Regional Doctoral Program in Theoretical and Experimental Particle Physics
7th Autumn PhD School & Workshop "Frontiers of QCD" - Tbilisi, Georgia

Bound and Scattering States in a Finite Volume including the Coulomb Interaction

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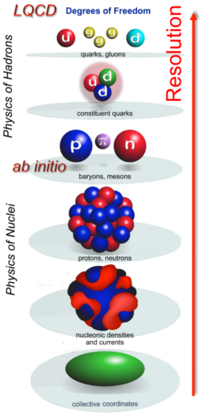
Motivation

We aim at deriving closed forms for the energy shifts concerning bound and scattering states of two-body charged systems on a cubic box of side L.

Applications



courtesy of V. Durant (2019)

- **Nuclear Lattice EFT** based on ChEFT for nucleons and pions PRL 104, 142501 (2010), PRL 117, 132501 (2016), Nature 528, 111-114 (2015), LNP 957, Springer (2019), Phys. Lett. B 772, 839-848 (2017) or hyperons Nucl. Phys. A 936, 29-44 (2015).
- **Lattice Quantum Chromodynamics**
  - for hadrons at the physical quark masses PRD 76, 114508 (2007), PRD 82, 094508 (2010), PRD 86, 034507 (2012), PRL 111, 252001 (2013), PRD 91, 074516 (2015) $\rightsquigarrow$ The accuracy with which the properties of the mesons are calculated requires the embedding of the strong interactions within the full SM.
  - for light nuclei PRD 81, 054505 (2010), PRL 106, 162001 (2011), PRD 85, 054511 (2012) $\rightsquigarrow$ QED plays a critical role in the stability and structure of nuclei **even if** calculations for $A \geq 5$ nuclei are still performed at unphysical quark masses
- **Phenomenological models** for α particles within α-conjugate nuclei PRD 90, 034507 (2014), PRD 92, 014506 (2014), G.S. EPJ A 54, 232 (2018), G.S. (in preparation)

Overview

33 years of finite volume corrections for the energy of lattice eigenstates

... for two-body systems:

- $\ell = 0$ eigenstates Commun. Math. Phys. 104, 177 (1986), Commun. Math. Phys. 105, 153 (1986), Nucl. Phys. B 354, 531-578 (1991) and $\ell > 0$ eigenstates PRD 83, 114508 (2011), PRL 107, 112001 (2011), Ann. Phys. 327, 1450 (2012);
- particles with spin J. High Energy Phys. 08, 024 (2008), PRD 92, 074509 (2015);
- moving frames Nucl. Phys. B 450, 397 (1995), Nucl. Phys. B 727, 218 (2005) , PRD 84, 091503 (2011), EPJ A 48, 114 (2012), PRD 86, 094513 (2012), J. Phys. G 41, 015105 (2014);
- generalized boundary conditions PLB 609, 73 (2005), EPJ A 47, 139 (2011), PRD 89, 074509 (2014), PRC 93, 054002 (2016), PRC 93, 054304 (2016), PRC 95, 074512 (2017);
- perturbative QED corrections for $\ell = 0$ eigenstates PRD 90, 074511 (2014) and for $\ell > 0$ eigenstates $\rightsquigarrow$ **our task !**

... for three-body systems:

- $\ell = 0$ eigenstates EPJ A 48, 67 (2012), Phys. Rev. Lett. 114, 091602 (2015);
- moving frames PRD 95, 034501 (2017);
- twisted boundary conditions A. Agadjanov, Bethe Forum, BCTP Bonn (2019);

... for N-body systems: PLB 779, 9-15 (2018)

Finite volume energy corrections

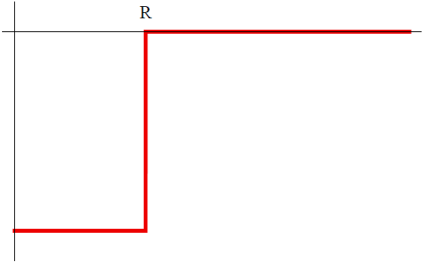
We review the derivation for bound states of two-body systems with angular momentum ℓ in a cubic box of size L^3 under periodic boundary conditions (PBC)

The Schrödinger equation for a two-body system in relative coordinates

$$\hat{H} = -\frac{1}{2\mu}\nabla_r^2 + V(r)$$

$$\hat{H}|\psi_B\rangle = -\frac{\kappa_0^2}{2\mu}|\psi_B\rangle$$

with a finite range interaction
$\rightsquigarrow$ low energy universality



$$V(r) = 0 \ \text{ for } \ r > R$$

$$\psi_B(\mathbf{r}) = \frac{u_\ell(r)}{r} Y_\ell^m(\theta, \varphi) \ \rightsquigarrow \ \left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - 2\mu V(r) - \kappa^2\right) u_\ell(r) = 0$$

courtesy of S. König (2013)

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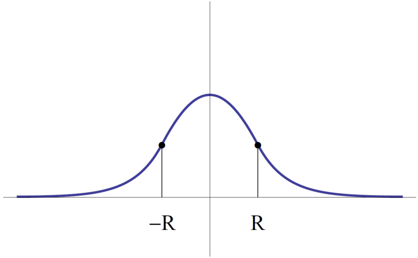

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courtesy of S. König (2013)

Finite volume energy corrections

Outside the interaction region, the bound state wavefunction can be replaced by Riccati-Hankel wavefunctions $\hat{h}_\ell^+(z)$ regular at infinity for complex arguments

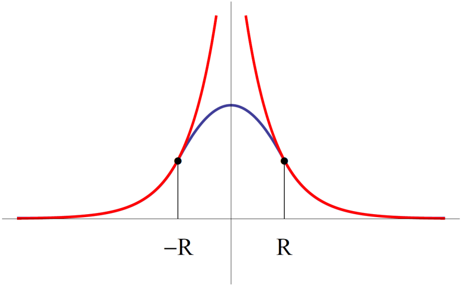
$$\hat{h}_0^+(z) = \mathrm{e}^{\mathrm{i}z}$$

$$\hat{h}_1^+(z) = \left(1 + \frac{\mathrm{i}}{z}\right)\mathrm{e}^{\mathrm{i}(z-\pi/2)}$$

$$\hat{h}_2^+(z) = \left(1 + \frac{3\mathrm{i}}{z} - \frac{3}{z^2}\right)\mathrm{e}^{\mathrm{i}(z-\pi)}$$

