Effects of gravity in metastable vacuum decay

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Plan of the talk

- Euclidean approach to metastable vacuum decay
- Creation of wormholes in tunneling transitions with gravity
- Negative mode problem in tunnelling transitions with gravity
- Concluding remarks

Instantons and tunnelling

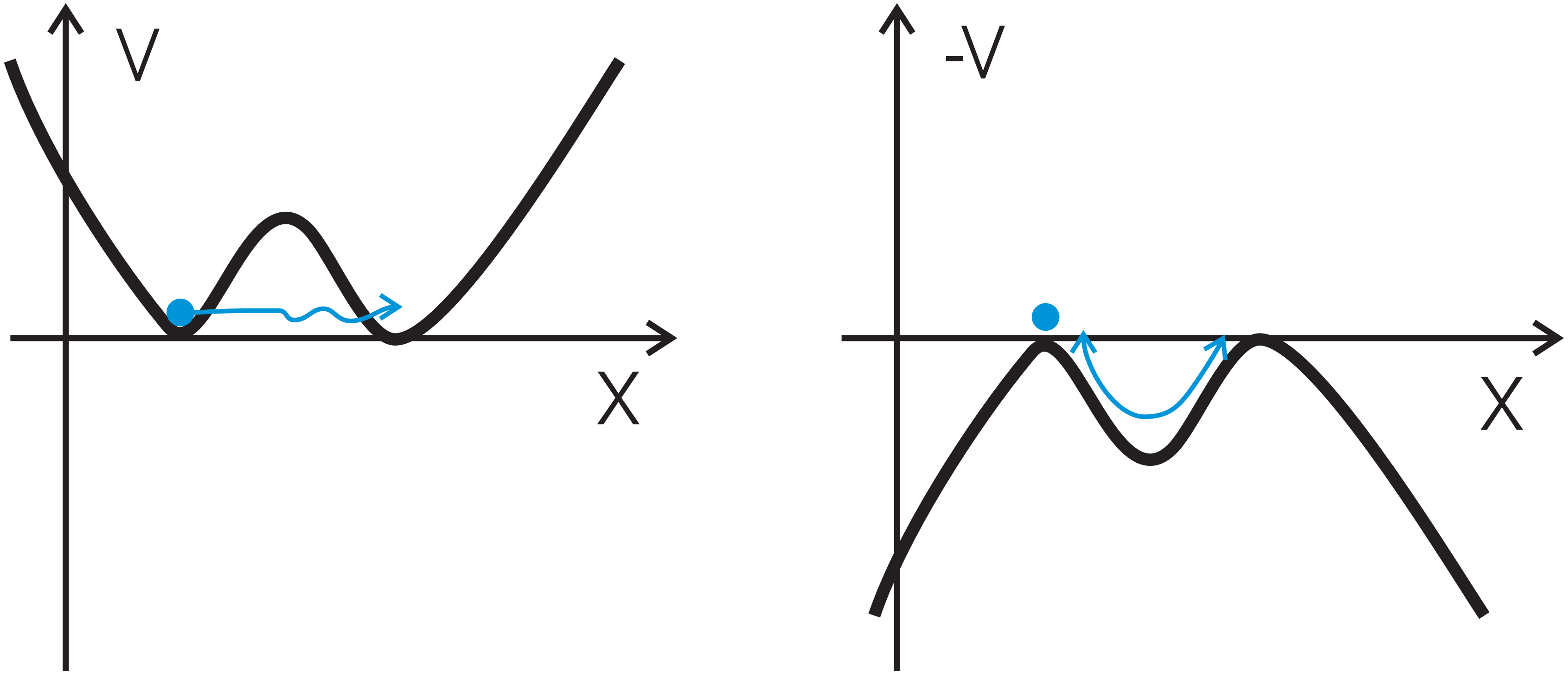


Figure 1: Tunneling in symmetric double well potential.

$$\Delta E \propto \mathrm{e}^{-S_E[x_{instanton}]} \tag{1}$$

Bounces and false vacuum decay

Coleman (1977)

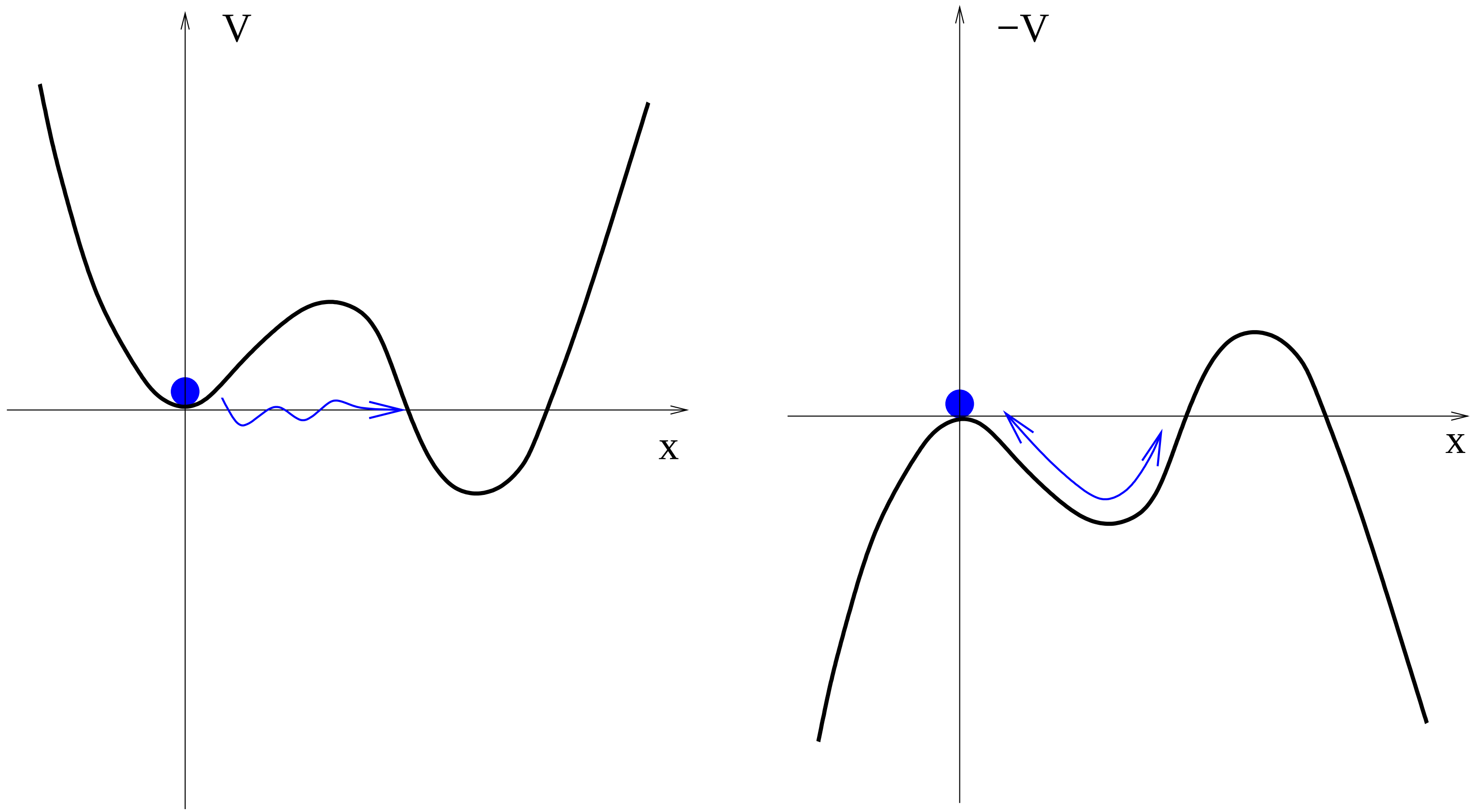


Figure 2: Tunneling in asymmetric double well potential.

Euclidean approach to tunneling

Coleman (1977)

In the semiclassicall approximation, summing multibounce configuration one finds correction to ground state energy $E_0 = \hbar\omega/2$ in the following form

$$E = E_0 - \hbar K \mathrm{e}^{-S/\hbar}[1 + O(\hbar)] \ , \tag{2}$$

where, $S = \int d\eta[\frac{1}{2}(\frac{dx}{d\eta})^2 + V(x)]$ is the Euclidean action on the bounce solution $\bar{x}(\eta)$ and pre-exponential factor K is given by (Gaussian) integration of the exponential of quadratic action of linear perturbations $S^{(2)} = \frac{1}{2}\int d\eta \delta x[-\partial_\eta^2 + V'']\delta x$,

$$K = \frac{1}{2}(\frac{S}{2\pi\hbar})^{1/2}(\frac{det'[-\partial_\eta^2 + V''(\bar{x})]}{det[-\partial_\eta^2 + \omega^2]})^{-1/2} \ . \tag{3}$$

There is exactly one tunnelling negative mode in the spectrum of linear perturbations about the bounce solution, since (translational) zero energy wave function $\psi_0 \sim \frac{d\bar{x}}{d\eta}$ of corresponding Schrödinger equation has a node. i.e. $K = i\Gamma$.

Finally, the decay probability per unit time of the unstable state is given by

$$\begin{aligned}\Gamma &= -2\mathrm{Im}E/\hbar \\ &= \left(\frac{S}{2\pi\hbar}\right)^{1/2}\left|\frac{det'[-\partial_\lambda^2 + V''(\bar{x})]}{det[-\partial_\lambda^2 + \omega^2]}\right|^{-1/2} \mathrm{e}^{-S/\hbar}\,[1 + O(\hbar)]\,.\end{aligned} \tag{4}$$

In the 1988 NPB article “Quantum Tunneling And Negative Eigenvalues,” Coleman arrives to strong conclusion: “There may exist solutions in other ways like bounces and which have more than one negative eigenvalue, but, even if they do exist, they have nothing to do with tunnelling.”

These quantum mechanical results could be generalized for

- Field theory in flat space-time   Coleman, Callan and Coleman (1977)
- Field theory with gravity   Coleman and De Luccia (1980)

Special classical solutions in Euclidean quantum gravity

Hawking and Moss (1982)

1. The Hawking-Moss solution is a 4-sphere corresponding to scalar field sitting on the top of the potential barrier

$$\phi(\eta) = \phi_{top}, \;\; \rho(\eta) = \mathcal{H}_{top}^{-1} \, sin(\mathcal{H}_{top}\eta) \;, \tag{5}$$

with $\mathcal{H}_{top} = \sqrt{\kappa V(\phi_{top})/3}$. Note that the Euclidean time η varies in finite interval $\eta = (0, \eta_f)$.

