Non-perturbative QCD corrections to electro-weak observables: the case of muon's g-2

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Outline

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Motivation - muon anomalous magnetic moment $(g-2)_\mu$

- intrinsic property of spin-1/2 muon

$$\langle \mu(\vec{p}') \mid J_\nu^{\mathrm{em}}(0) \mid \mu(\vec{p}) \rangle = -e\, \bar{u}(\vec{p}') \left(F_1(q^2)\, \gamma_\nu + i \frac{F_2(q^2)}{4m} \left[\gamma_\nu \,,\, \gamma_\rho \right] q_\rho \right) u(\vec{p})$$

$$a_\mu = (g-2)_\mu / 2 = F_2(0)$$

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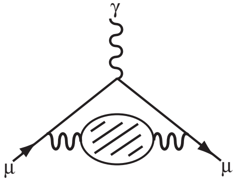
- It's so precise! $\quad a_\mu(\,\mathrm{Standard\ Model}\,) = 11659182.3\,(4.3)\cdot 10^{-10}$
- What precisely is so precise?
  persistent discrepancy 3.5 standard deviations ($2 \sim 4$)
- gold-plated observable in search for physics beyond the Standard Model

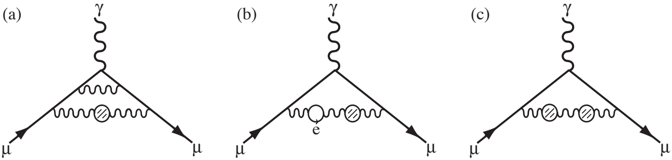
$$(g-2)_\ell^{\mathrm{new\ physics}} \sim m_\ell^2/M_{\mathrm{new\ physics}}^2$$

$$m_e^2 : m_\mu^2 : m_\tau^2 = 1 : 4\cdot 10^4 : 10^7$$

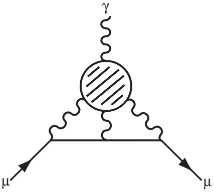
$(g-2)_\mu$ in perturbative QED+EW and non-perturbative QCD

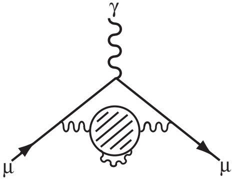


α^2_{QED} HVP LO



α^3_{QED} HVP NLO





α^3_{QED} HLbL and FSR

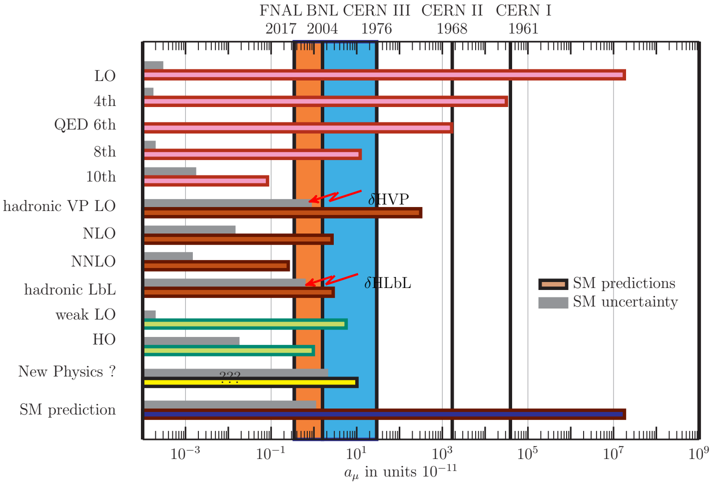
Motivation — $(g-2)_\mu$ Standard Model contributions

SM Contribution	Value $\pm$ Error	Ref
QED (incl. 5-loops)	116584718.951 ± 0.080	[1]
HVP LO	6931 ± 34	[2]
HVP NLO	-98.4 ± 0.7	[2,3]
HVP NNLO	12.2 ± 0.1	[3]
HLbL	105 ± 26	[4]
Weak (incl. 2-loops)	153.6 ± 1.0	[5]
SM Total (0.37 ppm)	116591823 ± 43	
Experiment (0.54 ppm)	116592089 ± 63	[6]
Difference (Exp − SM)	266 ± 76	$3.5\,\sigma$

([Prog.Part.Nucl.Phys. 104 (2019) 46-96, EPJ Web Conf. 166 (2018) 00022], in units of 10^{-11})

[1] Phys.Rev.Lett. 109, 111808 (2012)
[2] Eur.Phys.J. C77 (2017) no.12, 827
[3] Phys.Lett. B734, 144 (2014)
[4] Adv.Ser.Direct.High Energy Phys. 20 (2009) 303-317
[5] Phys.Rev. D88, 053005 (2013)
[6] Phys.Rev. D73, 072003 (2006)