If $u_\ell(r) = i^\ell \gamma\, \chi^+_{\ell,\kappa}(r)$ we define

$$\gamma = \left(\int_0^{+\infty} \mathrm{d}r |\chi^+_{\ell,\kappa_0}(r)|^2\right)^{-1/2}$$



where γ is the asymptotic normalization constant and $\chi^+_{\ell,\kappa_0}(r) \xrightarrow{r\to\infty} \hat{h}_\ell^+(i\kappa_0 r)$

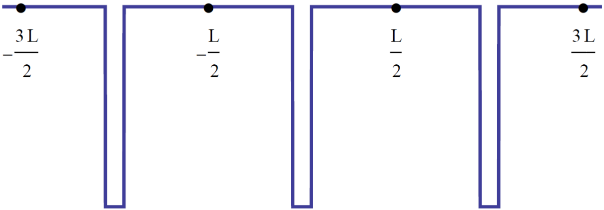
$$\left(\frac{\mathrm{d}}{\mathrm{d}r^2} - \frac{\ell(\ell+1)}{r^2} + 0 - \kappa_0^2\right) u_\ell(r) = 0 \;\text{ for }\; r > R \;\rightsquigarrow\; u_\ell(r) = i^\ell \gamma\, \hat{h}_\ell^+(i\kappa_0 r)$$

courtesy of S. König (2013)

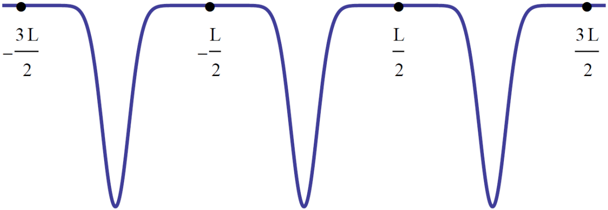
Finite volume energy corrections

Periodic Boundary Conditions imply the creation of infinitely many copies of $V(r)$

In particular, the interaction potential $V(r) = V_0\theta(R - r)$ gives



whereas the potential $V(r) = V_0 \exp(-r^2/R^2)$ gives



courtesy of S. KÖNIG (2013)

Finite volume energy corrections

The comparison between Schrödinger equation in finite and ∞ volume yields

$$\hat{H}_L|\psi_0\rangle = -E_B(L)|\psi_0\rangle \qquad \hat{H}|\psi_B\rangle = -E_B(L)|\psi_B\rangle$$

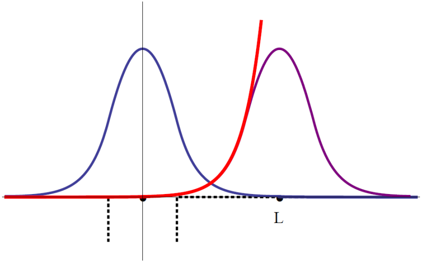
Also the wavefunction ψ_0 has to fulfil periodicity: $\rightsquigarrow \psi_0(\mathbf{r}+\mathbf{n}L) = \psi_0(\mathbf{r})$

Considering the following ansatz,

$$\psi_0(\mathbf{r}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} \psi_B(\mathbf{r}+\mathbf{n}L)$$

$$\hat{H}_L|\psi_0\rangle = -E_B(\infty)|\psi_0\rangle + |\eta\rangle$$

$$\eta(\mathbf{r}) = \sum_{\mathbf{n}\neq\mathbf{n}'} V(\mathbf{r}+\mathbf{n}L)\psi_B(\mathbf{r}+\mathbf{n}'L)$$



the result of S. König et al. where $\Delta m_B \equiv E_B(\infty) - E_B(L)$ is recovered

$$\Delta m_B = \frac{\langle\psi_0|\eta\rangle}{\langle\psi_0|\psi_0\rangle} = \sum_{|\mathbf{n}|=1} \int \mathrm{d}^3r\, \psi_B^*(\mathbf{r}) V(\mathbf{r}) \psi_B(\mathbf{r}+\mathbf{n}L) + \mathcal{O}(\mathrm{e}^{-\sqrt{2}\kappa_0 L})$$

courtesy of S. König (2013)

Finite volume energy corrections

LO finite volume energy corrections for relative two-body bosonic states with reduced mass μ, angular momentum ℓ and belonging to the Γ irrep of the cubic group $\mathcal{O}$ are given by PRL 107, 112011 (2011)

$$\Delta E_B^{(\ell,\Gamma)} \equiv E_B^{(\ell,\Gamma)}(\infty) - E_B^{(\ell,\Gamma)}(L) = \beta\left(\frac{1}{\kappa_0 L}\right)|\gamma|^2 \frac{e^{-\kappa_0 L}}{\mu L} + \mathcal{O}\left(e^{-\sqrt{2}\kappa_0 N}\right)$$

where κ_0 is the binding momentum and $\beta(x)$ is a polynomial:

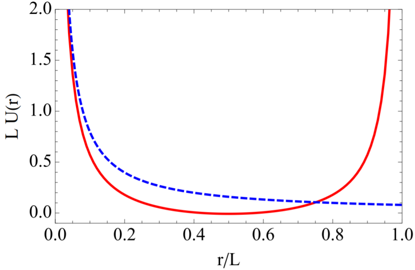
ℓ	Γ	$\beta(x)$
0	A_1^+	-3
1	T_1^-	$+3$
	T_2^+	$30x + 135x^2 + 315x^3 + 315x^4$
2	E^+	$-\frac{1}{2}(15 + 90x + 405x^2 + 945x^3 + 945x^4)$
	A_2^-	$315x^2 + 2835x^3 + 122285x^4 + 28350x^5 + 28350x^6$
3	T_2^-	$-\frac{1}{2}(105x + 945x^2 + 5355x^3 + 19530x^4 + 42525x^5 + 42525x^6)$
	T_1^-	$-\frac{1}{2}(14 + 105x + 735x^2 + 3465x^3 + 11340x^4 + 23625x^5 + 23625x^6)$

The asymptotic behaviour of the corrections with the side of the cubic box is exponential, the decay constant being proportional to the binding momentum

Finite Volume Energy Corrections with QED

In absence of QED the FV artifacts on the eigen-energies are exponentially suppressed in L. As soon as QED is turned on, FV effects become proportional to the inverse of L.

