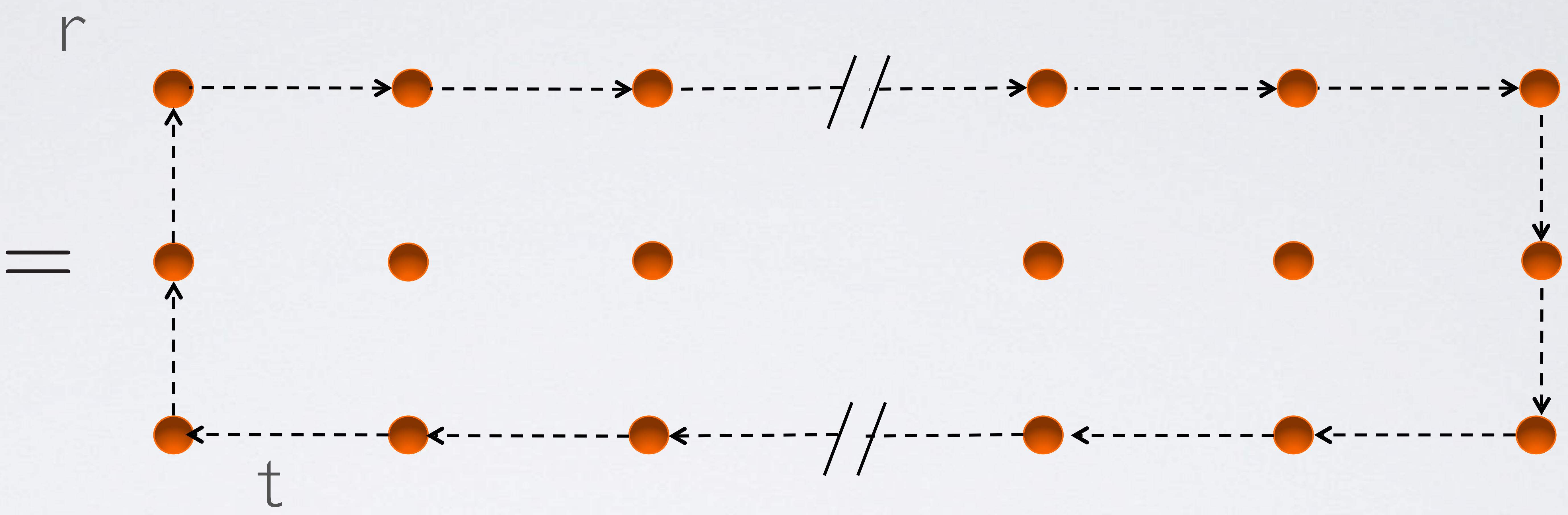
- Because the quarks are infinitely massive, they do not propagate spatially, i.e. they stay put

$$G(x,t) \approx U^\dagger(x,t-a)U^\dagger(x,t-2a)\cdots U^\dagger(x,a)U^\dagger(x,0)$$



- The time-propagation from 0 to T of an infinitely massive quark/anti-quark pair separated by distance R can be ascertained by looking at expectation values of the Wilson loop of dimension $R \, x \, T$