Motivation — $(g-2)_\mu$ error budget



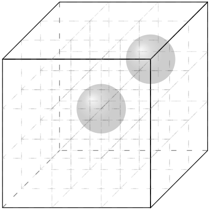
[Jegerlehner, EPJ Web Conf. 166 (2018) 00022]

@ δHVP: "*Lattice QCD is the ultimate tool to get QCD predictions in future.*"

@ δHLbL: "*Also in this case lattice QCD for me is the ultimate approach, although tough to be achieved with limited computing resources.*"

Outline

Lattice calculation - overview



- discretized finite box in Euclidean space, lattice spacing $a \sim 0.05 - 0.1\,\mathrm{fm}$, volume $L^3 \times T$, $L \sim$ several fm, $T = (2 \sim 3)\,L$

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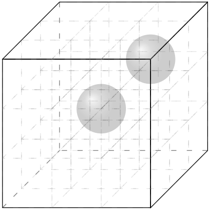

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  but approaching *physical point* simulations

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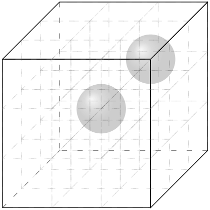

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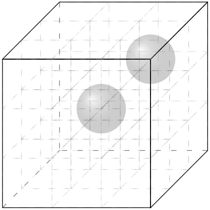

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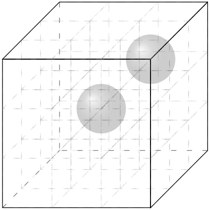

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- strong and electroweak IB from dedicated calculations

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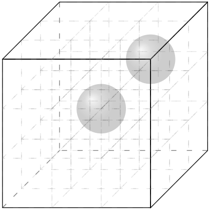

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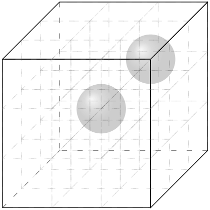

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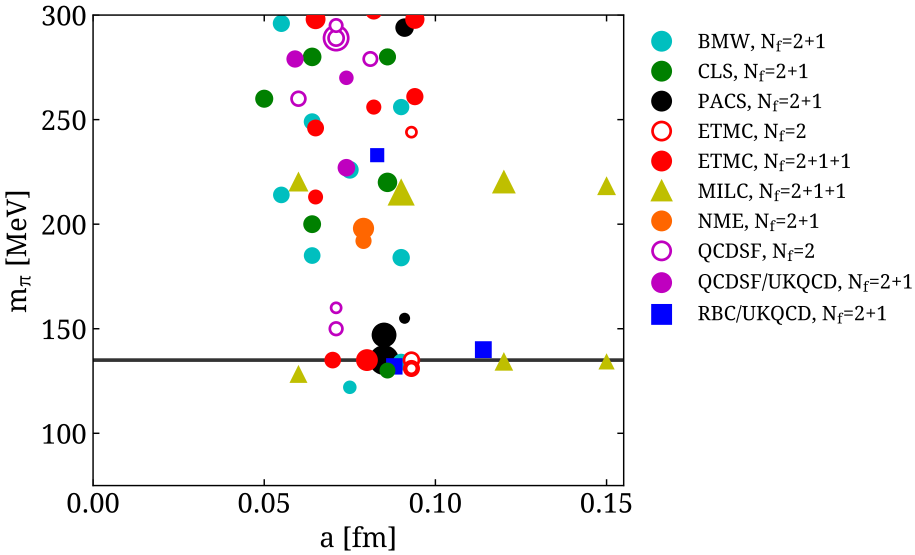
- finite time extent (+ signal-to-noise ratio); IR / UV cutoff $2\pi/L \lesssim Q \lesssim \pi/a$
- chiral, continuum and infinite volume limit to control all systematics

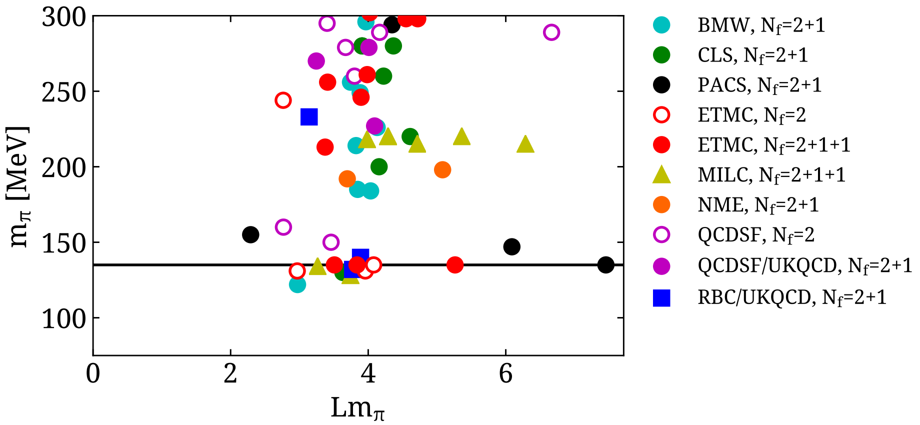
Lattice QCD - ensemble landscape (1)