The Coulomb interaction is the leading contribution of QED in elastic scattering at low energies.



courtesy of S. Beane et al. (2014)

Its FV counterpart for unit opposite charges is

$$U(\mathbf{r}, L) = \frac{\alpha}{\pi L} \sum_{\mathbf{n} \neq 0} \frac{1}{|\mathbf{n}|^2} e^{2\pi i \frac{\mathbf{n} \cdot \mathbf{r}}{L}}$$

In particular, QED affects:

- the masses of the individual particles PRD 90, 054503 (2014) $\rightsquigarrow$ for π^+'s:

$$\Delta M_\pi \approx \frac{1}{L} + \frac{2}{M_\pi + L^2} + \ldots$$

- the energy of the resulting many-body system PRD 90, 074511 (2014) $\rightsquigarrow$ for $\ell = 0$ bound states:

$$\Delta E \approx -\frac{\alpha \mathcal{I}}{\pi L} + \ldots$$

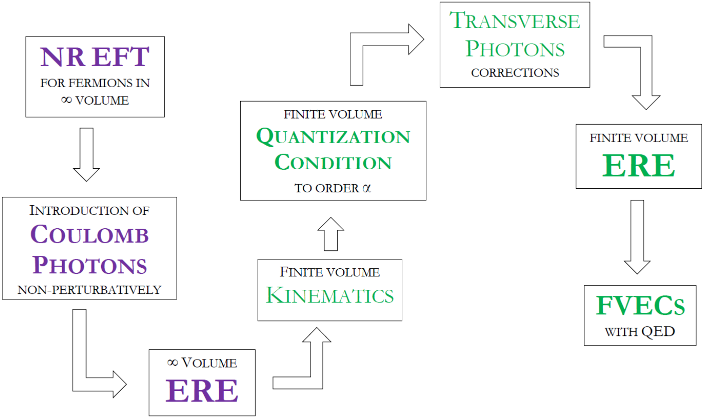
with $\mathcal{I} = -8.9136$

Consequences

Ampère's Law and Gauss Law are no more satisfied by a gauge field obeying PBCs. $\rightsquigarrow$ a uniform backg. charge density is introduced $\equiv$ the removal of the zero modes of the photons

Finite volume energy corrections with QED

For the derivation of the two-body finite volume energy corrections (FVECs) in presence of QED we adopt S. Beane et al. PRD 90, 074511 (2014) procedure and apply it also to strong potentials coupling to higher partial waves $\ell = 0, 1, 2, \ldots$



Key ingredient: the generalized Effective Range Expansion for strong & Coulomb interactions...

Non relativistic EFT for S-wave scattering in ∞ volume

Let's consider a non-relativistic pionless effective field theory for spinless **fermions** with mass M and S-wave interaction Nucl. Phys. A 665, 137-163 (2000)

$$\mathcal{L}^{(0)} = \psi^\dagger \left(\hbar \partial_t + \frac{\hbar^2 \nabla^2}{2M} \right) \psi - \frac{C_0}{2} (\psi\psi)^\dagger (\psi\psi)$$

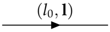
The induced two-fermion interaction in the CoM frame reads

$$V^{(0)}(\mathbf{p}, \mathbf{q}) \equiv \langle \mathbf{q}, -\mathbf{q} | \hat{\mathcal{V}}^{(0)} | \mathbf{p}, -\mathbf{p} \rangle = C_0$$

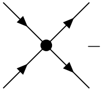
$\pm\mathbf{p}$ (resp. $\pm\mathbf{q}$) $\Rightarrow$ momentum of the incoming (resp. outcoming) particles

$\langle \mathbf{r} | \mathbf{p} \rangle \Rightarrow$ plane-waves, eigenfunctions of $\hat{H}_0 = \hat{P}^2/M$

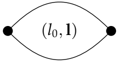
▶ Feynman rules (momentum space)



$\dfrac{i}{l_0 - \frac{\mathbf{l}^2}{2M} + i\varepsilon}$



$-iC_0$



$\int_{\mathbb{R}} \frac{dl_0}{(2\pi)} \int_{\mathbb{R}^3} \frac{d^3 l}{(2\pi)^3}$

▶ Two-body free retarded $(+)$ and advanced $(-)$ Green's functions at energy $E \equiv \mathbf{p}^2/M$ in the CoM frame in momentum and coordinate space:

$$G_0^{(\pm)}(E, \mathbf{r}, \mathbf{r}') \equiv \langle \mathbf{r}' | \hat{G}_0^{(\pm)}(E) | \mathbf{r} \rangle$$
$$G_0^{(\pm)}(E, \mathbf{p}, \mathbf{p}') \equiv \langle \mathbf{p}' | \hat{G}_0^{(\pm)}(E) | \mathbf{p} \rangle$$

where

$$\hat{G}_0^{(\pm)}(E) = \frac{1}{E - \hat{H}_0 \pm i\varepsilon},$$

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The induced two-fermion interaction in the CoM frame reads

$$V^{(0)}(\mathbf{p}, \mathbf{q}) \equiv \langle \mathbf{q}, -\mathbf{q} | \hat{\mathcal{V}}^{(0)} | \mathbf{p}, -\mathbf{p} \rangle = C_0$$

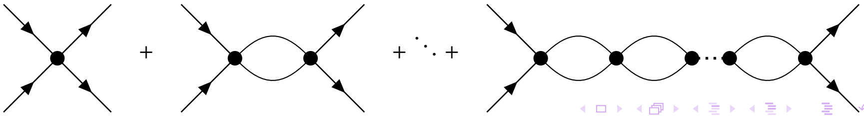
- Two-body free Green's functions at energy $E \equiv \mathbf{p}^2/M$:

$$G_0^{(\pm)}(E, \mathbf{r}, \mathbf{r}') \equiv \langle \mathbf{r}' | \hat{G}_0^{(\pm)}(E) | \mathbf{r} \rangle$$
$$G_0^{(\pm)}(E, \mathbf{p}, \mathbf{p}') \equiv \langle \mathbf{p}' | \hat{G}_0^{(\pm)}(E) | \mathbf{p} \rangle$$

where

$$\hat{G}_0^{(\pm)}(E) = \frac{1}{E - \hat{H}_0 \pm i\varepsilon},$$

The amplitude of the fermion-fermion scattering process is the superposition of all the contributions fom the bubble diagrams:



Two-fermion S-wave scattering states in ∞ volume

The translation of the diagrams into the language of matrix element returns the scattering amplitude (T-matrix) of the fermion-fermion process,

$$iT_S(\mathbf{p},\mathbf{q}) = i\langle \mathbf{q}, -\mathbf{q}|\hat{\mathcal{V}}^{(0)}(\mathbb{1} + \hat{G}_0^E\hat{\mathcal{V}}^{(0)} + \hat{G}_0^E\hat{\mathcal{V}}^{(0)}\hat{G}_0^E\hat{\mathcal{V}}^{(0)} + \ldots)|\mathbf{p}, -\mathbf{p}\rangle$$

which that can be conveniently rewritten as a Geometric series

$$T_S(\mathbf{p},\mathbf{q}) = \frac{C_0}{1 - C_0 G_0^E(\mathbf{0},\mathbf{0})} \qquad \text{and} \qquad G_0^E(\mathbf{r},\mathbf{r}',\mu)\Big|_{\mathbf{r}=\mathbf{r}'=0}^{\mathrm{PDS}} = -\frac{M}{4\pi}(i|\mathbf{p}|+\mu)\ ,$$

where μ is the renormalization mass in the PDS scheme **Nucl. Phys. B 534, 329 (1998)**