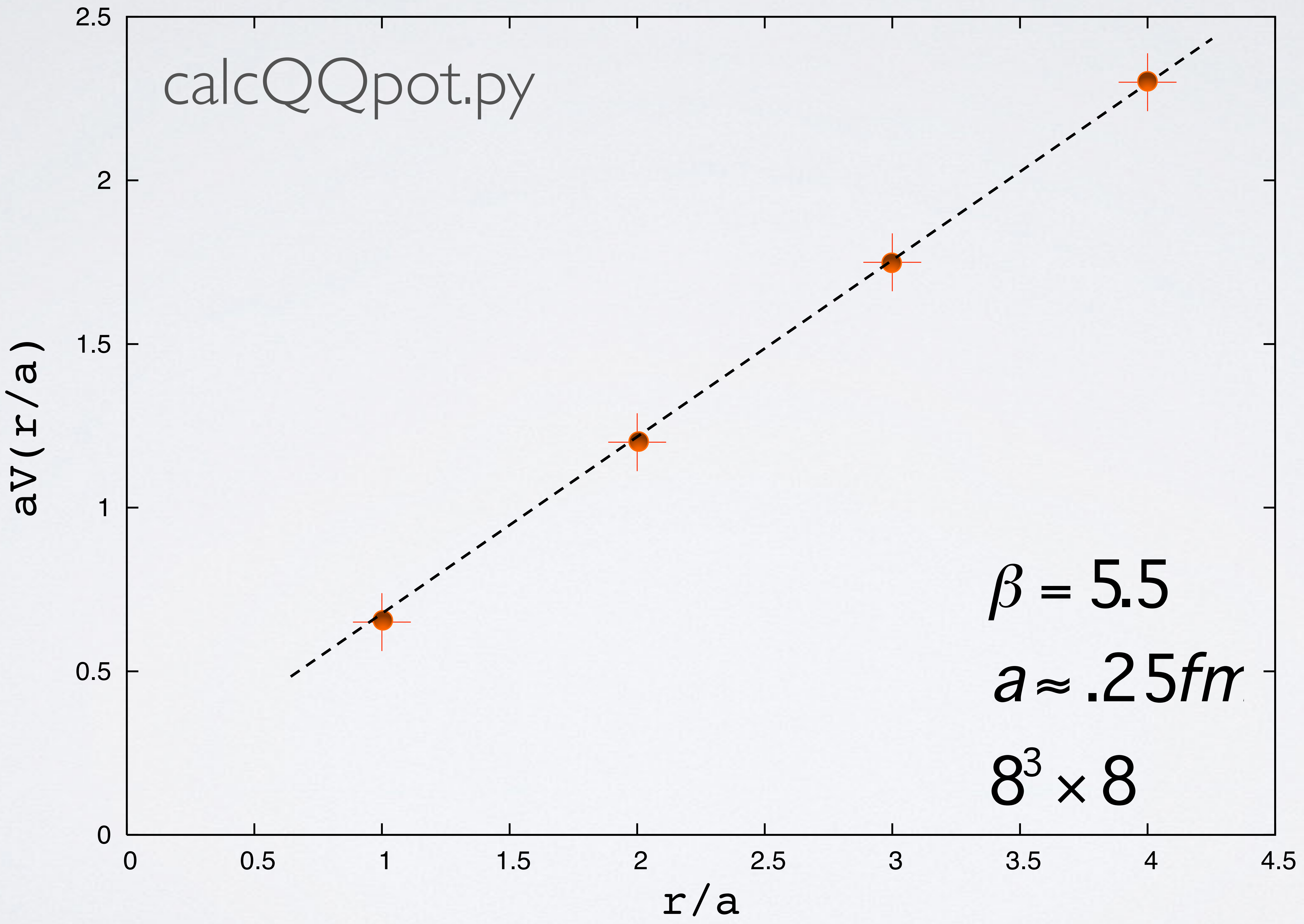
$$W(R=2a,T)=$$



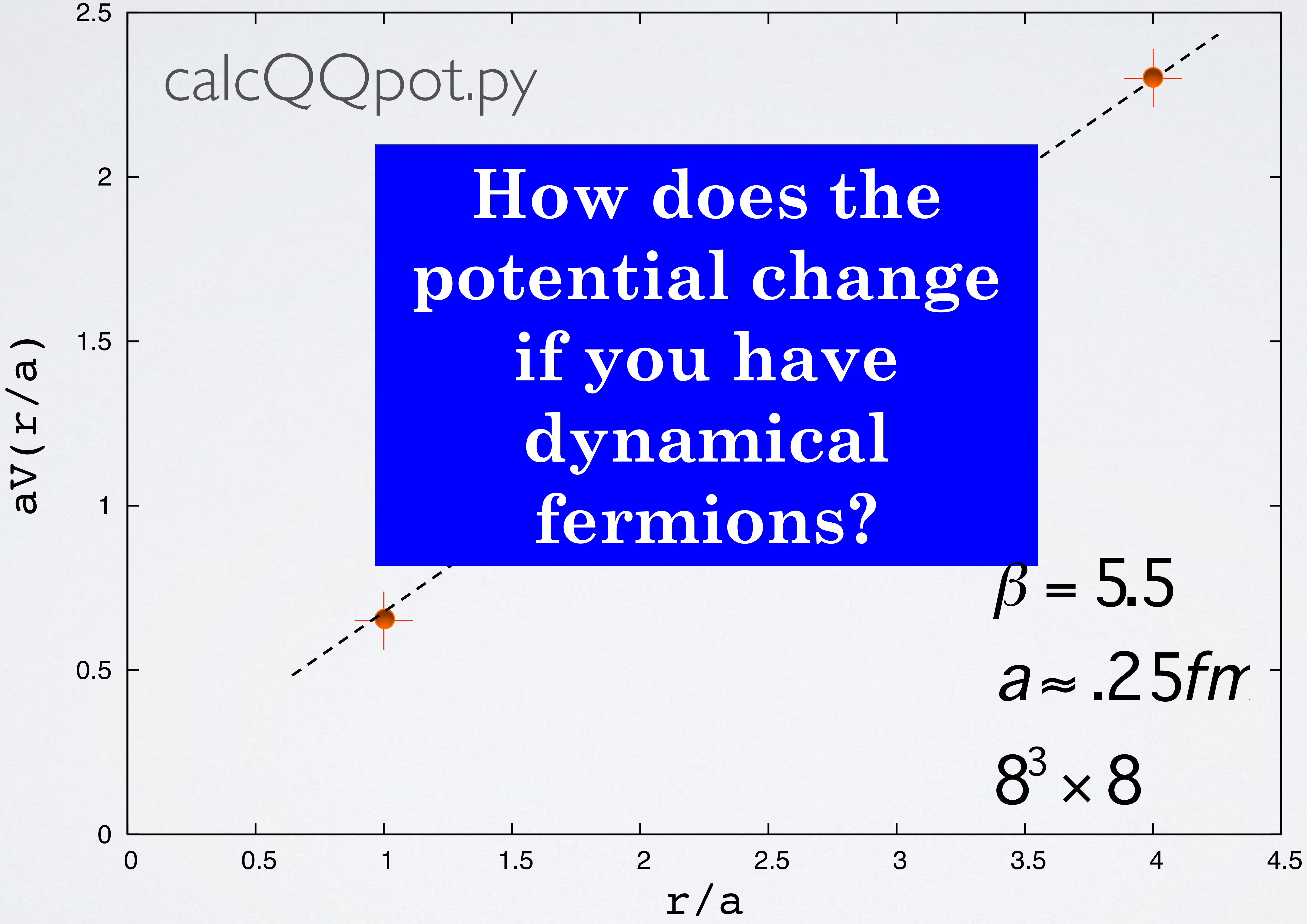
- The expectation value of the Wilson loop at large times behaves as

$$\langle \mathrm{Re}\,\mathrm{tr}\, W(R,T)\rangle \xrightarrow{T\gg 1} Ce^{-aV(R)T}$$

MY RESULTS FOR THE POTENTIAL



MY RESULTS FOR THE POTENTIAL



How does the potential change if you have dynamical fermions?

$\beta = 5.5$

$a \approx .25fm$

$8^3 \times 8$

Problem #2 (easy):

Download the python routines that generate gauge ensembles and calculate QQb-potential. Play with the codes to try to make sense of the concepts presented in this lecture.