Coleman and De Luccia (1980)

2. The Coleman-De Luccia bounce is a deformed 4-sphere. It starts with some $\phi = \phi_0$ at $\eta = 0$ close to ϕ_-, stops at $\eta = \eta_f$ close to ϕ_+ and obeys the regularity conditions

$$\rho(0) = \dot{\phi}(0) = 0, \;\; \rho(\eta_f) = \dot{\phi}(\eta_f) = 0 \;. \tag{6}$$

Bousso and Linde (1998), Balek and Demetrian (2004)

Hackworth and Weinberg (2005)

B.-H. Lee, C. H. Lee, W. Lee and C. Oh (2010, 2012)

3. Oscillating bounces and instantons, solutions in which the scalar field passes over the barrier more then once.

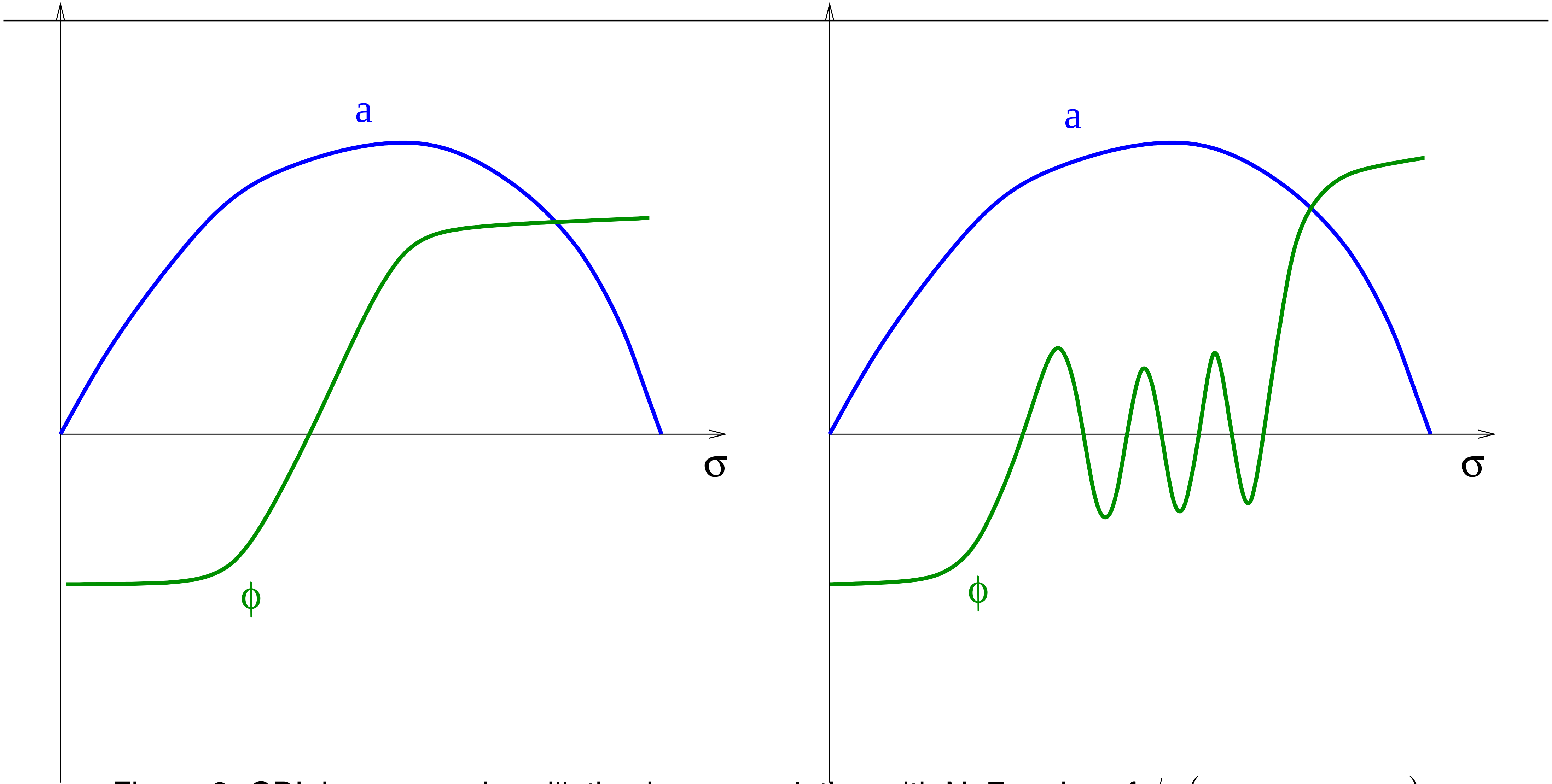


Figure 3: CDL bounce and oscillating bounce solution with N=7 nodes of ϕ, $(\sigma \equiv \eta, a \equiv \rho)$.

dS $\longrightarrow$ dS tunneling: Four Bounces

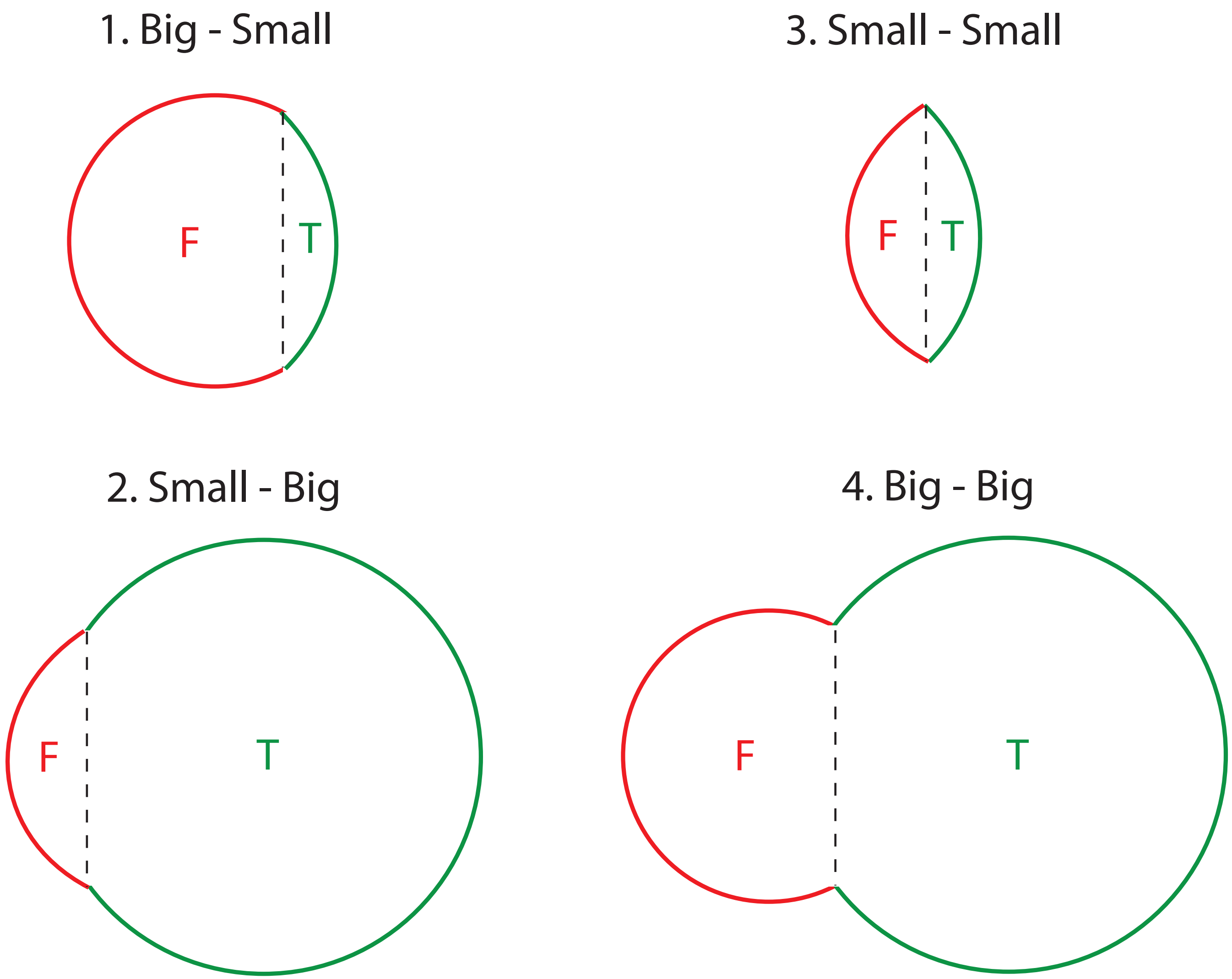


Figure 4: Four apriori possible instantons in dS to dS transitions.

Scalar field with minimal coupling and NEC violation

We will start with a simple model of a scalar field ϕ with a potential $V(\phi)$ minimally coupled to gravity and described by the action

$$S = \int d^4x\sqrt{-g}\Big(\frac{1}{2\kappa}R - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\Big) \ , \tag{7}$$

where κ is the reduced Newton's constant. We will consider homogeneous and isotropic universes, described by the metric

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}^K dx^i dx^j = -dt^2 + a^2(t)[\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2\sin^2(\theta)d\varphi^2]. \tag{8}$$

In what follows we will only be interested in the $K = +1$ case, but for clarity we will write K out explicitly in this section. The energy momentum tensor is given by

$$T_{00} = \rho_s \ , \quad T_{ij} = a^2\gamma_{ij}^K p_s \tag{9}$$

where the energy density and the pressure are given respectively by

$$\rho_s = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + V \ , \quad p_s = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 - V \ . \tag{10}$$

The null energy condition (NEC),

$$T_{\mu\nu} n^{\mu} n^{\nu} > 0 \ , \tag{11}$$

with n_{μ} being a null vector, $n_{\mu} n^{\mu} = 0$, then reduces to the requirement

$$\rho_s + p_s > 0 \ . \tag{12}$$

The equations of motion (Friedmann equations) can be written in the form

$$H^2 = \frac{\kappa}{3}\rho_s - \frac{K}{a^2} \ , \tag{13}$$

$$\frac{dH}{dt} = -\frac{\kappa}{2}(\rho_s + p_s) + \frac{K}{a^2} \ , \tag{14}$$

where $H \equiv (da/dt)/a$. Tunnelling can be described by performing an analytic continuation to Euclidean time, with $t = -i\bar{\lambda}$. Then the metric and scalar field are of the form

$$d\bar{s}_E^2 = d\bar{\lambda}^2 + \bar{\rho}^2 d\Omega_3^2 \ , \quad \bar{\phi} = \bar{\phi}(\bar{\lambda}) \ , \tag{15}$$

where $\bar{\rho}(\bar{\lambda}) \equiv a(it)$.

Note that the Euclidean version of the NEC condition Eq. (12) reverses sign:

$$\rho_s^E + p_s^E < 0 \ , \tag{16}$$

with

$$\rho_s^E = \frac{1}{2}\left(\frac{d\phi}{d\bar{\lambda}}\right)^2 - V \ , \quad p_s^E = \frac{1}{2}\left(\frac{d\phi}{d\bar{\lambda}}\right)^2 + V \ . \tag{17}$$

The Euclidean versions of the Friedmann equations read

$$H_E^2 = -\frac{\kappa}{3}\rho_s^E + \frac{K}{a^2} \ , \tag{18}$$

$$\frac{dH_E}{d\bar{\lambda}} = \frac{\kappa}{2}(\rho_s^E + p_s^E) - \frac{K}{a^2} \ , \tag{19}$$

where $H_E = (d\bar{\rho}/d\bar{\lambda})/\bar{\rho}$.

At the putative neck of an instanton, i.e. at a local minimum of $\bar{\rho}(\bar{\lambda})$, we have $H_E = 0$ and would need $\frac{dH_E}{d\bar{\lambda}} > 0$, which, in view of Eq. (19), is impossible if the “NEC” condition Eq. (16) is fulfilled.

Thus we can see that ($O(4)-$ symmetric) instantons in theories whose Lorentzian counterpart satisfies the NEC cannot have a neck.

Scalar field with non-minimal coupling to gravity

We will be interested in the modified gravity theory defined by the Euclidean action

$$S_E = \int d^4x\sqrt{g}\left(-\frac{1}{2\kappa}f(\phi)R + \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + V(\phi)\right) + S_m(\psi_m, g_{\mu\nu}) \ , \tag{20}$$

where the matter action S_m depends on matter fields ψ_m, which we assume to couple to the physical metric $g_{\mu\nu}$, $f(\phi) = 1 - \kappa\xi\phi^2$ and ξ is dimensionless parameter.

Assuming $O(4)-$symmetry, $ds^2 = N^2(\eta)d\eta^2 + \rho(\eta)^2 d\Omega_3^2, \phi = \phi(\eta)$, in proper time gauge, $N \equiv 1$, the equations of motion can be written in a form convenient for numerical integration:

$$\ddot{\phi} + 3\frac{\dot{\rho}}{\rho}\dot{\phi} - \frac{\kappa\xi\phi}{1-\kappa\xi(1-6\xi)\phi^2}[4V - 6\xi\phi\frac{dV}{d\phi} + (1-6\xi)\dot{\phi}^2] = \frac{dV}{d\phi} \ . \tag{21}$$

$$\ddot{\rho} = -\frac{\tilde{\kappa}\rho}{3}\Big((1 - \frac{3\xi}{1-\kappa\xi(1-6\xi)\phi^2})\dot{\phi}^2 + \frac{1-\kappa\xi(1+6\xi)\phi^2}{1-\kappa\xi(1-6\xi)\phi^2}V + 6\xi\frac{\dot{\rho}}{\rho}\phi\dot{\phi} - \frac{3\xi(1-\kappa\xi\phi^2)}{1-\kappa\xi(1-6\xi)\phi^2}\phi\frac{dV}{d\phi}\Big) \ , \tag{22}$$

where $^{\cdot} \equiv d/d\eta$ and $\tilde{\kappa} \equiv \frac{\kappa}{1-\kappa\xi\phi^2}$ is the ”effective gravitational constant”.