Pion mass m_π versus lattice spacing a [G. Koutsou @ ETMC]

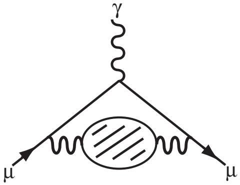


Lattice QCD - ensemble landscape (2)

Lattice size $m_\pi L$ versus spatial lattice size L [G. Koutsou @ ETMC]



LO HVP in lattice QCD



$$a_\mu^{\rm LO\,HVP} = \alpha_{\rm QED}^2 \int_0^\infty \frac{dQ^2}{Q^2}\, w\left(Q^2/m_\mu^2\right)\, \left(\Pi(Q^2) - \Pi(0)\right)$$

- integral over Euclidean momenta [Phys.Rev.Lett. 91 (2003) 052001]
- weight function w from QED perturbation theory

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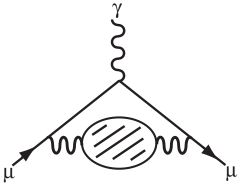

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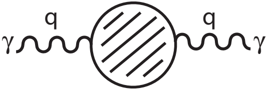
$$\Pi_{\mu\nu}(Q) = \sum_x \langle J_\mu(x)\, J_\nu(y)\rangle\, \mathrm{e}^{iQ(x-y)}$$

$$\Pi_{\mu\nu}(Q) = \left(Q_\mu\, Q_\nu - Q^2\delta_{\mu\nu}\right)\, \Pi(Q^2)\,(\quad + \text{lattice artefacts})$$

$$J_\mu = \frac{2}{3}\,\bar{u}\,\gamma_\mu u - \frac{1}{3}\,\bar{d}\,\gamma_\mu\, d - \frac{1}{3}\,\bar{s}\,\gamma_\mu\, s + \frac{2}{3}\,\bar{c}\,\gamma_\mu\, c$$

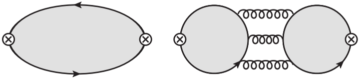
- local current ($+Z_V$) or conserved vector current

LO HVP in lattice QCD



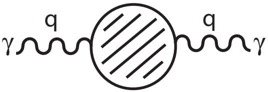
$$\Pi_{\mu\nu}(Q) = \left(Q_\mu\, Q_\nu - Q^2\delta_{\mu\nu}\right)\, \Pi(Q^2)$$

- $\int_0^\infty dQ^2 \leftrightarrow$ discretized lattice momenta $Q_\mu = n_\mu\, 2\pi/L,\; n_\mu = 0, \ldots, L/a - 1$
- $\Pi(Q^2)$ for $(2\pi/L)^2 \lesssim Q^2 \lesssim (\pi/a)^2$
- determine $\Pi(0)$ and curvature for $Q^2 < Q^2_{\mathrm{min}}$
- quark-connected and quark-disconnected diagrams



- (momentum dependent) signal-to-noise ratio of $\Pi(Q^2)$ from finite gauge statistics

LO HVP in lattice QCD - example data



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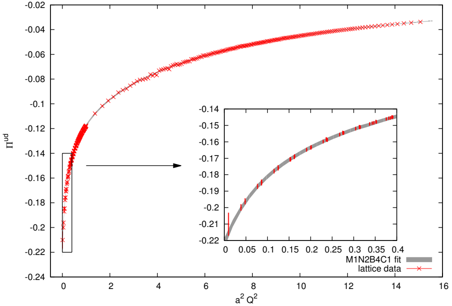
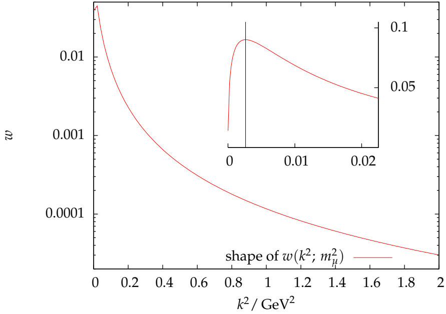
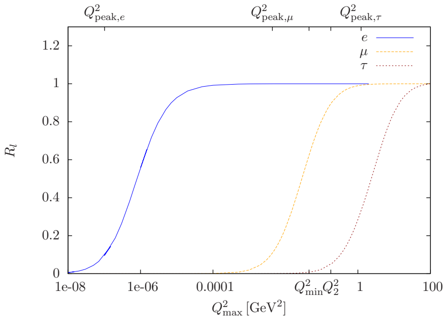




left: hadronic vacuum polarization function (light quarks, connected) for $a = 0.078\,\mathrm{fm}$, $L = 2.5\,\mathrm{fm}$, $m_{PS} = 274\,\mathrm{MeV}$; right: weight function