Some things to note:

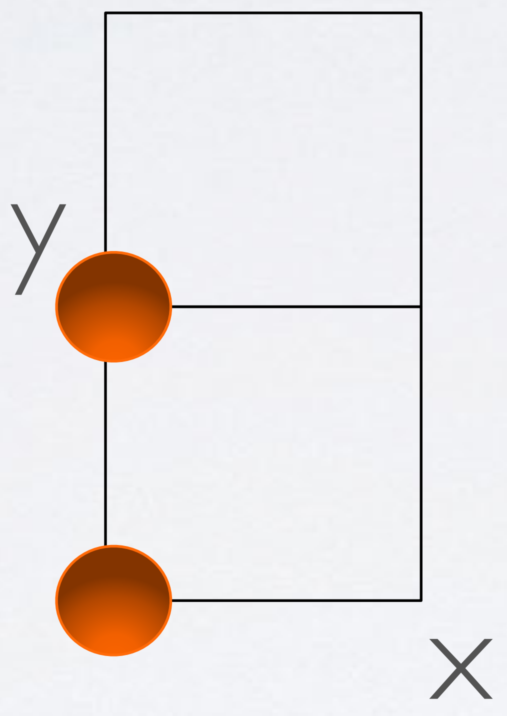
If you run ***generateGaugeFields.py***, the default option is to start from a "hot" (i.e. random) configuration. The configurations will be stored in the folder ***cnfgs***.

The routine ***calcQQpot.py*** needs thermalized configurations to perform measurements.

In the python routines, each configuration is stored as a list of 3x3 arrays: U[x][y][z][t][mu], where mu goes from 0 to 3.

FOR THE MASOCHIST IN YOU…

- The *calcQQpot.py* routine I've given you only calculates separation distances along axis lines, e.g.



- Try modifying the routine to make measurements along diagonals (be careful of boundary conditions!)

LET'S ADD FERMIONS!!!

- Hold your horses…
- Two rules in physics:
  - Don't drink and derive
  - Be careful when mixing fermions with computers

Fermions and Computers–why can't they get along?

- Fermions obey anti-commutation relations:

$$\{a_\lambda, a^\dagger_\mu\} = \delta_{\lambda,\mu} \ .$$

- Bosons, on the other hand, *commute*:

$$[a_\lambda, a^\dagger_\mu] = \delta_{\lambda,\mu} \ .$$

    - Fortunately, regular c-numbers also commute!
    - So we can represent bosonic fields with c-numbers.

- But we can calculate slater determinants, or explicitly anti-symmetrize fermionic wavefuntions, etc . . . But this hits the "Curse of Dimensionality"
- Calculations in a thermal bath: Grand-Canonical ensemble: Particle number not conserved (in principle infinite)

Let's first talk about bosonic states

Bosonic case:

$$[a_\lambda, a_\mu^\dagger] = \delta_{\lambda,\mu}, \quad [a_\lambda, a_\mu] = [a_\lambda^\dagger, a_\mu^\dagger] = 0 \; .$$

$$a_j^\dagger|0\rangle = |j\rangle$$ (*j* is combined index for all degrees of freedom)

$$a_j|j\rangle = |0\rangle$$ ($|0\rangle$ is the "vacuum state")

$$(a_j^\dagger)^n|0\rangle = |n_j\rangle$$ (*n*- particles each w/ *j* quantum number(s))

Application of $a^\dagger$ or a moves you throughout the *Fock* space:

$$\mathcal{F} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \mathcal{H}_{\Lambda_{max}}$$

$a^\dagger$ or a provide a basis for all operators in the Fock space

$\Longrightarrow$ can define states that span entire Fock space!