We will now assume that the potential $V(\phi)$ is positive and has two non-degenerate local minima at $\phi = \phi_{\rm tv}$ and $\phi = \phi_{\rm fv}$, with $V(\phi_{\rm fv}) > V(\phi_{\rm tv})$, as well as a local maximum for some $\phi = \phi_{\rm top}$, with $\phi_{\rm fv} < \phi_{\rm top} < \phi_{\rm tv}$. The Euclidean solution describing vacuum decay satisfies the boundary conditions

$$\phi(0) = \phi_0, \qquad \dot{\phi}(0) = 0, \qquad \rho(0) = 0, \qquad \dot{\rho}(0) = 1 \ , \tag{23}$$

at $\eta = 0$ and

$$\phi(\eta_{max}) = \phi_m, \qquad \dot{\phi}(\eta_{max}) = 0, \qquad \rho(\eta_{max}) = 0, \qquad \dot{\rho}(\eta_{max}) = 1 \ , \tag{24}$$

at some $\eta = \eta_{max}$. This assumes the following Taylor series at $\eta \to 0$

$$\phi(\eta) = \phi_0 + \frac{(1-\kappa\xi\phi_0^2)\frac{\partial V}{\partial\phi}|_{\phi=\phi_0} + 4\kappa\xi\phi_0 V(\phi_0)}{8(1-\kappa\xi\phi_0^2(1-6\xi))}\eta^2 + O(\eta^4) \ , \tag{25}$$

$$\rho(\eta) = \eta - \frac{\kappa V(\phi_0) - \frac{3}{2}\kappa\xi\phi_0\frac{\partial V}{\partial\phi}|_{\phi=\phi_0}}{18(1-\kappa\xi\phi_0^2(1-6\xi))}\eta^3 + O(\eta^5) \ , \tag{26}$$

and similar power law behaviour as $x \to 0$, where $x = \eta_{max} - \eta$.

Numerical results

For our numerical examples, we will consider the potential double well potential parameterized as follows:

$$V(\phi) = \Lambda + \frac{1}{2}\mu\phi^2 + \frac{1}{3}\beta_3\phi^3 + \frac{1}{4}\beta_4\phi^4 + A\mathrm{e}^{-\alpha\phi^2} \ . \tag{27}$$

We have chosen the following values for the constants appearing in S_E,

$$\kappa = 0.1 \ , \ \xi = 3 \ , \ \Lambda = 0.1 \ , \ \mu = 1.0 \ , \ \beta_3 = -0.25 \ , \ \beta_4 = 0.1 \ , \ A = 3.0 \ , \ \alpha = 2.0 \ . \tag{28}$$

When $|\phi|$ is too large, the effective gravitational constant $\tilde{\kappa}$

$$\tilde{\kappa} \equiv \frac{\kappa}{1 - \kappa\xi\phi^2} \ , \tag{29}$$

becomes negative, and a region of “anti-gravity” is reached. In what follows, we will solely be concerned with the regions of ordinary-sign gravity.

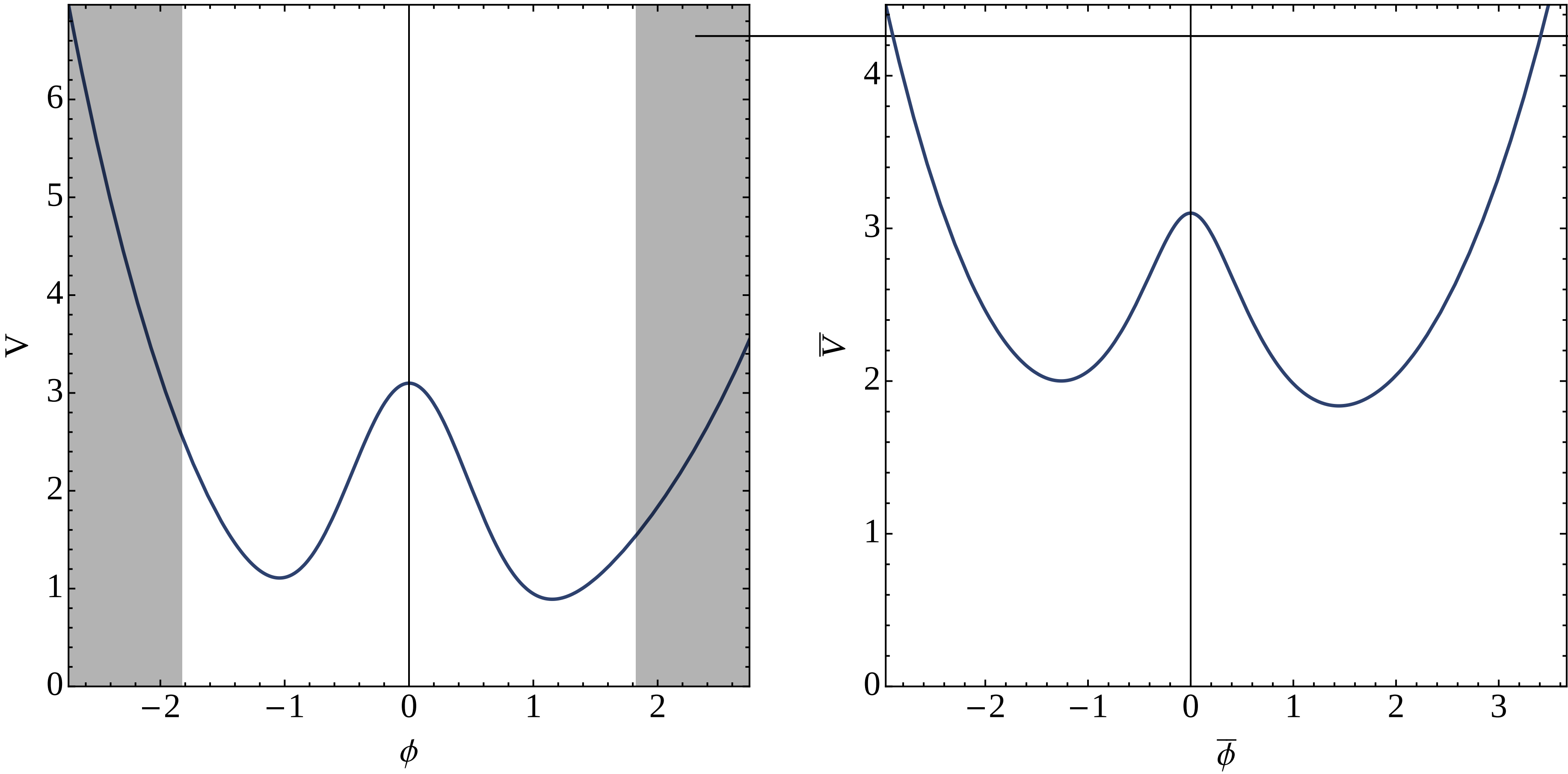


Figure 5: The scalar field potential $V(\phi)$ in Jordan frame (left) and the corresponding potential $\bar{V}(\bar{\phi})$ in Einstein frame (right).

When $|\phi|$ is too large, the effective gravitational constant $\tilde{\kappa}$ becomes negative, and a region of "anti-gravity" is reached. These regions are shaded in the plot of $V(\phi)$ – in our discussion, we will solely be concerned with the regions of ordinary-sign gravity.

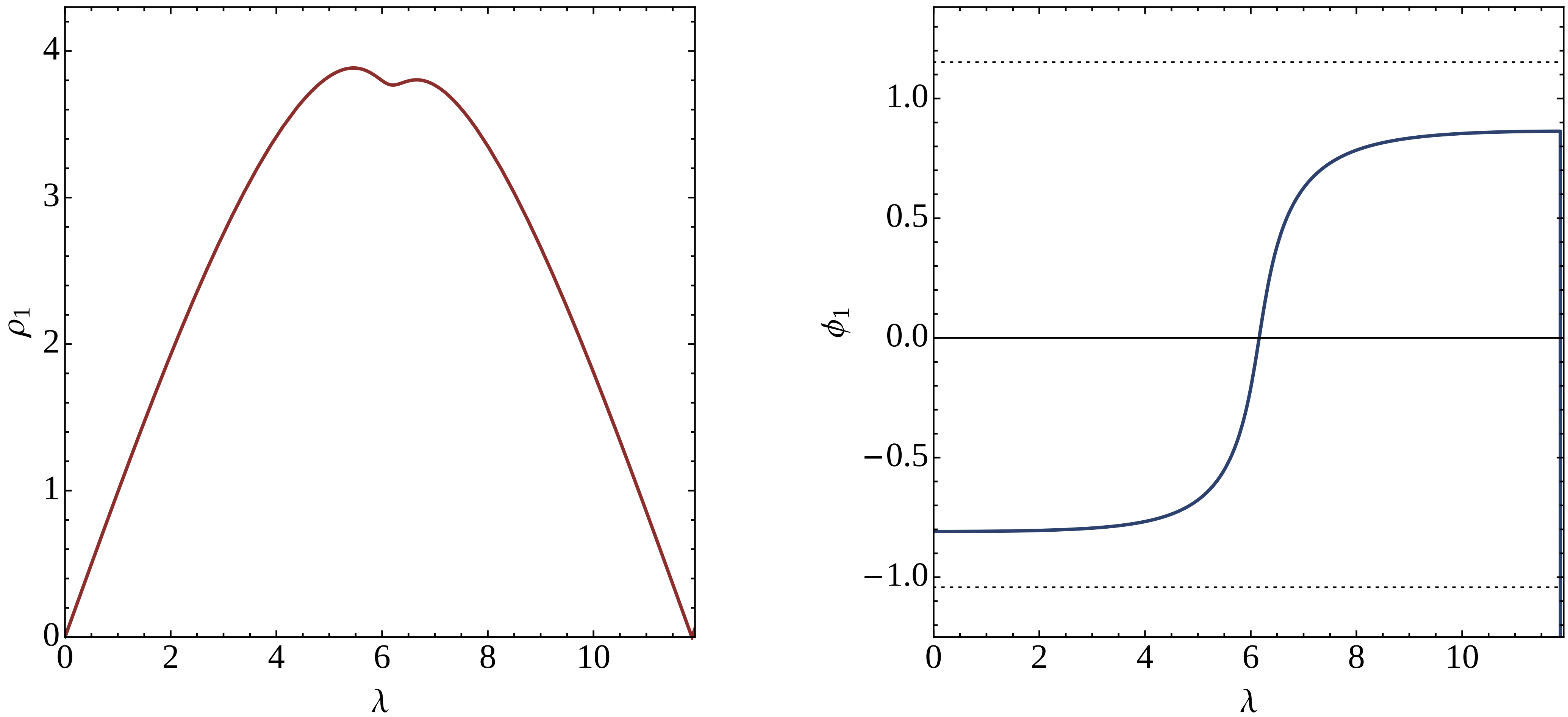


Figure 6: The field profiles (scale factor on the left, scalar field on the right) for our example of an instanton with a neck $(\lambda \equiv \eta)$.