LO HVP integral saturation



- example for $N_f = 2+1+1$ tmLQCD with $a = 0.061\,\mathrm{fm}$, $m_\pi = 318\,\mathrm{MeV}$, $L = 2.9\,\mathrm{fm}$ [EPJ Web Conf. 118 (2016) 01029]
- $Q^2_{\mathrm{min}} = 0.045\,\mathrm{GeV}^2 \gg Q^2_{\mathrm{peak},\mu} \approx 0.0026\,\mathrm{GeV}^2$

$$a_l^{\mathrm{LO\,HVP}} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2}\, w\left(Q^2/m_l^2\right)\, \Pi_{\mathrm{R}}(Q^2)$$

- shown is the ratio

$$R_l(Q^2_{\mathrm{max}}) = \frac{a_l^{\mathrm{LO\,HVP}}(Q^2_{\mathrm{max}})}{a_l^{\mathrm{LO\,HVP}}(100\,\mathrm{GeV}^2)}\,,$$

- weight function w peaked at $Q^2_{\mathrm{peak},l} = (\sqrt{5}-2)\, m_l^2$
- quantized lattice momenta (periodic boundary conditions), $Q^2_{\mathrm{min}} \sim (2\pi/L)^2$

Model-independent parametrization of HVP [Phys.Rev. D86 (2012) 054509]

- rewrite $\Pi(Q^2)$ using the spectral function $\rho(t)$

$$\Pi(Q^2) = \Pi(0) - Q^2\,\Phi(Q^2)\,, \quad \Phi(Q^2) = \int_{4m_\pi^2}^{\infty} dt\, \frac{\rho(t)}{t(t+Q^2)}$$

- with $\tau = 1/t$, $d\nu(\tau) = \rho(1/\tau)\,d\tau$, $\rho(t) = \frac{1}{\pi}\Im\Pi(t)$, $\rho(t) \geq 0$ for $4m_\pi^2 \leq t < \infty$

$$\text{Stieltjes function} \qquad \Phi(z = Q^2) = \int_0^{1/R} \frac{d\nu(\tau)}{1+\tau z}$$

analytic in complex plane except on negative real axis (cut) for $z \leq -R$

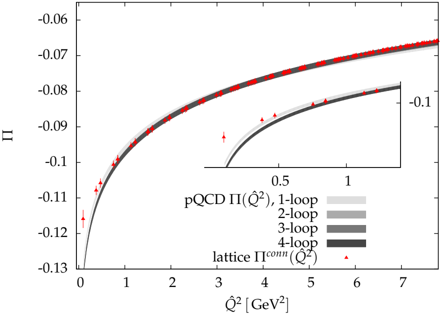
- multipoint Padé approximants converge to Φ

$$R_{\lfloor P/2 \rfloor}^{\lfloor (P-1)/2 \rfloor}(Q^2) = a_0 + \sum_{n=1}^{\lfloor P/2 \rfloor} \frac{a_n}{b_n + Q^2}$$

with $a_0 = 0$ for P even and

$$a_n > 0\,, \qquad n \in \{1, \ldots, \lfloor P/2 \rfloor\}$$
$$b_{\lfloor P/2 \rfloor} > b_{\lfloor P/2 \rfloor - 1} > \cdots > b_1 \geq R$$

Matching to perturbative QCD



$$\Pi(q^2;\,\bar{\mu}) = \sum_{l=0}^{L} \left(\frac{\alpha_s(\bar{\mu};\,\Lambda^{(2)}_{\overline{\mathrm{MS}}})}{\pi} \right)^l \Pi^{(l)}(q^2;\,\bar{\mu})$$

$$\Pi^{(l)}(q^2;\,\bar{\mu}) = C^{(l)}_{m\le n}(-q^2/\bar{\mu}^2)\,\log(-4z)^m\,\frac{1}{z^n}$$

$$z = q^2/(4\bar{m}(\bar{\mu})^2)\,.$$

- $N_f = 2$, tmLQCD, light quark contribution
- $a = 0.063\,\mathrm{fm}$, $m_\pi = 325\,\mathrm{MeV}$, $L = 2\,\mathrm{fm}$