Bosonic coherent state

$$|\phi\rangle \equiv e^{\sum_i \phi_i a_i^\dagger}|0\rangle \quad \forall\ \phi_i \in \mathbb{C}$$

$$\begin{aligned}
&= \sum_{n_1,n_2,\dots} \frac{(\phi_1 a_1^\dagger)^{n_1}}{n_1!}\frac{(\phi_2 a_2^\dagger)^{n_2}}{n_2!}\cdots|0\rangle \\
a_1|\phi\rangle &= \sum_{n_1,n_2,\dots} \frac{a_1(\phi_1 a_1^\dagger)^{n_1}}{n_1!}\frac{(\phi_2 a_2^\dagger)^{n_2}}{n_2!}\cdots|0\rangle \\
&= \phi_1 \sum_{n_1',n_2,\dots} \frac{(\phi_1 a_1^\dagger)^{n_1'}}{n_1'!}\frac{(\phi_2 a_2^\dagger)^{n_2}}{n_2!}\cdots|0\rangle \\
&= \phi_1|\phi\rangle \qquad \text{(similarly for } \langle\phi|a_k^\dagger = \langle\phi|\phi_k^*)
\end{aligned}$$

In general, $a_i|\phi\rangle = \phi_i|\phi\rangle \implies |\phi\rangle$ is an eigenstate of a_i w/ eigenvalue ϕ_i.

Note: $[a_i, a_j]|\phi\rangle = 0 \implies [\phi_i, \phi_j] = 0$. No problem since $\phi_{i,j} \in \mathbb{C}$!

Other properties of the Bosonic coherent state

- $|\phi\rangle$ clearly spans the enitire Fock space
- Has closure relation

$$1 = \int_{-\infty}^{\infty}\left[\prod_j \frac{d\phi_j^* d\phi_j}{2\pi i}\right] e^{\sum_k \phi_k^* \phi_k} |\phi\rangle\langle\phi|$$

$$= 1_0 \otimes 1_1 \otimes 1_2 \otimes \cdots \otimes 1_{\Lambda_{max}} \qquad \text{(overcomplete)}$$

- Overlap of two coherent states:

$$\langle\psi|\phi\rangle = e^{\sum_i \psi_i^* \phi_i} \neq 0$$

- Matrix element of *normal-ordered* operator

$$\langle\psi|a_k^\dagger a_j|\phi\rangle = \psi_k^* \phi_j \langle\psi|\phi\rangle = \psi_k^* \phi_j e^{\sum_i \psi_i^* \phi_i}$$

$$\langle\psi| : f(a_k^\dagger, a_j) : |\phi\rangle = f(\psi_k^*, \phi_j) e^{\sum_i \psi_i^* \phi_i}$$

Trace of an operator A

$$\begin{aligned}
\mathrm{tr}A \equiv \sum_n \langle n|A|n\rangle &= \sum_n \int_{-\infty}^{\infty} \left[\prod_j \frac{d\phi_j^* d\phi_j}{2\pi i}\right] e^{\sum_k \phi_k^* \phi_k} \langle n|\phi\rangle\langle\phi|A|n\rangle \\
&= \int_{-\infty}^{\infty} \left[\prod_j \frac{d\phi_j^* d\phi_j}{2\pi i}\right] e^{\sum_k \phi_k^* \phi_k} \sum_n \langle\phi|A|n\rangle\langle n|\phi\rangle \\
&= \int_{-\infty}^{\infty} \left[\prod_j \frac{d\phi_j^* d\phi_j}{2\pi i}\right] e^{\sum_k \phi_k^* \phi_k} \langle\phi|A|\phi\rangle
\end{aligned}$$

Some comments on bosonic coherent states

- Our case is called the “Canonical Coherent State”–other types of coherent states are relevant for signal processing, image processing, etc. . .
- “Classical Electric Field”–coherent state of photons in the classical limit
- “Gaussian wave packets” $\langle x|\phi\rangle$ minimize the uncertainty principle (see Schiff, QM, 1955)

Can we do the same for fermionic states?

We want $|\xi\rangle$ s.t. $a_i|\xi\rangle = \xi_i|\xi\rangle$, where ξ_i is an eigenvalue of operator a_i. If we have this, then we also have

$$\{a_i, a_j\}|\xi\rangle = 0 \implies \{\xi_i, \xi_j\} = 0.$$

So ξ_i CANNOT be a c-number!