Recalling the full amplitude for S-wave scattering together with the zero angular momentum ($\ell = 0$) effective range expansion (ERE),

$$T_S(\mathbf{p},\mathbf{q}) = -\frac{4\pi}{M}\frac{1}{|\mathbf{p}|\cot\delta_0 - i|\mathbf{p}|} \qquad |\mathbf{p}|\cot\delta_0 = -\frac{1}{a} + \frac{1}{2}r_0\mathbf{p}^2 + v_2\mathbf{p}^4 + v_3\mathbf{p}^6 \ldots,$$

one finds an expression for the scattering amplitude and the effective range

$$a = \frac{4\pi C_0}{M} \qquad \text{and} \qquad r_0 = 0\ .$$

Remark: a zero r_0 was expected, since the potential is proportional to $\delta(\mathbf{r}')\delta(\mathbf{r})$

Non relativistic EFT for P-wave scattering in ∞ volume

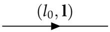
Let's consider a non-relativistic pionless effective field theory for spinless fermions with mass M and P-wave interaction Nucl. Phys. A 712, 37-58 (2002)

$$\mathcal{L} = \psi^\dagger \left[i\hbar\partial_t + \frac{\hbar^2\nabla^2}{2M} \right] \psi + \frac{D_0}{8} (\psi \overleftrightarrow{\nabla}_i \psi)^\dagger \cdot (\psi \overleftrightarrow{\nabla}_i \psi) \, .$$

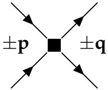
Analogously, the induced two-fermion interaction in the CoM frame reads

$$V^{(1)}(\mathbf{p}, \mathbf{q}) \equiv \langle \mathbf{q}, -\mathbf{q} | \hat{\mathcal{V}}^{(1)} | \mathbf{p}, -\mathbf{p} \rangle = D_0 \, \mathbf{p} \cdot \mathbf{q} \, ,$$

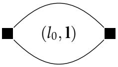
▶ Feynman rules (momentum space)



$\frac{i}{l_0 - \frac{\mathbf{l}^2}{2M} + i\varepsilon}$

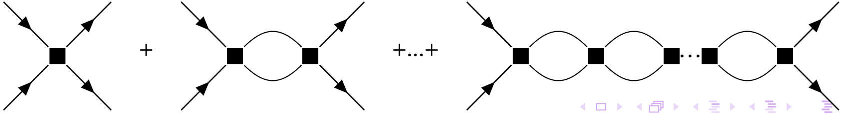


$-iD_0 \mathbf{p} \cdot \mathbf{q}$



$\int_{\mathbb{R}} \frac{\mathrm{d}l_0}{(2\pi)} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3 l}{(2\pi)^3}$

The amplitude of the fermion-fermion scattering process is again the superposition of all the contributions fom the bubble diagrams:



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which that can be conveniently rewritten as a Geometric series

$$T_S(\mathbf{p},\mathbf{q}) = \mathbf{q}\cdot\frac{D_0}{\mathbb{1} - D_0\mathbb{T}_S}\mathbf{p} \qquad \text{and} \qquad \mathbb{T}_{ij}^{\text{PDS}} = -\delta_{ij}\frac{M}{4\pi}\left(\frac{i|\mathbf{p}|^3}{3} + \mu\frac{\mathbf{p}^2}{2}\right),$$

where μ is the renormalization mass in the PDS scheme and $\mathbb{T} = \nabla\otimes\nabla' G_0^E(3;\mathbf{r},\mathbf{r}')|_{\mathbf{r},\mathbf{r}'=0}^{\text{PDS}}$
Recalling the full amplitude for P-wave scattering together with the zero angular momentum ($\ell = 1$) effective range expansion,

$$T_S(\mathbf{p},\mathbf{q}) = -\frac{12\pi}{M}\frac{\mathbf{p}\cdot\mathbf{q}}{|\mathbf{p}|^3\cot\delta_1 - i|\mathbf{p}|^3} \qquad \text{and} \qquad |\mathbf{p}|^3\cot\delta_1 = -\frac{1}{a} + \frac{1}{2}r_0\mathbf{p}^2 + v_2\mathbf{p}^4 + \ldots,$$

one finds an expression for the scattering amplitude and the effective range

$$a = \frac{M}{4\pi}\frac{D_0}{3} \qquad \text{and} \qquad r_0 = 0\,.$$

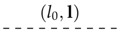
Remark: a zero effective range was expected, since the potential is proportional to $\nabla'\delta(\mathbf{r}')\cdot\nabla\delta(\mathbf{r})$.

Non relativistic EFT with QED in ∞ volume

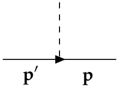
We introduce QED in a non-relativistic fashion. The Lagrangian for NR Pauli spinor fields Ψ associated to particles with mass M and unit charge e is provided by [1]

$$\mathcal{L}^{\text{QED}} = -\frac{1}{2}\left(\mathbf{E}^2 - \mathbf{B}^2\right) + \Psi^\dagger\left(i\partial_t - e\phi + \frac{\mathbf{D}^2}{2M}\right)\Psi + \Psi^\dagger\left[d_1\frac{e}{8M^3}\{\mathbf{D}^2, \boldsymbol{\sigma}\cdot\mathbf{B}\}\right]\Psi$$
$$+\Psi^\dagger\left[c_1\frac{\mathbf{D}^4}{8M^3} + c_2\frac{e}{2M}\boldsymbol{\sigma}\cdot\mathbf{B} + c_3\frac{e}{8M^2}\nabla\cdot\mathbf{E} + c_4\frac{e}{8M^2}\mathrm{i}\mathbf{D}\times\boldsymbol{\sigma}\right]\Psi + \ldots$$

where $\mathbf{D} = \nabla + ie\mathbf{A}$ is the covariant derivative and ϕ is the scalar potential. In the spinless fermion case $\Psi \to \psi$. Plugging $\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}$ in $\mathcal{L}^{\text{QED}}$, and considering the $-e\phi\psi^\dagger\psi$ term the following Feynman rules are recovered



$$\frac{e^2}{\mathbf{l}^2 + \lambda^2}$$



e

where $\lambda \equiv$ IR regulator

where the dotted line indicates a Coulomb photon.