- perturbative expansion up to 4-loop order [JHEP 1207 (2012) 017]
- $\bar{\mu} = 2\,\mathrm{GeV}$ $\overline{MS}$ renormalization scale
- $\bar{m}(\bar{\mu})$ renormalized $\overline{MS}$ quark mass
- matching at $Q^2 = 4\,\mathrm{GeV}^2$ by single additive constant $(\Pi^{\mathrm{pQCD}}(0) - \Pi^{\mathrm{lat}}(0))$

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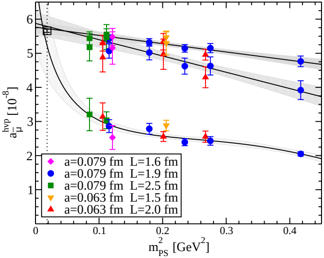
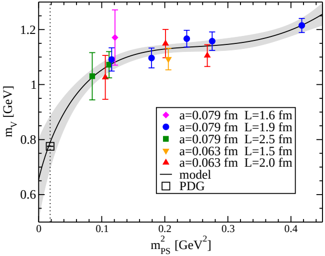
Pion mass dependence and external scales problem





- integration requires lattice scale

$$a_\mu^{\rm hlo} = \alpha_{\rm QED}^2 \int_0^\infty \frac{d\hat{Q}^2}{\hat{Q}^2} w(\hat{Q}^2/(am_\mu)^2)\Pi_{\rm lat}(\hat{Q}^2)\,, \quad \hat{Q} = aQ$$

- effective dimension of lattice QCD observable X (at fixed gauge coupling g)

$$d_{\rm eff}[X] \equiv -\frac{a}{X}\left.\frac{\partial X}{\partial a}\right|_g$$

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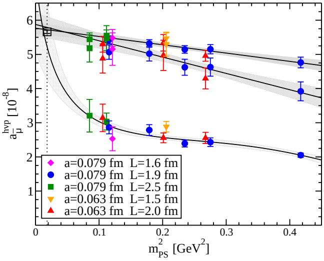

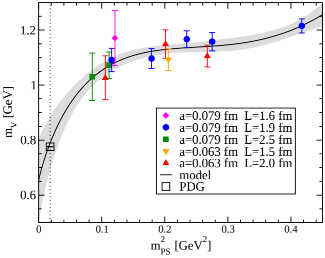

- effective dimension of "dimensionless" $a_\mu^{\rm hlo}$

$$d_{\rm eff}[a_\mu^{\rm hlo}] = -2\left(\int_0^\infty \frac{dQ^2}{Q^2} w(Q^2/m_\mu^2) Q^2 \frac{d\Pi_R}{dQ^2}\right)\left(\int_0^\infty \frac{dQ^2}{Q^2} w(Q^2/m_\mu^2)\Pi_R(Q^2)\right)^{-1}$$
$$\approx -2$$

- modified observables with hadronic scale H (m_ρ , f_π, ...), $d_{\rm eff}[a_\mu^{\rm hlo}] = 0$

$$a_\mu^{\rm hlo} = \alpha^2 \int_0^\infty \frac{dQ^2}{Q^2}\, w\left(\frac{Q^2}{H^2}\cdot\frac{H_{\rm phys}^2}{m_\mu^2}\right)\Pi_R(Q^2) = \alpha^2\int_0^\infty \frac{d\hat{Q}^2}{\hat{Q}^2} w\left(\frac{\hat{Q}^2}{\hat{H}^2}\cdot\frac{H_{\rm phys}^2}{m_\mu^2}\right)\Pi_{\rm lat}(\hat{Q}^2)$$

Finite volume / anisotropy effects [Phys.Rev. D93 (2016) no.5, 054508]

- WTI in finite volume with $T \neq L$ and at non-zero lattice spacing

$$\sum_{\mu} \hat{q}_{\mu} \, \Pi_{\mu\nu}(\hat{q}) = 0 \qquad \hat{q}_{\mu} = \frac{2}{a} \sin\left(a q_{\mu}/2\right)$$

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- but $\Pi_{\mu\nu}(0) \neq 0$; $\Pi_{\mu\nu}(0) = \delta_{\mu\nu} \, \left(\Pi_s(0) + \delta_{\mu 4} \, \left(\Pi_4(0) - \Pi_s(0)\right)\right)$