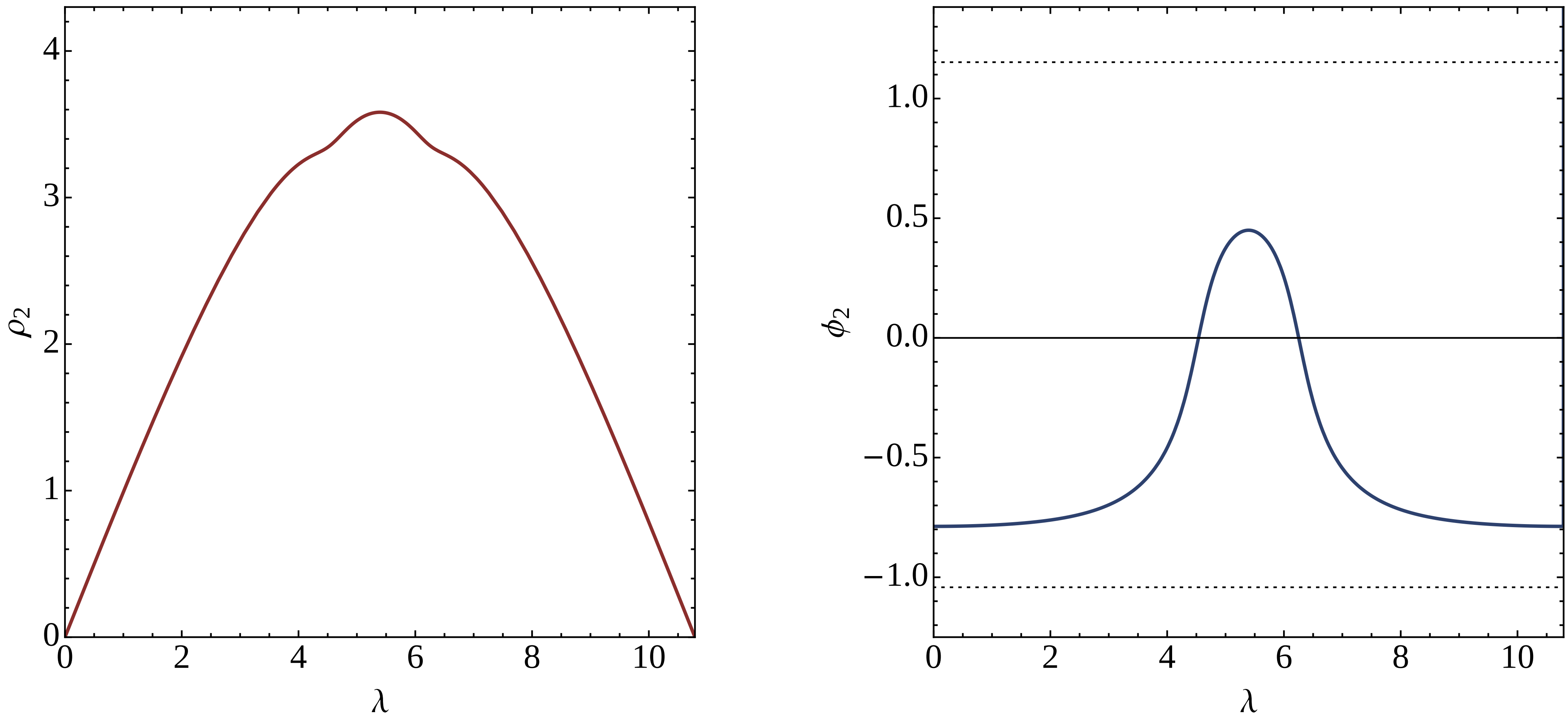


Figure 7: The field profiles (scale factor on the left, scalar field on the right) for our oscillating instanton example. The scalar field profile now leads to a hump in the scale factor, rather than a neck $(\lambda \equiv \eta)$.

Bubble materialization

In order to obtain the bubble shape at the moment of materialisation, we have to analytically continue the Euclidean metric

$$ds^2 = d\eta^2 + \rho^2(\eta)\left[d\psi^2 + \sin^2(\psi)(d\theta^2 + \sin^2\theta d\varphi^2)\right] \tag{30}$$

into Lorentzian signature. This procedure is not single valued. Using analytic continuation

$$\psi = \frac{\pi}{2} + it \ , \quad \eta = r \ , \tag{31}$$

we obtain the bubble geometry

$$ds^2 = -\rho^2(r)dt^2 + dr^2 + \tilde{\rho}^2(t,r)d\Omega_2^2 \ , \tag{32}$$

where

$$\tilde{\rho}(t,r) \equiv \cosh(t)\rho(r) \ . \tag{33}$$

We see that the function ρ indeed determines the spatial geometry of the bubble at the moment of materialisation, $t = 0$, and thus the neck region becomes a wormhole.

Negative mode problem: Selected publications

1. G. Lavrelashvili, V. A. Rubakov and P. G. Tinyakov, “Tunneling Transitions With Gravitation: Breaking Of The Quasiclassical Approximation,” (1985).

2. T. Tanaka and M. Sasaki, “False vacuum decay with gravity: Negative mode problem,” (1992).

3. A. Khvedelidze, G. Lavrelashvili and T. Tanaka, “On cosmological perturbations in closed FRW model with scalar field and false vacuum decay,” (2000).

4. S. Gratton and N. Turok, “Homogeneous modes of cosmological instantons,” (2001).

5. J. C. Hackworth and E. J. Weinberg, “Oscillating bounce solutions and vacuum tunneling in de Sitter spacetime,” (2005).

6. G. Lavrelashvili, “The Number of negative modes of the oscillating bounces,” (2006).

7. H. Lee and E. J. Weinberg, “Negative modes of Coleman - De Luccia bounces,” (2014).

8. M. Koehn, G. Lavrelashvili and J. L. Lehners, “Towards a Solution of the Negative Mode Problem in Quantum Tunnelling with Gravity,” (2015).

9. R. Gregory, K. M. Marshall, F. Michel and I. G. Moss, “Negative modes of Coleman - De Luccia and black hole bubbles,” (2018)

10. S. Bramberger, M.Chitishvili, and G.Lavrelashvili, ”Aspects of the negative mode problem in quantum tunneling with gravity”, (2019), arXiv:1906.07033 [gr-qc].

Negative mode problem

The Euclidean action of system composed of scalar field minimally coupled to gravity is

$$S = \int d^4x\sqrt{g}\left[-\frac{1}{2\kappa}R + \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + V(\phi)\right] , \tag{34}$$

where $\kappa = 8\pi G$ is the reduced Newton's constant. For the $O(4)-$symmetric metric ansatz we will use

$$ds^2 = d\eta^2 + \rho^2(\eta)d\Omega_3^2 = a^2(\tau)(d\tau^2 + d\Omega_3^2) , \tag{35}$$

where η is (Euclidean) proper time, τ- conformal time, $\rho(\eta)$ is the scale factor $^{\cdot} \equiv d/d\eta, ' \equiv d/d\tau$, and $d\Omega_3^2$ is metric of unit three-sphere,

$$d\Omega_3^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2(\theta)d\varphi^2) . \tag{36}$$

We assume that potential $V(\phi)$ has characteristic asymmetric double-well shape with local minimum at some $\phi = \phi_-$, local maximum ϕ_{top} and global minimum ϕ_+.

We expand the metric and the scalar field over a $O(4)-$symmetric background

$$\begin{aligned} ds^2 &= a(\tau)^2\left[(1+2A(\tau))d\tau^2+\gamma_{ij}^{\mathcal{K}}(1-2\Psi(\tau))dx^i dx^j\right], \\ \phi &= \varphi(\tau)+\Phi(\tau), \end{aligned} \tag{37}$$

where τ is the conformal time, a and φ are the background field values and A, Ψ and Φ are small perturbations. Here $\gamma_{ij}^{\mathcal{K}}$ is the three-dimensional metric on a constant curvature section and as usual $\mathcal{K}=\pm 1, 0$.

Under the infinitesimal shift $\tau \to \tau+\lambda$ the gauge transformations are

$$\delta\Psi=-\mathcal{H}\lambda\,, \qquad \delta\Phi=\varphi'\lambda\,, \qquad \delta A=\lambda'+\mathcal{H}\lambda\ \ , \tag{38}$$

where $\mathcal{H}\equiv a'/a$.

Expanding the total action to second order in perturbations and using the background equations of motion, we find

$$S = S^{(0)}[a, \varphi] + S^{(2)}[A, \Psi, \Phi] \ , \tag{39}$$

where $S^{(0)}$ is the action of the background solution and $S^{(2)}[A, \Psi, \Phi]$ is the quadratic action for scalar $O(4)-$symmetric perturbations given by the Lagrangian:

$$ {}^{(s)}\mathcal{L} = \frac{1}{2\kappa} a^2 \sqrt{\gamma} \Big[-6\Psi'^2 + 6\mathcal{K}\Psi^2 + \kappa(\Phi'^2 + a^2 \frac{\delta^2 V}{\delta\varphi\delta\varphi}\Phi^2 + 6\varphi'\Psi'\Phi) $$
$$ -(2\kappa\varphi'\Phi' - 2\kappa a^2 \frac{\delta V}{\delta\varphi}\Phi + 12\mathcal{H}\Psi' + 12\mathcal{K}\Psi)A - 2(\mathcal{H}' + 2\mathcal{H}^2 + \mathcal{K})A^2 \Big] . $$

Note that the variation with respect to A gives the first order (constraint) equation

$$ 2\kappa\varphi'\Phi' - 2\kappa a^2 \frac{\delta V}{\delta\varphi}\Phi + 12\mathcal{H}\Psi' + 12\mathcal{K}\Psi + 4(\mathcal{H}' + 2\mathcal{H}^2 + \mathcal{K})A = 0 \ . $$

To obtain unconstrained (physical) degree of freedom one should impose gauge condition and solve constraints.

G.L., Rubakov, Tinyakov (1985)

The I approach (Lagrangian):

Fixing the gauge with the condition $\Psi = 0$ and eliminating A with the help of the constraint equation we obtain the unconstrained quadratic action in the form

$$S^{(2)}_{LRT} = \int \frac{a^4\sqrt{\gamma}}{2Q_{LRT}} \left[\frac{\mathcal{H}^2}{a^2}\Phi'^2 - \frac{\kappa\varphi'}{3}\frac{\delta V}{\delta\varphi}\Phi'\Phi \right.$$
$$\left. + \left(\frac{\kappa a^2}{6} (\frac{\delta V}{\delta\varphi})^2 + Q_{LRT}\frac{\delta^2 V}{\delta\varphi\delta\varphi} \right) \Phi^2 \right] d\tau d^3x, \tag{40}$$

with

$$Q_{LRT} := \mathcal{H}^2 - \frac{\kappa{\varphi'}^2}{6} = \mathcal{K} - \frac{\kappa a^2}{3}V \ . \tag{41}$$

Tanaka and Sasaki (1992)

The II approach (Hamiltonian):

Fixing the gauge by $\Phi = 0$ and eliminating Π_Φ (matter degrees of freedom), after some canonical transformation one gets the quadratic part of the Euclidean action ($\mathcal{K}$ is the curvature parameter):

$$S^{(2)} = \frac{(1-4\mathcal{K})}{2} \int \left[(\frac{dq}{d\tau})^2 + U q^2 \right] \sqrt{\gamma} d^3 x d\tau \ , \tag{42}$$

with a potential U depending on the background fields

$$U = \frac{\kappa}{2} {\varphi'}^2 + \varphi' (\frac{1}{\varphi'})'' + 1 - 4\mathcal{K} \ . \tag{43}$$

We see that quadratic action for the homogeneous harmonic has "wrong" overall sign. To overcome this problem it was suggested that analytic continuation $q \to -iq$ is performed while integrating over this mode.

Khvedelidze, G.L., Tanaka (2000)

The III approach (Hamiltonian):

Fixing the gauge by $A = 0$, $\Pi_\Psi = 0$ one obtains unconstrained quadratic action for one physical dynamical degree of freedom as

$$S^{(2)} = \int d\tau d^3x \frac{a^2\sqrt{\gamma}}{2Q} \Big[\Phi'^2 - \frac{\kappa\varphi'}{3\mathcal{K}}(a^2\frac{\delta V}{\delta\varphi} - 3\varphi'\mathcal{H})\Phi'\Phi + \left(\frac{\kappa}{6\mathcal{K}}(a^2\frac{\delta V}{\delta\varphi} - 3\varphi'\mathcal{H})^2 + Q\left(a^2\frac{\delta^2 V}{\delta\varphi\delta\varphi} + \frac{3}{2}\kappa\varphi'^2\right)\right)\Phi^2\Big] , \quad (44)$$

with

$$Q := 1 - \frac{\kappa}{6\mathcal{K}}\varphi'^2 . \quad (45)$$

Assuming $Q > 0$ along the bounce and introducing a new variable $q = a/\sqrt{Q}\ \Phi$, after integration by parts we obtain

$$S^{(2)} = \frac{1}{2}\int \left(q'^2 + W[a(\tau), \varphi(\tau)]q^2\right) d\tau\sqrt{\gamma}d^3x . \quad (46)$$

Khvedelidze, G.L. and Tanaka (2000)

G.L. (2000)

Gratton and Turok (2001)

Corresponding Schrödinger equation

$$-\frac{d^2}{d\tau^2}q + W[a(\tau), \varphi(\tau)]q = Eq \ , \tag{47}$$

was shown to have one boundstate, i.e the Coleman-De Luccia bounce has exactly one negative mode.