Enter Grassmann numbers

Grassmann numbers

Define set of Grassmann numbers

$$\eta_1, \eta_2, \ldots, \eta_n, \eta_1^*, \eta_2^*, \ldots, \eta_n^*$$

such that $\{\eta_i, \eta_j\} = \{\eta_i^*, \eta_j^*\} = \{\eta_i^*, \eta_j\} = 0 \implies \eta_i^2 = 0$ (nilpotent).

Grassmann functions

Define set of Grassmann numbers

$$\left.\begin{aligned} f(\eta_i) &= f_0 + f_1\eta_i \\ g(\eta_i) &= g_0 + g_1\eta_i \end{aligned}\right\} \implies f(\eta_i) + g(\eta_i) = (f_0 + g_0) + (f_1 + g_1)\eta_i$$

We have a *Grassmann Algebra*!

Also! $\{\eta_i, a\} = \{\eta_j, a^\dagger\} = 0$

Integration and Differentiation, Grassmann style!

Rules for integrating and differentiating

$$\frac{\partial}{\partial \eta_i}\eta_j = \delta_{ij}$$

$$\int d\eta_i \; \eta_j = \delta_{ij}$$

$$\int d\eta_i = 0$$

Note: no limits on the integration! Purely formal manipulations.

Weird, huh?

Fermionic coherent state

$$|\xi\rangle \equiv e^{-\sum_i \xi_i a_i^\dagger}|0\rangle \quad \forall\ \xi_i \text{ Grassmann}$$

$$\begin{aligned}
&= \prod_i (1-\xi_i a_i^\dagger)|0\rangle \\
a_1|\xi\rangle &= a_1(1-\xi_1 a_1^\dagger)\prod_{i=2}(1-\xi_i a_i^\dagger)|0\rangle \\
&= \xi_1 a_1 a_1^\dagger \prod_{i=2}(1-\xi_i a_i^\dagger)|0\rangle \\
&= \xi_1(1-a_1^\dagger a_1)\prod_{i=2}(1-\xi_i a_i^\dagger)|0\rangle \\
&= \xi_1 \prod_{i=2}(1-\xi_i a_i^\dagger)|0\rangle \\
&= \xi_1(1-\xi_1 a_1^\dagger)\prod_{i=2}(1-\xi_i a_i^\dagger)|0\rangle \\
&= \xi_1 \prod_i (1-\xi_i a_i^\dagger)|0\rangle \\
&= \xi_1|\xi\rangle\ .
\end{aligned}$$

Properties of fermionic coherent states

- Define $\langle\xi| = \langle 0|e^{-\sum_i a_i \xi_i^*}$ and have that $\langle\xi|a_k^\dagger = \langle\xi|\xi_k^*$.
- Overlap $\langle\eta|\xi\rangle = e^{-\sum_i \eta_i^* \xi_i}$
- Matrix element of normal ordered operator

$$\langle\eta| : f(a_j^\dagger, a_i) : |\xi\rangle = f(\eta_j^*, \xi_i)e^{-\sum_i \eta_i^* \xi_i}$$

- Completeness relation:

$$1 = \int \left[\prod_i d\xi_i^* d\xi_i\right] e^{-\sum_i \xi_k^* \xi_k} |\xi\rangle\langle\xi|$$

Trace of operator A

$$
\begin{aligned}
\mathrm{tr}A &= \sum_n \langle n|A|n\rangle \\
&= \int \left[\prod_j d\xi_j^* d\xi_j\right] e^{-\sum_k \xi_k^* \xi_k} \sum_n \langle n|\xi\rangle\langle\xi|A|n\rangle \\
&= \int \left[\prod_j d\xi_j^* d\xi_j\right] e^{-\sum_k \xi_k^* \xi_k} \langle -\xi|A \sum_n |n\rangle\langle n||\xi\rangle \\
&= \int \left[\prod_j d\xi_j^* d\xi_j\right] e^{-\sum_k \xi_k^* \xi_k} \langle -\xi|A|\xi\rangle \ .
\end{aligned}
$$

Is there a physical interpretation of fermionic coherent states?

None that I know of.