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- $\bar{\Pi}_{\mu\nu}(\hat{q}) = P^T_{\mu\kappa} \, \left(\Pi_{\kappa\lambda}(\hat{q}) - \Pi_{\kappa\lambda}(0)\right) \, P^T_{\lambda\nu}$
  with $P^T_{\mu\nu} = \delta_{\mu\nu} - \hat{q}_{\mu} \, \hat{q}_{\nu} / \hat{q}^2$

Finite volume / anisotropy effects [Phys.Rev. D93 (2016) no.5, 054508]

- WTI in finite volume with $T \neq L$ and at non-zero lattice spacing

$$\sum_\mu \hat{q}_\mu \, \Pi_{\mu\nu}(\hat{q}) = 0 \qquad \hat{q}_\mu = \frac{2}{a} \sin\left(a q_\mu / 2\right)$$

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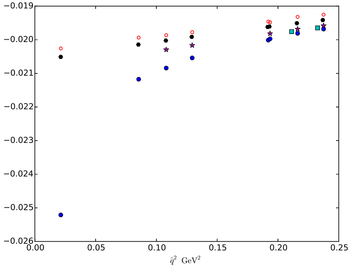

five irreducible sub-structures

$$\begin{aligned}
A_1 &: \textstyle\sum_i \bar{\Pi}_{ii} = (3\hat{q}^2 - \vec{\hat{q}}^{\,2})\bar{\Pi}_{A_1} \;, \\
A_1^{44} &: \bar{\Pi}_{44} = (\vec{\hat{q}}^{\,2})\bar{\Pi}_{A_1^{44}} \;, \\
T_1 &: \bar{\Pi}_{4i} = -(\hat{q}_4 \hat{q}_i)\bar{\Pi}_{T_1} \;, \\
T_2 &: \bar{\Pi}_{ij} = -(\hat{q}_i \hat{q}_j)\bar{\Pi}_{T_2} \;, \; i \neq j \;, \\
E &: \bar{\Pi}_{ii} - \textstyle\sum_i \bar{\Pi}_{ii}/3 = (-\hat{q}_i^2 + \vec{\hat{q}}^{\,2}/3)\bar{\Pi}_E
\end{aligned}$$

$m_\pi = 220\,\mathrm{MeV}$, $L = 3.8\,\mathrm{fm}$, $m_\pi L = 4.2$

Time momentum representation

- subtracted vacuum polarization in terms of time-dependent vector-vector correlator

$$\Pi(Q^2) - \Pi(0) = \frac{1}{Q^2} \int_0^\infty dt\, G(t) \left[Q^2 t^2 - 4 \sin^2 (Q\, t/2) \right]$$

$$G(t) = \sum_{\vec{x}} \langle J_k(t, \vec{x})\, J_k(0) \rangle$$

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- integration with Euclidean time-dependent kernel [Eur.Phys.J. A47 (2011) 148]

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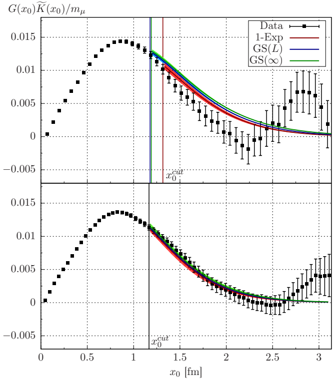
- a_μ^{hvp} in terms of lattice spectrum of vector states

$$G(t) = \sum_{n\geq 0} |A_n|^2 \left(\mathrm{e}^{-E_n t} + \mathrm{e}^{-E_n(T-t)}\right)$$

Finite - T / volume correction in TMR



[JHEP 1710 (2017) 020]

- isospin components of G for asymptotic time

$$G(t) = G^{(I=0)}(t) + G^{(I=1)}$$

$$I = 0 : \omega - \pi\pi\pi \quad / \quad I = 1 : \rho - \pi\pi$$

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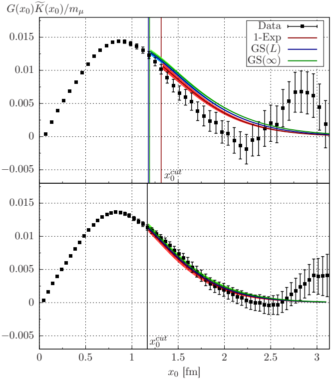

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$$G^{(I=1)}(t) = \int_0^\infty d\omega\, \omega^2\, \rho(\omega^2)\, \mathrm{e}^{-\omega t}$$

$$\rho(\omega) = \frac{1}{48\pi^2} \left(1 - \frac{4m_\pi^2}{\omega^2}\right) |F_\pi(\omega)|^2$$