Remark: In low-momentum NR QED transverse photons are negligible, since their coupling is proportional to the fermion's velocity.

[1] see also T. Kinoshita and M. Nio PRD 53, 4909-4929 (1996) and Phys. Lett. B 167, 437-442 (1986)

Two-fermion scattering states in ∞ volume with QED

We begin with the **repulsive** case. The potential in momentum space becomes

$$\langle \mathbf{q}|\hat{V}_{\mathrm{C}}|\mathbf{p}\rangle = \frac{e^2}{(\mathbf{q}-\mathbf{p})^2-\lambda^2}$$

and two solutions of the Coulomb Schrödinger equation are given by

$$\psi_{\mathbf{p}}^{(\pm)}(\mathbf{r}) = e^{-\frac{1}{2}\pi\eta}\Gamma(1\pm i\eta)M(\mp i\eta,1;\pm ipr - i\mathbf{p}\cdot\mathbf{r})e^{i\mathbf{p}\cdot\mathbf{r}}$$

where $M \equiv {}_1F_1$, $\langle\psi_{\mathbf{p}}^{(\pm)}|\psi_{\mathbf{q}}^{(\pm)}\rangle = (2\pi)^3\delta(\mathrm{q}-\mathrm{p})$ and $\psi_{\mathbf{p}}^{(+)}(\mathbf{r}) = \psi_{-\mathbf{p}}^{(-)*}(\mathbf{r})$

$(+) \rightsquigarrow$ outgoing spherical waves in the future

$(-) \rightsquigarrow$ incoming spherical waves in the past

$\eta = \frac{\alpha M}{2|\mathbf{p}|}$ regulates the viability of the perturbative treatment of the QED corrections.
The above Coulomb wavefunctions admit an expansion into spherical waves:

$$\psi_{\mathbf{p}}^{(+)}(\mathbf{r})\frac{4\pi}{pr}\sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell}\mathrm{i}^{\ell}\mathrm{e}^{i\sigma_\ell}F_\ell(\eta,pr)Y_\ell^m(\hat{\mathbf{r}})Y_\ell^{m*}(\hat{\mathbf{p}})$$

where $\sigma_\ell = \arg\Gamma(\ell+1+\mathrm{i}\eta)$ is the Coulomb phase shift and

$$F_\ell(\eta,pr) = \frac{2^\ell \mathrm{e}^{-\pi\eta/2}|\Gamma(\ell+1+\mathrm{i}\eta)|}{(2\ell+1)!}(pr)^{\ell+1}\mathrm{e}^{ipr}M(\ell+1+\mathrm{i}\eta,2\ell+2,-2\mathrm{i}pr)$$

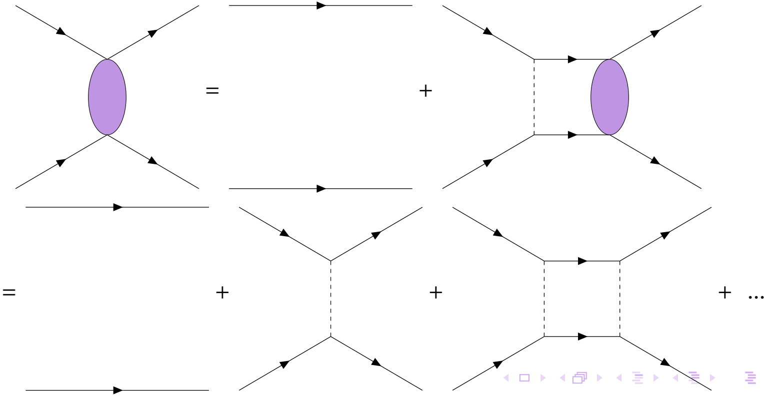
Two-fermion scattering states in ∞ volume with QED

In the non relativistic regime perturbation theory in α breaks down:

$$|\mathbf{p}| \sim \alpha M \quad \text{i.e.} \quad \eta \sim 1$$

The expression of the Coulomb Green's functions in terms of $\hat{V}_C$ and $\hat{G}_0^{(\pm)}$ yields the Dyson equation and the **Ladder** diagrammatic expansion

$$\hat{G}_C^{(\pm)}(E) = \frac{1}{E - \hat{H}_0 - \hat{V}_C \pm i\varepsilon} \quad \rightsquigarrow \quad \hat{G}_C^{(\pm)}(E) - \hat{G}_0^{(\pm)}(E) = \hat{G}_0^{(\pm)}(E)\hat{V}_C\hat{G}_C^{(\pm)}(E)$$

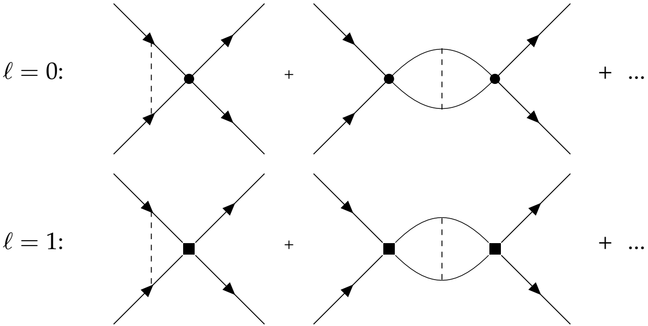


Two-fermion scattering states in ∞ volume with QED

An analogous expansion is recovered when the full Green's function operator is rewritten in terms of $\hat{G}_{\mathrm{C}}^{(\pm)}(E)$ and the strong potential $\hat{V}_{\mathrm{S}}$,

$$\hat{G}_{\mathrm{SC}}^{(\pm)}(E) = \frac{1}{E - \hat{H}_0 - \hat{V}_{\mathrm{C}} - \hat{V}_{\mathrm{S}} \pm i\varepsilon} \quad \rightsquigarrow \quad \hat{G}_{\mathrm{SC}}^{(\pm)} - \hat{G}_{\mathrm{C}}^{(\pm)} = \hat{G}_{C}^{(\pm)} \hat{V}_{\mathrm{S}} \hat{G}_{\mathrm{SC}}^{(\pm)}$$

As a result, multiple Coulomb photon insertions appear both in the external legs of the bubble diagrams and within the fermion loops themselves.