Note that by definition at the local maximum, φ_{top}, the second derivative of potential is negative, $V''(\varphi_{top}) < 0$.

For $-4 < \frac{V''(\varphi_{top})}{\mathcal{H}_{top}^2} < 0$ CDL bounce does not exists and HM has one negative modes.

For $\frac{V''(\varphi_{top})}{\mathcal{H}_{top}^2} < -4$ CDL comes to existence and has one negative mode, whereas HM gets extra negative modes.

Number of negative modes of the oscillating bounces

G.L.(2006)

Oscillating bounces were discused intensively by Hackworth and Weinberg (2005), with the aim to give them physical interpretation.

Introducing new variable $f = \sqrt{a}q$ and passing to the proper time σ, quadratic action of KLT can be written in the form

$$S_E^{(2)} = 2\pi^2 \int \Big(\frac{1}{2}\dot{f}^2 + \frac{1}{2} U[a(\sigma), \varphi(\sigma)]\ f^2 \Big) d\sigma \ . \tag{48}$$

So, spectrum of small perturbations about oscillating bounce solution is determined by the following Schrödinger equation

$$-\frac{d^2}{d\sigma^2} f + U[a(\sigma), \varphi(\sigma)] f = E f, \tag{49}$$

and the number of negative modes of the oscillating bounce solution is the number of bound states of these Schrödinger equation.

Potential U is given by

$$U[a(\sigma),\varphi(\sigma)] = \frac{1}{Q}\frac{\delta^2 V}{\delta\varphi\delta\varphi} - \frac{10\dot{a}^2}{a^2 Q} + \frac{12\dot{a}^2}{a^2 Q^2} + \frac{8}{a^2 Q} - \frac{6}{a^2} - \frac{3Q}{a^2} - \frac{\dot{a}^2}{4a^2}$$
$$+\frac{\kappa a^2}{2Q^2}(\frac{\delta V}{\delta\varphi})^2 - \frac{2\kappa a\dot{a}\dot{\varphi}}{Q^2}\frac{\delta V}{\delta\varphi} - \frac{\kappa}{6}(\dot{\varphi}^2 + V), \quad (50)$$

where $Q = 1 - \kappa a^2\dot{\varphi}^2/6$.

Numerical results

Let's parameterize the general quartic scalar field potential as follows:

$$V = V_0 + H^2(-\frac{\beta}{2}\varphi^2 - \frac{g}{3}\varphi^3 + \frac{\lambda}{4}\varphi^4) \ , \tag{51}$$

with $H^2 = \kappa V_0/3$.

Passing to the dimensionless variables

$$\tilde{\varphi} = \frac{\varphi}{v} \ , \tilde{a} = av \ , \tilde{\sigma} = \sigma v \ , \tilde{V}_0 = \frac{V_0}{v^4} \ , \tilde{H}^2 = \frac{H^2}{v^2} \ , \tilde{\kappa} = \kappa v^2 \ , \tag{52}$$

with $v^2 = 2\beta/\lambda$ we will get the dimensionless equations of motion with the rescaled potential

$$\tilde{V} = \tilde{V}_0 + \tilde{H}^2\beta(-\frac{1}{2}\tilde{\varphi}^2 - \frac{\tilde{g}}{3}\tilde{\varphi}^3 + \frac{1}{2}\tilde{\varphi}^4) \ , \tag{53}$$

where $\tilde{g} = gv/\beta$. In what follows we will use dimensionless variables and omit tildes.

Potential U in the Schrödinger equation Eq. (49) close to the $\sigma = 0$ behaves as

$$U = \frac{3}{4\sigma^2} + U_0 + O(\sigma^2) \ , \tag{54}$$

where constant U_0 depends on the initial value of scalar field and parameters of the background solution potential V. The regular branch of the wave function f behaves as

$$f = \sigma^{3/2}(1 + \frac{1}{8}(U_0 - E)\sigma^2 + O(\sigma^4)) \ . \tag{55}$$

Convenient way to determine the number of bound states of Schrödinger equation in a given potential is the investigation of the zero energy wave function. The number of nodes of zero energy wave function exactly counts the number of negative energy states (see e.g. Amann and Quittner (1995)).

Let’s describe our results in details on concrete example. For the parameters choice

$$\kappa = 0.001 \ , V_0 = 0.1 \ , \beta = 70.03 \ , g = \frac{1}{2\sqrt{2}} \ , \tag{56}$$

the potential V has local maximum at $\varphi_{\rm top} = 0$, metastable minimum at $\varphi_{\rm fv} = -0.6242212930$ and true vacuum at $\varphi_{\rm tv} = 0.8009979884$. There exists CDL bounce solution in this potential, oscillating bounces on top of it with up to $N = 7$ nodes and, as always, the HM solution with $\varphi \equiv \varphi_{\rm top}$. Numerical investigation supports quite intuitive and expected result: the bounce with N nodes has exactly N negative modes. Typical results are demonstrated for $N = 3$ case on $\rm Fig.8$ and $\rm Fig.9$. The zero energy wave function of Schröedinger equation Eq. (49) has in this case three nodes, which means that there are exactly three negative energy states for $N = 3$ oscillating bounce. We also found this states explicitly and determined their energies: $E_0 = -0.0013787$, $E_1 = -0.0004362$ and $E_2 = -0.0001207$. Corresponding HM solution has 8 homogeneous negative modes, which is consistent with chosen value of β.

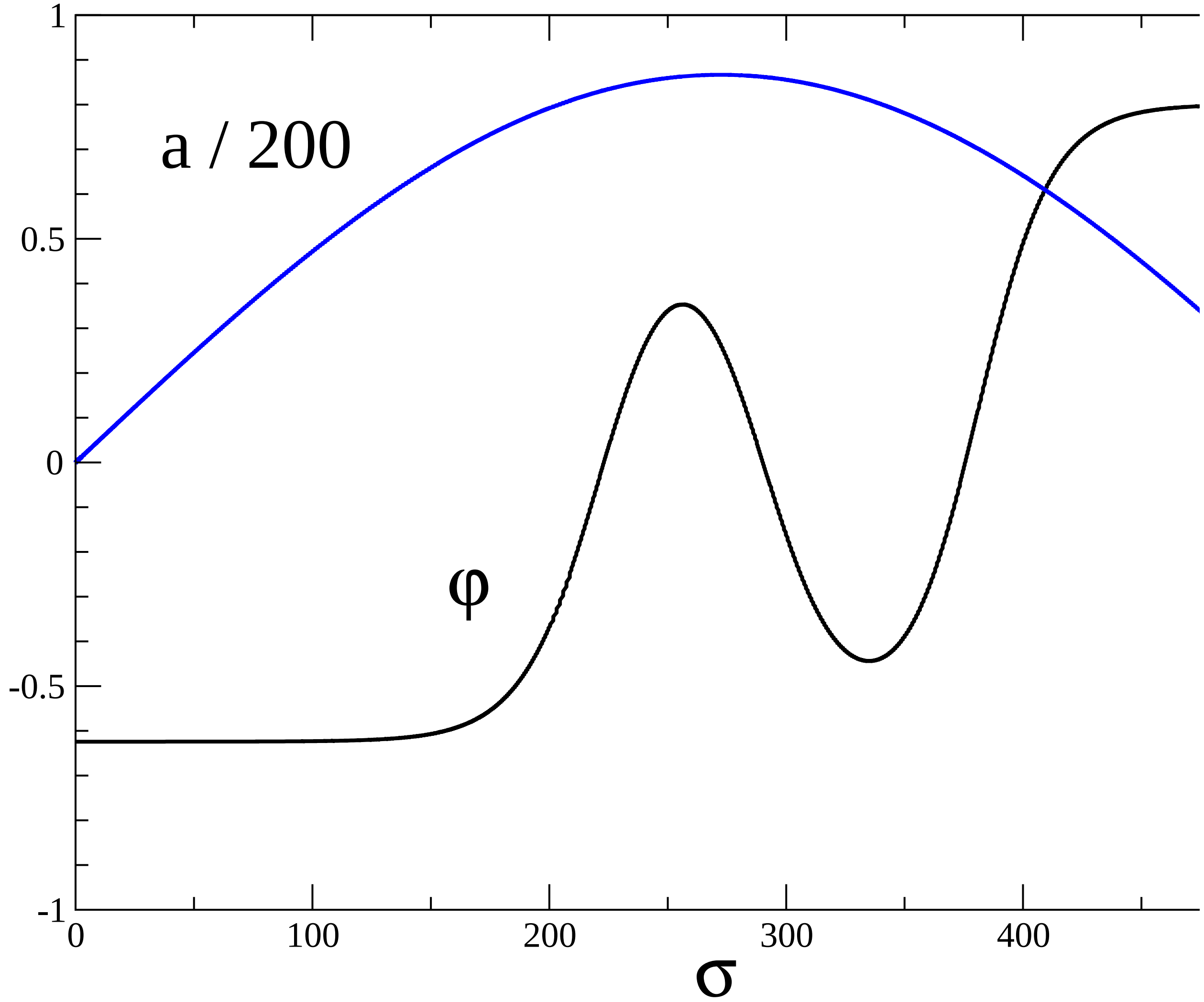


Figure 8: Oscillating bounce solution with three nodes of φ.

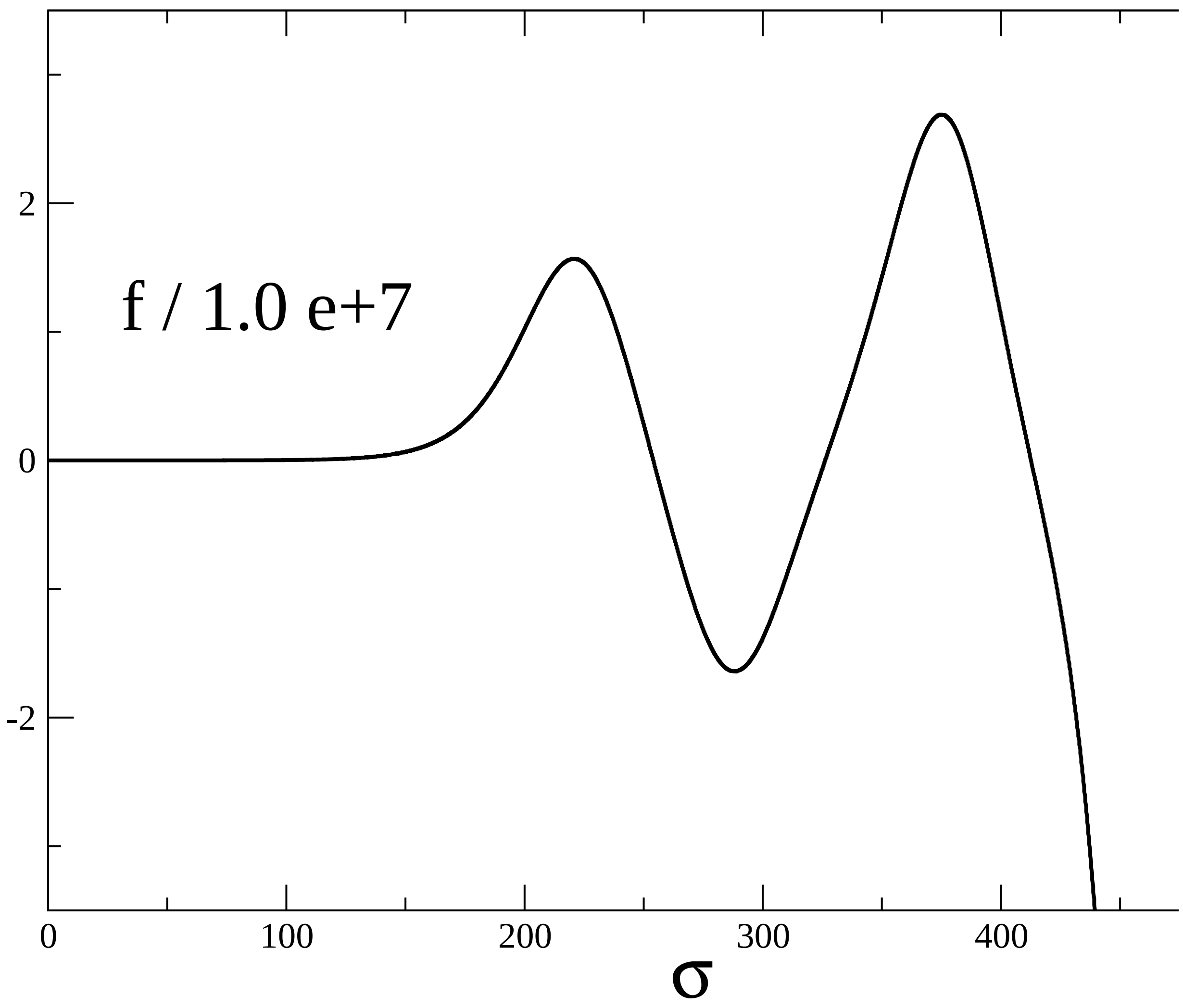


Figure 9: Zero energy wave function f of Schrödinger equation of linear perturbations about oscillating bounce solution with three nodes of φ.