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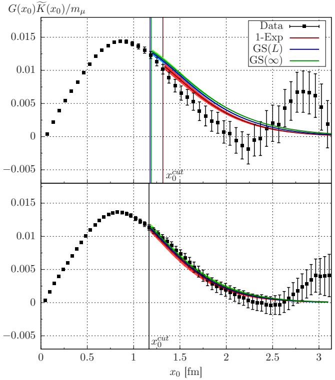

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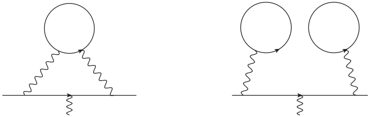
- infinite-volume F_π from finite-volume volume spectrum with Lellouch-Lüscher formalism [Commun.Math.Phys. 219 (2001) 31-44]

$$|A_n|^2 = \frac{2k^5}{3\pi E_n^2} \frac{|F_\pi(E_n)|^2}{q_n\, \phi'(q_n) + k_n \delta'_{\ell=1}(k_n)}$$

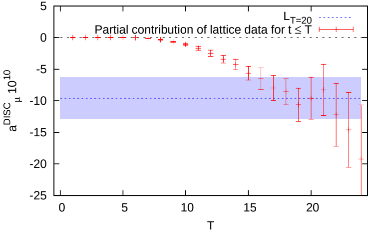
HVP disconnected contribution [Phys.Rev.Lett. 116 (2016) no.23, 232002]



- requires all-to-all quark propagation

$$C^{\mathrm{disc}}(t) = \frac{1}{TL^3} \left\langle \left[\sum_{\vec{x}} \mathrm{Tr}\left(J_j(\vec{x}, t) \right) \right] \times \left[\sum_{\vec{y}} \mathrm{Tr}\left(J_j(\vec{y}, 0) \right) \right] \right\rangle$$

- special techniques: exact fermionic low-modes, stochastic quark propagation, hierarchical probing



$$a_\mu^{\mathrm{LO\,HVP,disc}} = \lim_{T \to \infty} \sum_{t=0}^{T} \tilde{K}(t)\, C^{\mathrm{disc}}(t)$$

$$a_\mu^{\mathrm{LO\,HVP,disc}} = -9.6\,(3.3)\,(2.3) \times 10^{-10}$$

$a = 0.11\,\mathrm{fm}$, $L = 5.5\,\mathrm{fm}$, physical pion mass

Electromagnetic and strong isospin breaking

- $\mathcal{O}(\alpha_{\rm QED})$ QED correction to hadronic vector current correlator

$$\mathcal{S}_{\rm quark} \to \mathcal{S}_{\rm quark}^{\rm IB} = \bar\psi\, D\, \psi + \bar m_q\, \bar\psi\, \psi + e\, J_\mu^{\rm em}\, A_\mu$$

$$C(t) = \langle J_j(t)\, J_j(0)\rangle_{\rm QCD} \to C(t) + \alpha_{\rm QED}\, C^{\rm QED}(t) + \dots$$

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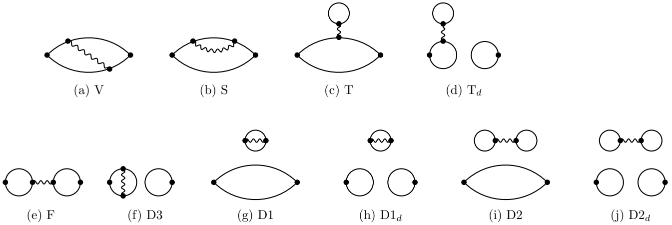

[PoS LATTICE2018 (2018) 134, Phys.Rev.Lett. 121 (2018) no.2, 022003]

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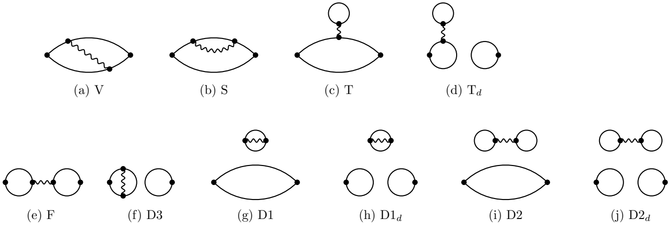


[PoS LATTICE2018 (2018) 134, Phys.Rev.Lett. 121 (2018) no.2, 022003]

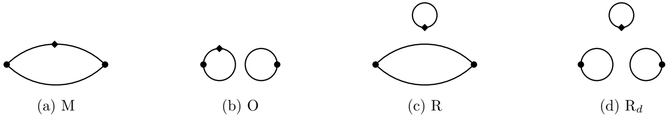
- photon propagation with QED$_L$ method [Prog.Theor.Phys. 120 (2008) 413-441]

$$\Delta_{\mu\nu} = \delta_{\mu\nu}\, \frac{1}{N} \sum_{k,\vec{k}\neq 0} \frac{e^{ik(x-y)}}{\hat{k}^2} \qquad \text{(Feynman gauge, excl. zero mode)}$$