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- The S-matrix element associated to the scattering process becomes

$$S(\mathbf{p}', \mathbf{p}) = \langle \chi_{\mathbf{p}'}^{(-)} | \chi_{\mathbf{p}}^{(+)} \rangle = (2\pi)^3 \delta(\mathbf{p}' - \mathbf{p}) - 2\pi i \delta(E' - E) T(\mathbf{p}', \mathbf{p})$$

where $|\chi_{\mathbf{p}}^{(\pm)}\rangle$ are the full eigenstates, $T(\mathbf{p}', \mathbf{p}) = T_C(\mathbf{p}', \mathbf{p}) + T_{SC}(\mathbf{p}', \mathbf{p})$ and

$T_C(\mathbf{p}', \mathbf{p}) = \langle \mathbf{p}' | \hat{V}_C | \psi_{\mathbf{p}}^{(+)} \rangle \rightsquigarrow$ purely Coulomb amplitude

$T_{SC}(\mathbf{p}', \mathbf{p}) = \langle \psi_{\mathbf{p}'}^{(-)} | \hat{V}_S | \chi_{\mathbf{p}}^{(+)} \rangle \rightsquigarrow$ Coulomb corrected Strong amplitude

In particular, both the two amplitudes admit a **partial wave expansion**:

$$T_C(\mathbf{p}', \mathbf{p}) = -\frac{4\pi}{M} \sum_{\ell=0}^{+\infty} (2\ell + 1) \left[\frac{e^{2i\sigma_\ell} - 1}{2i|\mathbf{p}|} \right] \mathcal{P}_\ell(\cos\theta)$$

$$T_{SC}(\mathbf{p}', \mathbf{p}) = -\frac{4\pi}{M} \sum_{\ell=0}^{+\infty} (2\ell + 1) e^{2i\sigma_\ell} \left[\frac{e^{2i\delta_\ell} - 1}{2i|\mathbf{p}|} \right] \mathcal{P}_\ell(\cos\theta)$$

where $\sigma_\ell = \arg\Gamma(1 + \ell + i\eta)$ and δ_ℓ is the strong contribution to the total phase shift

Two-fermion S-wave scattering states in ∞ volume with QED

T_{SC} for the $\ell = 0$ interaction $\propto$ the sum of a geometric series with ratio $C_0 G_C^{(+)}(E, \mathbf{0}, \mathbf{0})$,

$$T_{SC}(\mathbf{p}', \mathbf{p}) = \frac{C_\eta^2 C_0 e^{2i\sigma_0}}{1 - C_0 G_C^{(+)}(E, \mathbf{0}, \mathbf{0})},$$

- where C_η is the Sommerfeld factor

$$C_\eta^2 \equiv |\psi_{\mathbf{p}}^{(\pm)}(0)|^2 = \frac{2\pi\eta}{e^{2\pi\eta} - 1}$$

$$H(\eta) = \psi(i\eta) + \frac{1}{2i\eta} - \log(i\eta)$$

and $\psi(z)$ is the Digamma function.

In particular, $G_C^{(+)}(E, \mathbf{0}, \mathbf{0}) \equiv J_C(\mathbf{p}) = J_C^{\text{fin}}(\mathbf{p}) + J_C^{\text{div}}(\mathbf{p})$ proves to be UV divergent in three dimensions. Its explict computation gives Nucl. Phys. A 665, 137-163 (2000)

$$J_C^{\text{fin}}(\mathbf{p}) = -\frac{\alpha M^2}{4\pi} H(\eta) \qquad J_C^{\text{div}}(\mathbf{p})\Big|^{\text{PDS}} = \frac{\alpha M^2}{4\pi}\left[\frac{1}{\epsilon} + \log\frac{\mu\sqrt{\pi}}{\alpha M} + 1 - \frac{3}{2}\gamma_E\right] - \frac{\mu M}{4\pi}$$

The exploitation of the generalized **ERE** for S-wave scattering amplitudes,

$$C_\eta^2 |\mathbf{p}|(\cot\delta_0 - i) + \alpha M H(\eta) = -\frac{1}{a_C^{(0)}} + \frac{1}{2} r_0^{(0)} \mathbf{p}^2 + \ldots$$

yields $r_0^{(0)} = 0$ and an expression of $a_C^{(0)}$ in terms of the coupling constants

$$\frac{1}{a_C} = \frac{4\pi}{M C_0} + \mu - \alpha M \left[\frac{1}{\epsilon} + \log\frac{\mu\sqrt{\pi}}{\alpha M} + 1 - \frac{3}{2}\gamma_E\right].$$

Two-fermion P-wave scattering states in ∞ volume with QED

Analogously to the S-wave case, the Coulomb-corrected strong scattering amplitude is proportional to the sum of a geometric series,

$$T_{\rm SC}(\mathbf{p}',\mathbf{p}) = \nabla'\psi^{(-)*}_{\mathbf{p}'}(\mathbf{r}')\Big|_{\mathbf{r}'=0}\cdot\frac{D_0}{\mathbb{1}-D_0\mathbb{T}_{\rm SC}}\nabla\psi^{(+)}_{\mathbf{p}'}(\mathbf{r})\Big|_{\mathbf{r}=0}$$

albeit with a matrix ratio whose elements are diagonal and, in dimensional regularization, are given by

$$(\mathbb{T}_{\rm SC})_{ij}(d) = M\frac{\delta_{ij}}{d}\int_{\mathbb{R}^d}\frac{\mathrm{d}^d s}{(2\pi)^d}\frac{2\pi\eta(s)\,s^2}{e^{2\pi\eta(s)}-1}\frac{1+\eta(s)^2}{\mathbf{p}^2-\mathbf{s}^2+i\varepsilon}\equiv \mathbb{t}_{\rm SC}(d)\delta_{ij}\ ,$$

therefore, the T-matrix elements simplify into

$$T_{\rm SC}(\mathbf{p}',\mathbf{p}) = (1+\eta^2)C^2_\eta\frac{D_0e^{2i\sigma_1}\mathbf{p}\cdot\mathbf{p}'}{1-D_0\mathbb{t}_{\rm SC}} = (1+\eta^2)C^2_\eta\frac{D_0e^{2i\sigma_1}}{1-D_0\mathbb{t}_{\rm SC}}\mathbf{p}^2\cos\theta\ ,$$

The evaluation of $\mathbb{t}_{\rm SC}$ at the denominator in the PDS regularization scheme yields

$$\mathbb{t}_{\rm SC}(\mathbf{p}) = \mathbb{t}^{\rm fin}_{\rm SC}(\mathbf{p})+\mathbb{t}^{{\rm div},1}_{\rm SC}(\mathbf{p})+\mathbb{t}^{{\rm div},2}_{\rm SC}(\mathbf{p}) = \frac{\alpha^3M^4}{48\pi}\left[\frac{1}{3-d}+\zeta(3)-\frac{3}{2}\gamma_E+\frac{4}{3}+\log\frac{\mu\sqrt{\pi}}{\alpha M}\right]$$
$$-\frac{\alpha^2M^3}{32\pi}\frac{\mu}{3}\left(\pi^2-3\right)+\frac{\alpha M^2}{4\pi}\frac{\mathbf{p}^2}{3}\left[\frac{1}{3-d}+\frac{4}{3}-\frac{3}{2}\gamma_E+\log\frac{\mu\sqrt{\pi}}{\alpha M}\right]-\frac{\mu M}{4\pi}\frac{\mathbf{p}^2}{2}-\frac{\alpha M^2}{4\pi}\frac{\mathbf{p}^2}{3}H(\eta)(1+\eta^2)\ .$$

- The expression is UV divergent, but terms proportional to $\mathbf{p}^2$ give rise to finite $r_0^{(1)}$!