Number of negative modes of the oscillating instantons

L. Battarra, G. Lavrelashvili, J.-L. Lehners (2012)

Oscillating instantons were studied intensively by Korean group

B.-H. Lee, C. H. Lee, W. Lee, C. Oh (2012).

Analysis of spectrum of small perturbations about these oscillating instantons shows the same pattern: instanton solution with N nodes had exactly N homogeneous negative modes.

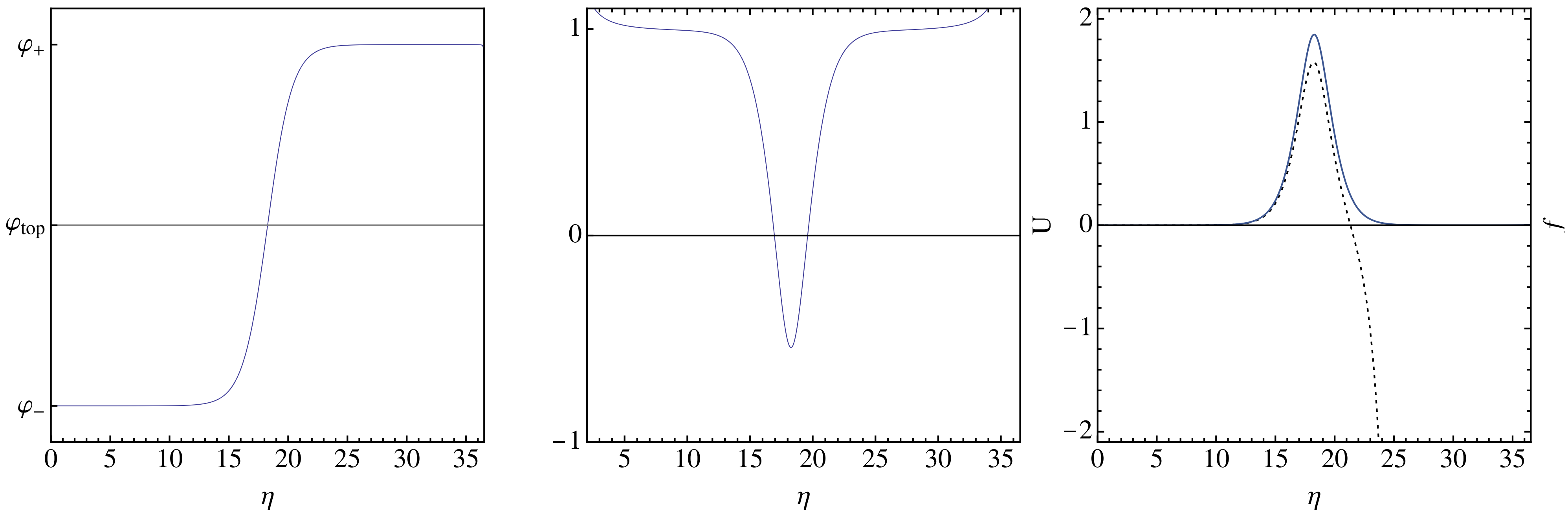


Figure 10: In the left panel, profile of the scalar field for the $N = 1$ oscillating instanton. In the central panel, potential for $O(4)$–symmetric perturbations. In the right panel, zero mode wavefunction (dotted line) and negative mode (solid line): the normalization of the wavefunctions is not imposed, so the overall scale of the vertical axis is irrelevant.

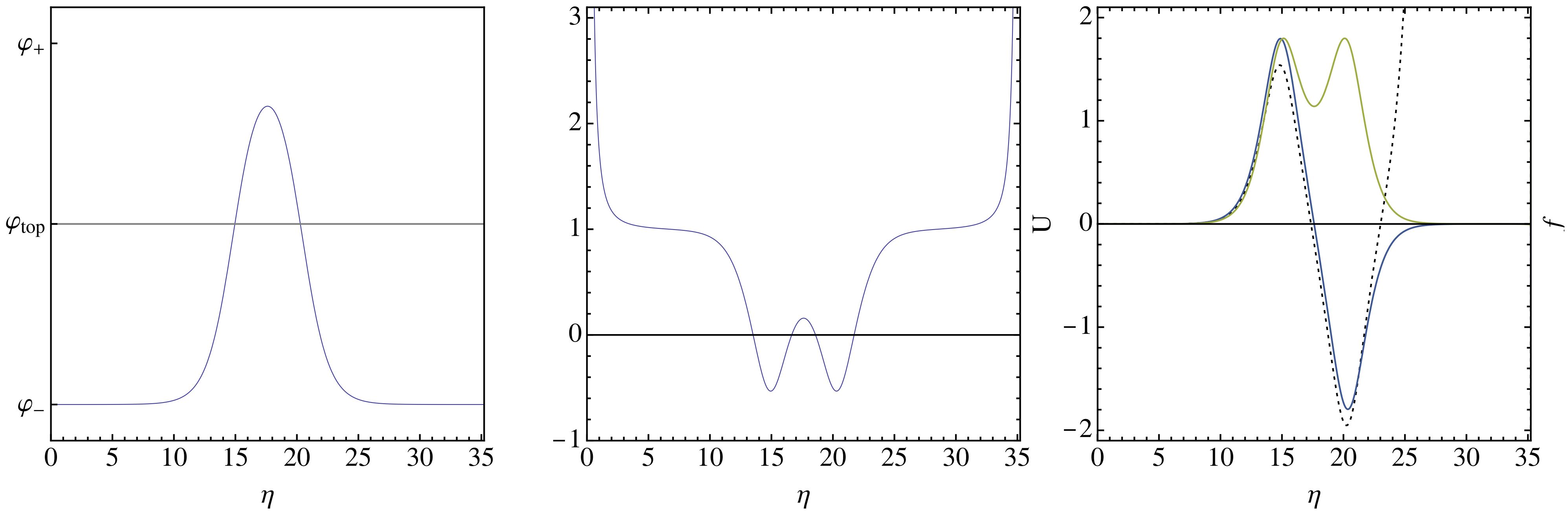


Figure 11: In the left panel, profile of the scalar field for the $N = 2$ oscillating instanton. In the central panel, potential for $O(4)$–symmetric perturbations. In the right panel, zero mode wavefunction (dotted line) and negative modes (solid lines): the normalization of the wavefunctions is not imposed, so the overall scale of the vertical axis is irrelevant.

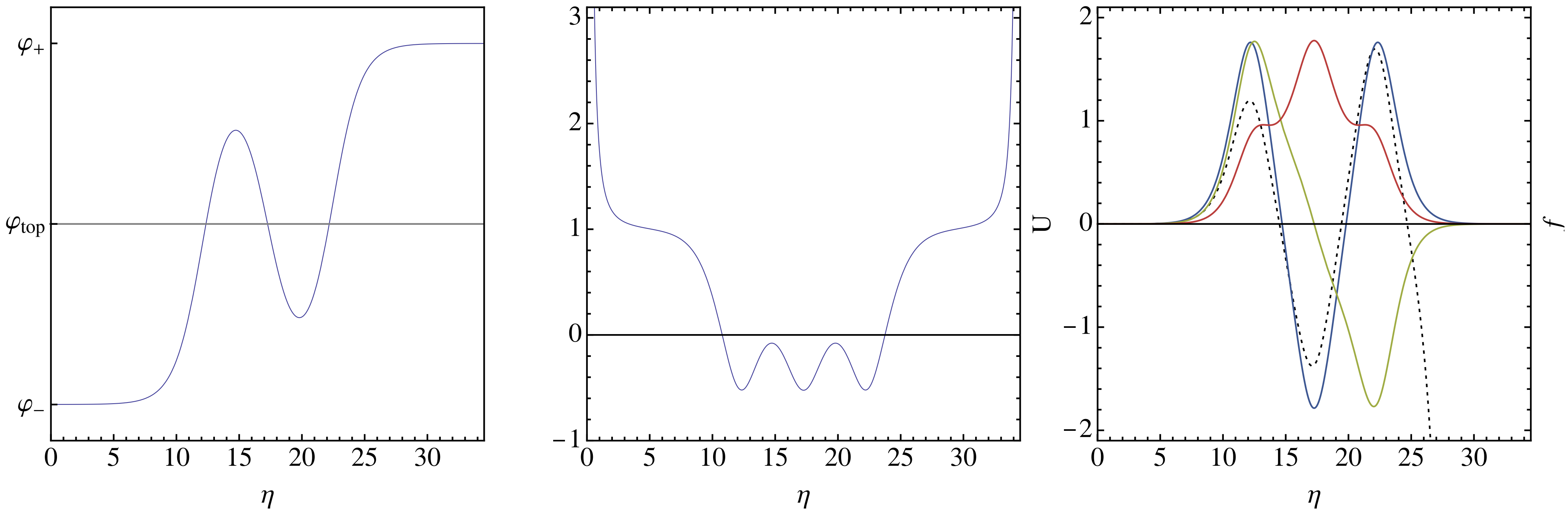


Figure 12: In the left panel, profile of the scalar field for the $N = 3$ oscillating instanton. In the central panel, potential for $O(4)$–symmetric perturbations. In the right panel, zero mode wavefunction (dotted line) and negative modes (solid lines): the normalization of the wavefunctions is not imposed, so the overall scale of the vertical axis is irrelevant. For $N = 4, 5, 6$ we found analogous results.

Recent developments

Hakjoon Lee and Erick Weinberg (2014)

Working in Lagrangian approach with gauge invariant variable

$$\chi \equiv \dot{\rho}\Phi - \rho\dot{\varphi}\Psi \tag{57}$$

and numerically solving with Mathematica pulsation equation for concrete potentials Lee and Weinberg arrived to conclusion that type A bounces have tunneling negative mode whereas type B bounces don't!

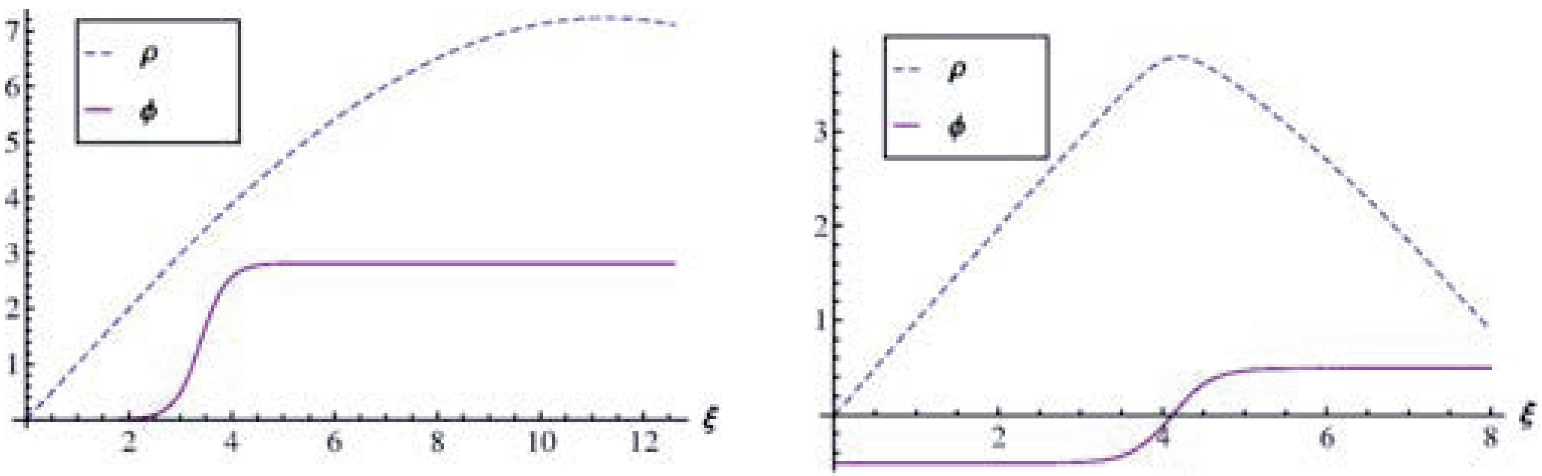


Figure 13: Type A bounces (left panel) and type B (right panel).