- Strong isospin correction from mass term insertions

$$\mathcal{S}_{\text{quark}} \to \mathcal{S}^{\text{IB}}_{\text{quark}} = \bar{\psi}\, D\, \psi + \bar{m}_q\, \bar{\psi}\, \psi + \Delta m_f\, \bar{\psi}_f\, \psi_f$$

$$C(t) = \langle J_j(t)\, J_j(0) \rangle_{\text{QCD}} \to C(t) + \sum_f \Delta m_f\, C^{\Delta m_f}(t) + \dots$$



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Outline

HVP LO lattice results update of [Prog.Part.Nucl.Phys. 104 (2019) 46-96]

Collaboration	$a_\mu^{\rm hvp}$	$(a_\mu^{\rm hvp})^{uds}$	$(a_\mu^{\rm hvp})^{ud}$	$(a_\mu^{\rm hvp})^{s}$	$(a_\mu^{\rm hvp})^{c}$	Method
$N_f = 2+1+1$:						
BMW 17 [1]	$711.0(7.5)(17.3)^{*\dagger}$	$696.3(7.5)(17.3)^{*\dagger}$		53.7(0)(4)	14.7(0)(1)	TMR
ETMC 19 [2]	$682(19)^{*\dagger}$			$53.1(2.5)^{\dagger}$	$14.72(56)^{\dagger}$	TMR
HPQCD 16 [3]	$667(6)(12)^{*\dagger}$		$599(11)^{*\dagger}$			Moments
HPQCD 14 [4]				53.4(6)	14.4(4)	Moments
ETMC 13 [5]	674(21)(18)	655(21)		53(3)	14.1(6)	Fits in Q^2
$N_f = 2+1$:						
PACS 19 [14]	$737(9)(^{13}_{18})$		673(9)(11)	52.1(2)(5)	11.7(2)(1.6)	Fits in Q^2, TMR
CLS/Mainz 19 [13]	$720.0(12.4)(9.9)^{*\dagger}$		674(12)(5)	54.5(2.4)(0.6)	14.66(0.45)(0.06)	TMR
RBC/UKQCD 18 [6]	$715.4(16.3)(9.2)^{*\dagger}$	$701.2(16.3)(9.2)^{*\dagger}$		53.2(4)(3)	14.3(0)(7)	TMR
	$692.5(1.4)(0.5)(0.7)(2.1)^{*\dagger}$					R-ratio, TMR
RBC/UKQCD 16 [7]				$53.1(9)(^{1}_{3})$		Hybrid
RBC/UKQCD 11 [8]		641(33)(32)				Fits in Q^2
Aubin & Blum 07 [9]		713(15) / 748(25)				Fits in Q^2
$N_f = 2$:						
CLS/Mainz 17 [10]	$654(32)(^{21}_{23})^{*}$	$639(32)(^{21}_{23})^{*}$	$588(32)(^{21}_{23})$	51.1(1.7)(0.4)	14.3(2)(1)	TMR
CLS/Mainz 11 [11]		618(64)				Fits in Q^2
ETMC 11 [12]			$572(16)^{*}$			Fits in Q^2

[1] Phys. Rev. Lett. 121 (2018) 022002
[2] Phys.Rev. D99 (2019) no.11, 114502
[3] Phys. Rev. D96 (3) (2017) 034516.
[4] Phys. Rev. D89 (11) (2014) 114501
[5] JHEP 02 (2014) 099
[6] Phys. Rev. Lett. 121 (2018) 022003
[7] JHEP 04 (2016) 063
[8] Phys. Rev. D85 (2012) 074504
[9] Phys. Rev. D75 (2007) 114502
[10] JHEP 10 (2017) 020
[11] JHEP 1203 (2012) 055
[12] Phys. Rev. Lett. 107 (2011) 081802
[13] Phys.Rev. D100 (2019) no.1, 014510
[14] Phys.Rev. D100 (2019) no.3, 034517

Summary and outlook

- HVP LO: convergence of lattice calculations with control of all leading systematics
- HLbL : Gilberto's talk
- White paper from $g-2$ Theory Initiative upcoming

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- HVP LO contribution to anomalous magnetic moment of e and τ
- HVP NLO $(g-2)_\ell$: with control on vacuum polarization and available QED integration kernels
- hadronic leading order correction to running of electro-weak couplings
- ...Lamb shift in muonic hydrogen

Thank you very much for your attention.