Two-fermion P-wave scattering states in ∞ volume with QED

Exploiting the last expression and the $\ell = 1$ component of the partial wave expansion,

$$|\mathbf{p}|^3(\cot\delta_1 - i) = -\frac{12\pi}{M}\frac{1 - D_0 \mathfrak{t}_{SC}(\mathbf{p})}{D_0 C_\eta^2(1+\eta^2)} \quad \text{and} \quad |\mathbf{p}|^3(\cot\delta_1 - i) = -\frac{12\pi\mathbf{p}^2}{M}\frac{e^{2i\sigma_1}\cos\theta}{T_{SC}(\mathbf{p}',\mathbf{p})}$$

together with the P-wave version generalized **ERE**, PRC 26, 2381-2396 (1982)

$$\mathbf{p}^2\left(1+\eta^2\right)\left[C_\eta^2|\mathbf{p}|(\cot\delta_1 - i) + \alpha M H(\eta)\right] = -\frac{1}{a_C^{(1)}} + \frac{1}{2}r_0^{(1)}\mathbf{p}^2 + \dots ,$$

an expression for the Coulomb P-wave scattering length $a_C^{(1)}$ is obtained,

$$\frac{1}{a_C^{(1)}} = \frac{12\pi}{MD_0} - \frac{\alpha^3 M^3}{4}\left[\frac{1}{3-d} + \zeta(3) - \frac{3}{2}\gamma_E + \frac{4}{3} + \log\frac{\mu\sqrt{\pi}}{\alpha M}\right] + \frac{\alpha^2 M^2 \mu}{8}\left(\pi^2 - 3\right) .$$

The components of $\mathfrak{t}_{SC}$ proportional to $\mathbf{p}^2$ generate a purely Coulomb nonzero effective range, i.e. vanishing as soon as the electrostatic interaction is turned off

$$r_0^{(1)} = \alpha M\left[\frac{2}{3-d} + \frac{8}{3} - 3\gamma_E + 2\log\frac{\mu\sqrt{\pi}}{\alpha M}\right] - 3\mu ,$$

S-wave scattering states in ∞ volume with QED: the attractive case

Now we consider the two fermions with **opposite** charges. As a result $\eta = -\alpha M/2|\mathbf{p}|$, and the $H(\eta)$ function in the ERE is replaced by

$$\overline{H}(\eta) = \psi(i\eta) + \frac{1}{2i\eta} - \log(-i\eta)$$

For convenience, we relabel the strong coupling constant: $C_0 \to \overline{C}_0$.

- The Green's function $G_C^{(+)}(E;\mathbf{0},\mathbf{0})$ becomes $\overline{G}_C^{(+)}(E;\mathbf{0},\mathbf{0}) = \overline{J}_C^b(\mathbf{p}) + \overline{J}_C^s(\mathrm{C})$ where

$$\overline{J}_C^s(\mathbf{p}) = -\frac{\mu M}{4\pi} - \frac{\alpha M^2}{4\pi}\left[\frac{1}{3-d} + \log\frac{\mu\sqrt{\pi}}{\alpha M} + \log(-i\eta) - \frac{1}{2i\eta} + \psi(-i\eta) + 1 + \frac{1}{2}\gamma_E\right]$$

and

$$\overline{J}_C^b(\mathbf{p}) = \sum_{n=1}^{+\infty}\sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell}\frac{|\phi_{n,\ell,m}(\mathbf{0})|^2}{E-E_n} = \frac{\alpha M^2}{4\pi}[\psi(i\eta)+\psi(-i\eta)+2\zeta(1)]$$

  - $\overline{J}_C^b(\mathbf{p})$ accounts for the bound states ∝ the associated *Laguerre* polynomials

$$\phi_{n,\ell,m}(\mathbf{r}) = \sqrt{\left(\frac{\alpha M}{n}\right)^3 \frac{n-\ell-1!}{n+\ell!\,2n}}\, e^{-\frac{\alpha M}{2n}r}\left(\frac{\alpha M r}{n}\right)^{\ell} L_{n-\ell-1}^{2\ell+1}\left(\frac{\alpha M}{n}r\right) Y_\ell^m(\theta,\varphi)$$

where $E_n = -\alpha^2 M/4n^2$ is the energy and $n \rightsquigarrow$ principal quantum number

S-wave scattering states in ∞ volume with QED: the attractive case

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$$\overline{J}_C^s(\mathbf{p}) = -\frac{\mu M}{4\pi} - \frac{\alpha M^2}{4\pi}\left[\frac{1}{3-d} + \log\frac{\mu\sqrt{\pi}}{\alpha M} + \log(-i\eta) - \frac{1}{2i\eta} + \psi(-i\eta) + 1 + \frac{1}{2}\gamma_E\right]$$

and

$$\overline{J}_C^b(\mathbf{p}) = \sum_{n=1}^{+\infty}\sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell} \frac{|\phi_{n,\ell,m}(\mathbf{0})|^2}{E-E_n} = \frac{\alpha M^2}{4\pi}[\psi(i\eta)+\psi(-i\eta)+2\zeta(1)]$$

The scattering length coincides with the repulsive one **except** the sign in front of α!

$$\frac{1}{\bar{a}_C^{(0)}(\mu)} = \frac{1}{\bar{a}_C^{(0)}} - \alpha M\left[\log\frac{\mu\sqrt{\pi}}{\alpha M} + 1 - \frac{3}{2}\gamma_E\right]$$