Koehn, G.L. and Lehners (2015)

Using KLT (Khvedelidze, G.L., Tanaka (2000)) approach,i.e. fixing the gauge by $A = 0$, $\Pi_\Psi = 0$ one obtains unconstrained quadratic action for one physical dynamical degree of freedom for $\mathcal{K} = +1$ as

$$S_E^{(2)}[\Phi] = 2\pi^2 \int \rho^3(\eta) d\eta \Big[\frac{1}{2Q(\eta)}\dot{\Phi}^2 + \frac{1}{2} U[\varphi(\eta), \rho(\eta)]\Phi^2\Big] , \tag{58}$$

where the factor Q was given by

$$Q := 1 - \frac{\kappa \rho^2 \dot{\varphi}^2}{6} , \tag{59}$$

and the potential U is expressed in terms of the bounce solution as

$$U[\varphi(\eta), \rho(\eta)] \equiv \frac{V''(\varphi)}{Q} + \frac{2\kappa\dot{\varphi}^2}{Q} + \frac{\kappa}{3Q^2}\Big(6\dot{\rho}^2\dot{\varphi}^2 + \rho^2 V'^2(\varphi) - 5\rho\dot{\rho}\dot{\varphi}V'(\varphi)\Big) . \tag{60}$$

The exact form of the fluctuation operator depends on the choice of a weight function, which can be specified by defining a norm.

With the natural choice of the norm

$$||\Phi||^2 \equiv \int d^4x \, \sqrt{g} \, \Phi^2 = 2\pi^2 \int d\eta \, \rho(\eta)^3 \, \Phi^2 \ . \tag{61}$$

The fluctuation equation diagonalizing the quadratic action Eq. (58) then has the form

$$-\frac{1}{Q}\frac{d^2\Phi}{d\eta^2} + \left(\frac{\dot{Q}}{Q^2} - \frac{3\dot{\rho}}{\rho Q}\right)\frac{d\Phi}{d\eta} + U\Phi = \lambda\Phi \ , \tag{62}$$

with the potential U given in Eq. (60). Note that the function $Q \to 1$ at the ends of the interval $[0, \eta_{max}]$, but for some bounces it can become negative for some interval of η.

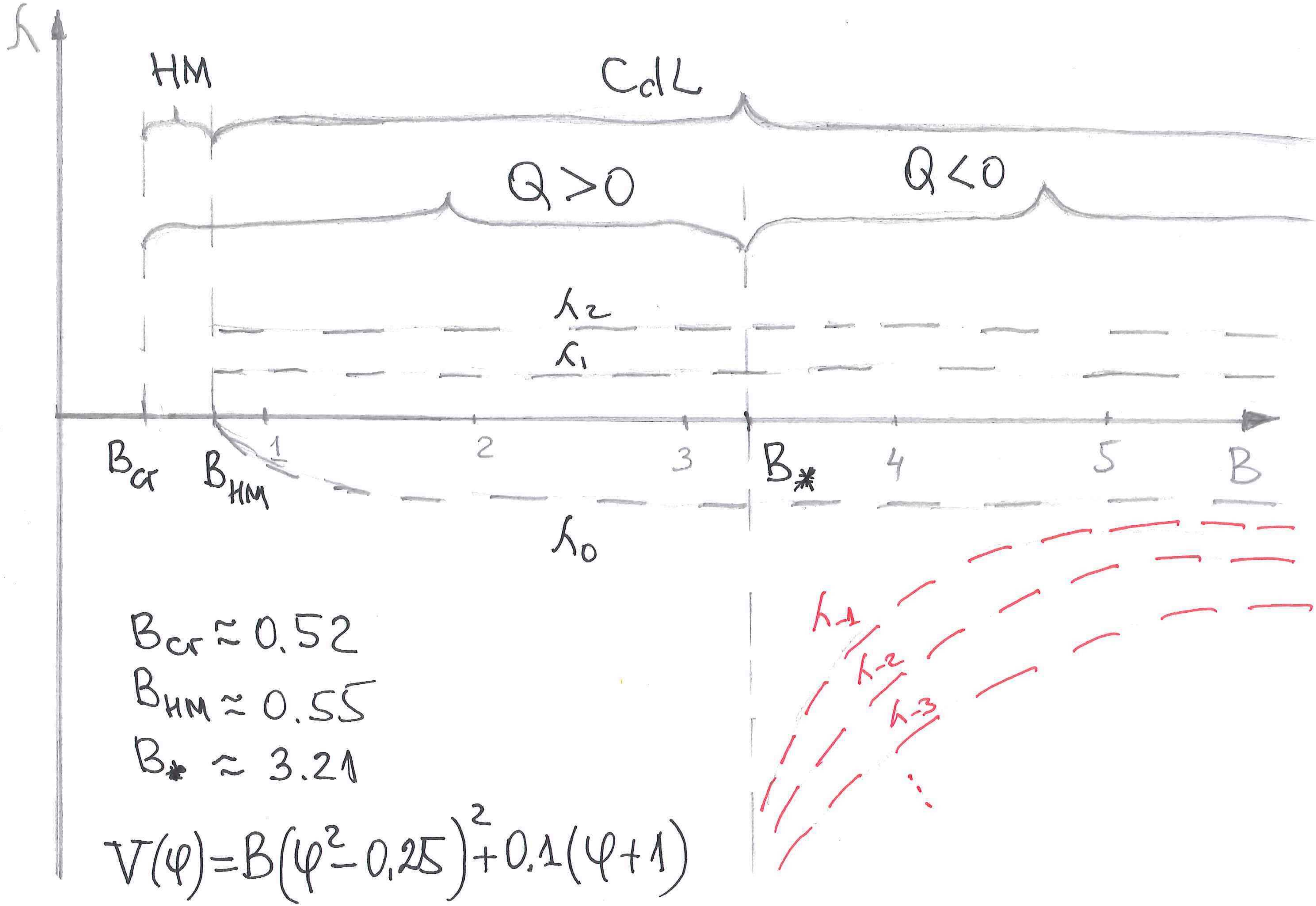
Our main results:

1. If $Q > 0$, one finds exactly one tunnelling negative mode for all bounces

2. If Q becomes negative in some interval

a). Solution of pulsation equation and first derivative is regular across $Q = 0$ points.

b). The one ”tunneling” negative mode continues to exist, but on top of it an infinite tower of additional negative modes arise with support in $Q < 0$ region.


λ
HM
CdL
Q>0
Q<0
λ2
λ1
λ0
λ-1
λ-2
λ-3
Bcr
BHM
B*
1
2
3
4
5
B
Bcr ≈ 0.52
BHM ≈ 0.55
B* ≈ 3.21
V(φ)=B(φ²−0,25)²+0,1(φ+1)

Examples of negative Q with parameters far from Planck scale

S. Bramberger, M.Chitishvili, and G.L., (2019).

Take the potential

$$V(\phi) = V_0 + \frac{\lambda}{8}(\phi^2 - \mu^2)^2 + \frac{\epsilon}{2\mu}(\phi + \mu) \tag{63}$$

Choose parameters

$$V_0 = 10^{-22}; \lambda = 10^{-19}; \epsilon = 10^{-30}; \mu = 0.4 \tag{64}$$

Then the potential is given in Figure 14 which has $V(\phi_{top})$ five orders of magnitude below the Planck scale.

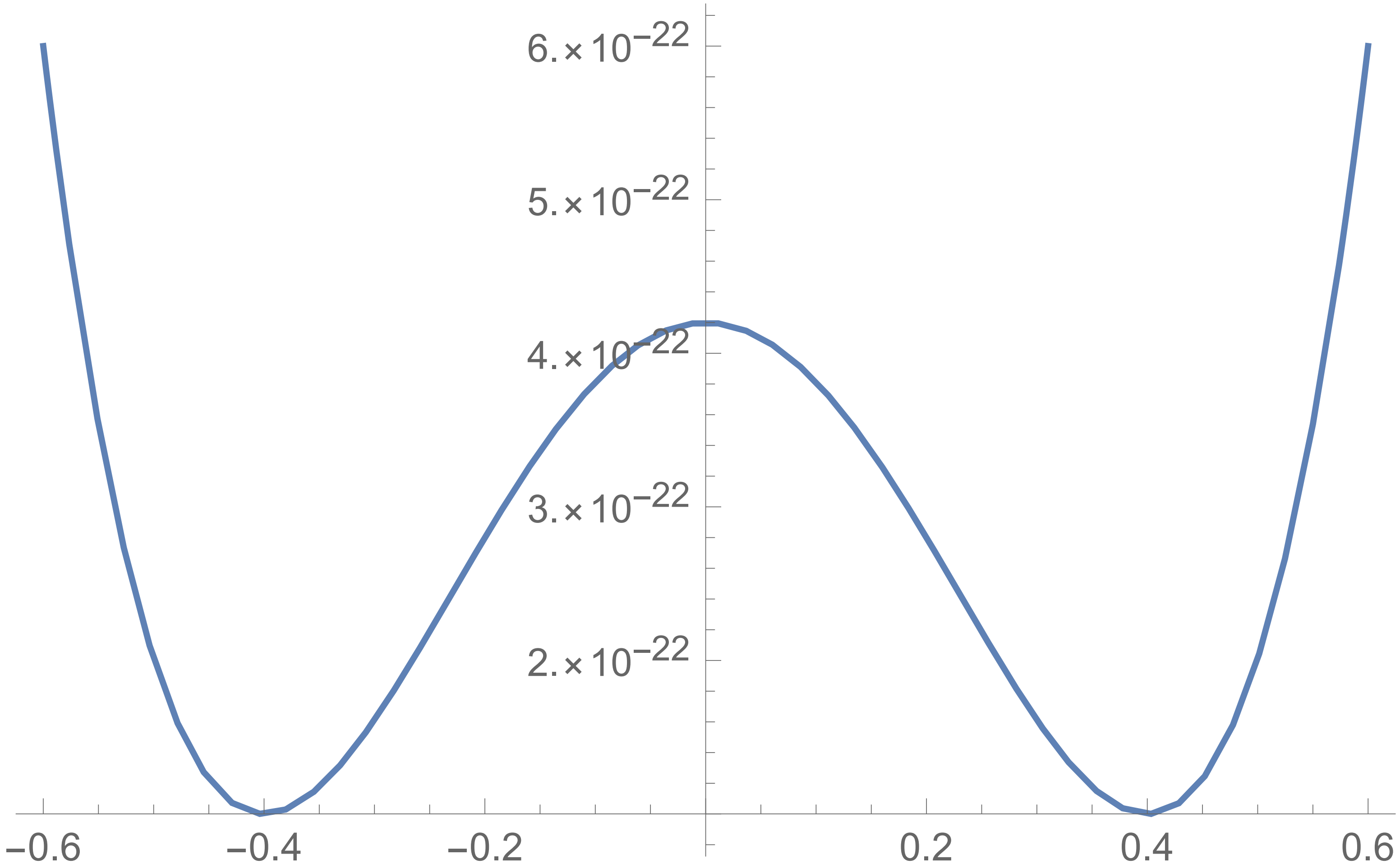


Figure 14: The potential.

The CdL bounce and the corresponding factor Q is shown in Figure 15. Clearly Q is negative while the potential is far from Planck.

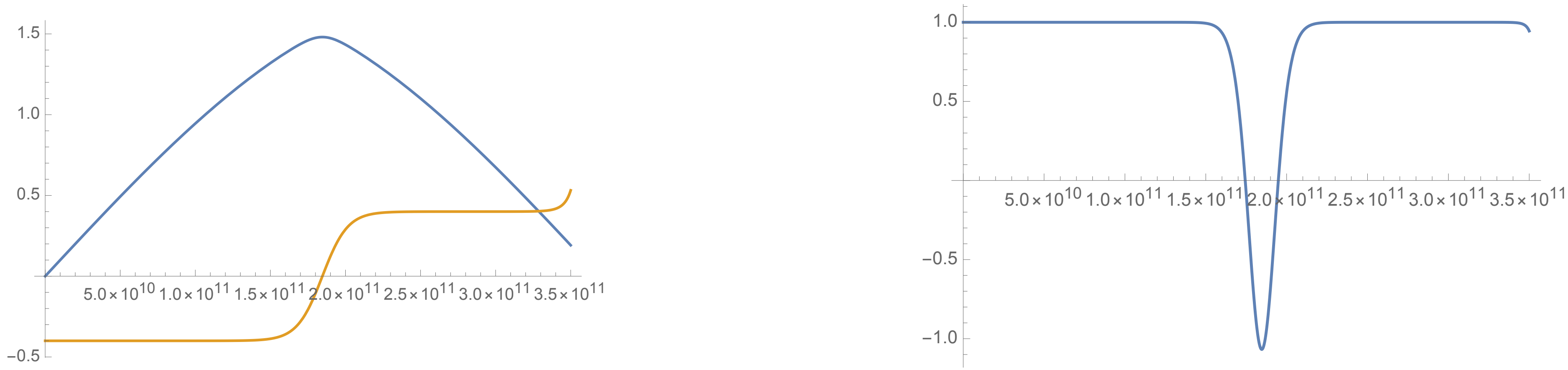


Figure 15: *Left:* Scale factor $a(t)/10^{11}$ in blue and scalar field $\phi(t)$ in orange.
Right: Corresponding factor Q

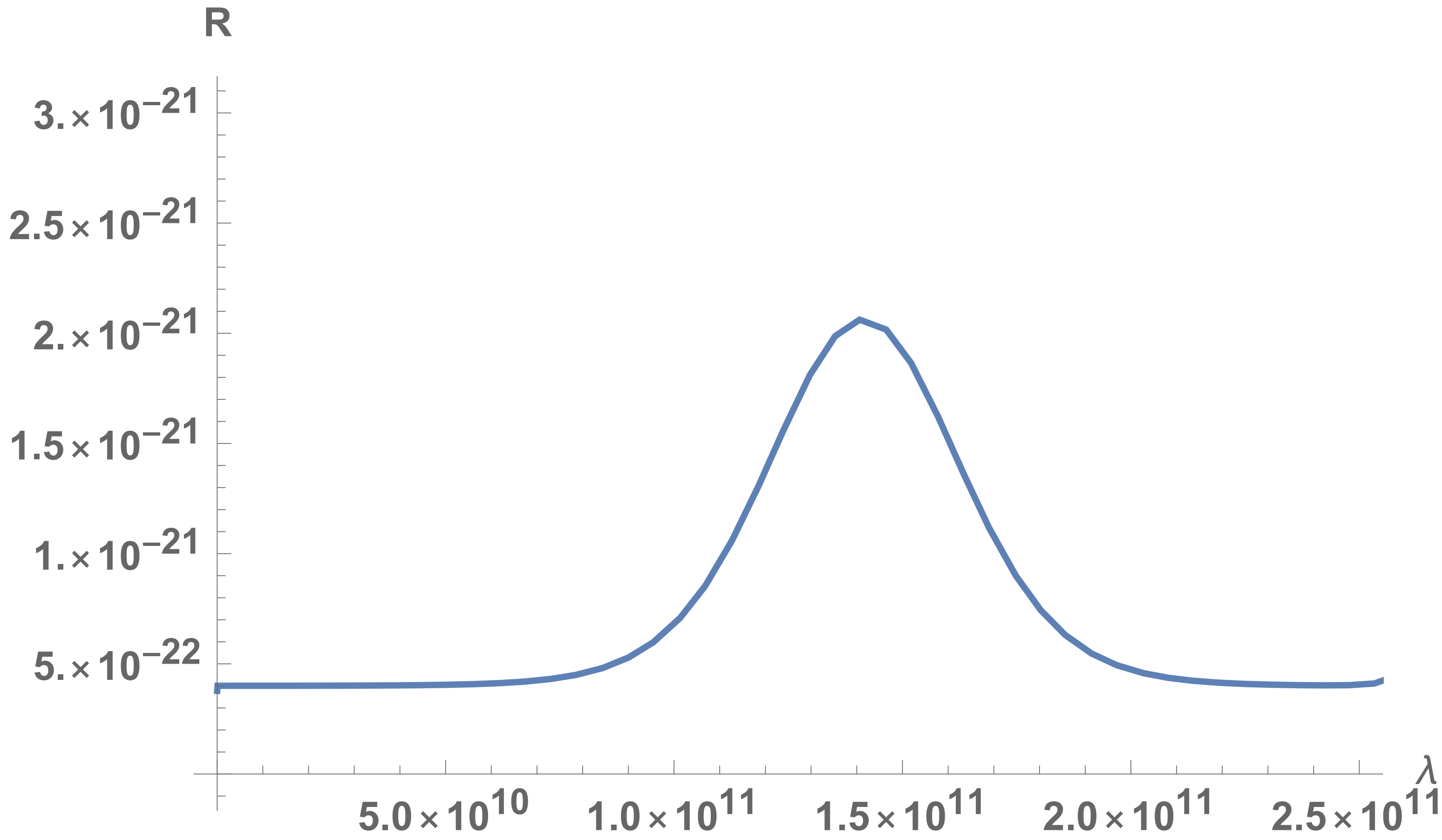


Figure 16: The four dimensional Ricci scalar close at the maximal radius of the bounce solution in fig 15

.

Higgs like potential

Markkanen, Rajantie and Stopyra (2018)

Taking into account current experimental values of the standard model parameters the instability scale of the Higgs potential, $\lambda(\mu_\Lambda) = 0$, is

$$\mu_\Lambda = 9.92 \cdot 10^9 Gev \tag{65}$$

It depends sensitively on the top and Higgs masses and at 1σ the range is $1.16 \cdot 10^9 Gev < \mu_\Lambda < 2.37 \cdot 2.37 \cdot 10^{11} Gev$. The top of the potential barrier lies at

$$\phi_{top} = 4.64 \cdot 10^{10} Gev \tag{66}$$

and the barrier height is

$$V_{top} = 3.46 \cdot 10^{38} Gev^4 = (4.31 \cdot 10^9 Gev)^4 \tag{67}$$

S. Bramberger, M.Chitishvili, and G.L., (2019).

Higgs potential can be modelled as

$$V_H = V_0 + \frac{\lambda_H(\phi)}{4}\phi^4 \tag{68}$$

$$\lambda_H = q\left[(ln\frac{\phi}{M_{Pl}})^4 - (ln\frac{\Lambda}{M_{Pl}})^4\right] \tag{69}$$

where q is fitting parameter, Λ is instability scale and $M_{Pl} = \sqrt{8\pi G} \approx 2.435 \cdot 10^{18} Gev$. Numerically we found that for $\Lambda < \Lambda_*$ Q is everywhere positive and for $\Lambda > \Lambda_*$ Q develops region with $Q < 0$. Choosing parameters $q = 10^{-7}$ and $V_0 = 10^{-12}$ we found $0.57 < \Lambda_* < 0.6$ in $M_{Pl} = 1$ units, see Figure 17. So, for realistic Higgs like potential negative mode problem shows up only at the Planck values of the instability scale.

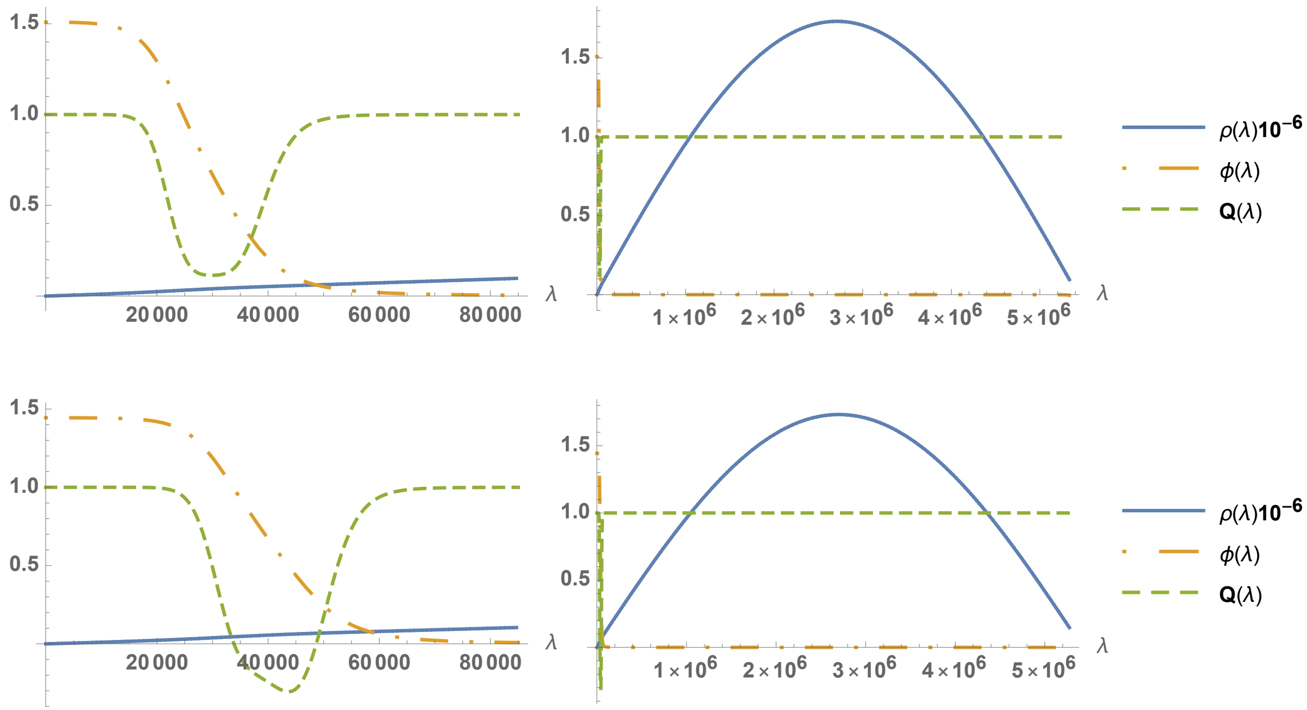


Figure 17: Here we show the values of the scalar field ϕ, scale factor ρ and the function Q for the Higgs like potential Eq. (68). The top figure shows the CdL instanton for $\Lambda = 0.57$ while the bottom one has $\Lambda = 0.6$. The images on the left are zoomed in versions of the full instantons shown on the right. $M_{Pl} = 1$ units are used.

Historical Summary of the negative mode problem

Year / Authors	Tunneling negative mode	Additional negative modes
1985, G.L., Rubakov Tinyakov	Not discussed	Infinitely many
1992, Tanaka, Sasaki	None	None
2000, Khvedelidze, G.L., Tanaka	One	Not discussed (only $Q > 0$ case)
2000, Gratton, Turok	One	Not discussed (only $Q > 0$ case)
2006, G.L.	N for the N-th oscillating bounce	Not discussed (only $Q > 0$ case)
2014, Lee, Weinberg	One / None	Infinitely many
2015, Koehn, G.L., Lehners	One	Infinitely many in $Q < 0$ case

Tab 1. Conclusion about the number of negative modes reached in different investigations.

Concluding remarks

I. We have shown that instantons with necks can be produced as a result of quantum tunnelling in the decay of a metastable vacuum in scalar field theories with non-minimal coupling to gravity (while they cannot be produced in the case of minimal coupling).

Challenge I:

- Spectrum of linear perturbations (number of negative modes)
- Lorentzian evolution of the bubble after materialization and global structure of space-time
- Existence of wormhole solutions in other NEC violating theories

II. It was found that with the proper reduction scheme CdL bounce has exactly one negative mode for $Q > 0$ backgrounds. The oscillating instantons and bounces with N nodes have exactly N homogeneous negative modes in their spectrum of linear perturbations. Existence of more than one negative modes makes obscure the relation of these oscillating bounce solutions to the false vacuum decay processes.

Challenge II:

- How to interpret an infinite tower of additional negative modes for $Q < 0$ cases: Their existence and significance remain mysterious even after 34 years.

Thank you for attention!