P-wave scattering states in ∞ volume with QED: the attractive case

As in the $\ell = 0$ case, $D_0 \to \overline{D}_0$ and $H(\eta) \to \overline{H}(\eta)$ in the ERE. The contribution of the bound states in the denominator of the scattering amplitude $\overline{T}_{SC}(\mathbf{p}', \mathbf{p})$ becomes

$$\bar{\mathfrak{t}}^{b}_{SC}(\mathbf{p}) = -\frac{\alpha^3 M^4}{24\pi}\zeta(3) + \frac{\alpha M^2}{3\pi}\frac{\mathbf{p}^2}{2}\left[\zeta(1) + \frac{1}{2}\psi(-i\eta) + \frac{1}{2}\psi(i\eta)\right](1+\eta^2)$$

and the overall function $\mathfrak{t}_{SC}$ is replaced by $\bar{\mathfrak{t}}^{b}_{SC}(\mathbf{p}) + \bar{\mathfrak{t}}^{old}_{SC}(\mathbf{p}) + \bar{\mathfrak{t}}^{new}_{SC}(\mathbf{p}) \equiv \bar{\mathfrak{t}}_{SC}(\mathbf{p})$:

$$\bar{\mathfrak{t}}_{SC}(\mathbf{p}) = -\frac{\alpha^3 M^4}{48\pi}\left[\frac{1}{3-d} + \zeta(3) - \frac{3}{2}\gamma_E + \frac{4}{3} + \log\frac{\mu\sqrt{\pi}}{\alpha M}\right] - \frac{\alpha^2 M^3}{32\pi}\frac{\mu}{3}(\pi^2 - 3)$$
$$-\frac{\alpha M^2}{4\pi}\frac{\mathbf{p}^2}{3}\left[\frac{1}{3-d} + \frac{4}{3} - \frac{3}{2}\gamma_E + \log\frac{\mu\sqrt{\pi}}{\alpha M}\right] - \frac{\mu M}{4\pi}\frac{\mathbf{p}^2}{2} + \frac{\alpha M^2 \mathbf{p}^2}{12\pi}\overline{H}(\eta)(1+\eta^2)$$

The scattering length assumes the same expression, **except** the sign in front of α

$$\frac{1}{\bar{a}^{(1)}_C} = \frac{12\pi}{M\overline{D}_0} + \frac{\alpha^3 M^3}{4}\left[\frac{1}{3-d} + \zeta(3) - \frac{3}{2}\gamma_E + \frac{4}{3} + \log\frac{\mu\sqrt{\pi}}{\alpha M}\right] + \frac{\alpha^2 M^2 \mu}{8}\left(\pi^2 - 3\right)$$

and the same conclusion can be drawn for the effective range parameter $\bar{r}^{(1)}_0$

$$\bar{r}^{(1)}_0 = -\alpha M\left[\frac{2}{3-d} + \frac{8}{3} - 3\gamma_E + 2\log\frac{\mu\sqrt{\pi}}{\alpha M}\right] - 3\mu$$

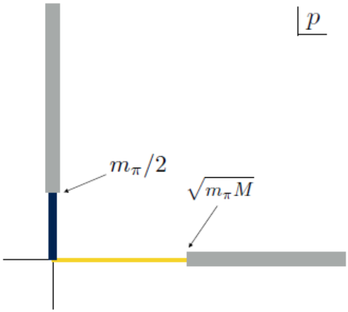
Non relativistic EFT with QED in the Finite Volume

The system is transposed into a discretized cubic box of side L

$\rightsquigarrow$ discretization of momentum $\rightsquigarrow$ $\mathbf{p} = 2\pi\mathbf{n}/L$ with $\mathbf{n} \in \mathbb{Z}^3$.

$\rightsquigarrow$ the analytic structure of the **scattering amplitude** changes

$\rightsquigarrow$ the validity of our generalized ERE is affected



COURTESY OF S. BEANE ET AL. (2014)

$T_{SC}(\mathbf{p}, \mathbf{p}')$

✓ Analytical along $\mathfrak{Im}\, p$

✗ Pion production cut with threshold 0 along $\mathfrak{Re}\, p$

$T^L_{SC}(\mathbf{p}, \mathbf{p}')$

✓ Analytical along $\mathfrak{Im}\, p$

✓ Pion production cut with threshold $\sqrt{2\pi M/L}$ along $\mathfrak{Re}\, p$ (no zero momentum modes)

Without QED: a QCD t-channel cut appears along the $\mathfrak{Im}\, p$ axis, with threshold $im_\pi/2$

the $\boldsymbol{\pi}$ **production** cut is lifted at $\sqrt{m_\pi M}$

Finite Volume Kinematics

In the finite volume both the **energies** of the bound or scattering states $\Delta E \equiv E^L - E$ and the **masses** $\Delta M \equiv M^L - M$ of the particles are affected.

$\rightsquigarrow$ without QED: ΔM are of order $e^{-m_\pi L}$ (negligible)

$\rightsquigarrow$ with QED: ΔM are of order L^{-1} (important)

As in PRD 90, 054503 (2014) shifted scattering parameters are introduced

$$\frac{1}{a_C^{\prime(\ell)}} = \frac{1}{a_C^{(\ell)}} - \frac{\alpha r_0^{(\ell)} M\mathcal{I}}{2\pi L} + \mathcal{O}\left(\alpha^2, \tfrac{\alpha}{L^2}\right) \quad r_0^{\prime(\ell)} = r_0^{(\ell)} + \frac{4\alpha r_1^{(\ell)} M\mathcal{I}}{\pi L} + \mathcal{O}\left(\alpha^2, \tfrac{\alpha}{L^2}\right)$$

- Making the dependence of C_0 on the total energy of the two-fermion system

$$E^* \equiv 2M + E = 2M + \frac{\mathbf{p}^2}{M}$$

in the CoM frame explicit, the $\ell = 0$ ERE for equal charges in FV becomes

$$C_\eta^2|\mathbf{p}|\cot\delta_0 + \alpha M H(\eta) = \frac{1}{a_C^{\prime(0)}} + \frac{1}{2} r_0^{\prime(0)} \mathbf{p}^2 + \ldots$$

$$= -\frac{4\pi}{M C_0^L(E^*)} + \alpha\left[\frac{1}{3-d} + \log\left(\frac{\mu\sqrt{\pi}}{\alpha M}\right) + 1 - \frac{3}{2}\gamma_E\right]$$

where the ΔM effects are incorporated in $a_C^{\prime(0)}$ and $r_0^{\prime(0)}$.

The Quantization Condition at $\ell = 0$

The QC determines the energy eigenvalues from the singularities of the two-point correlation function. The full Green's function at $\mathbf{r} = \mathbf{r}' = \mathbf{0}$ for $\ell = 0$ gives

$$\hat{G}_{SC}^{(\pm)} = \frac{\mathbb{1}}{\mathbb{1} - \hat{G}_C^{(\pm)}\hat{V}_S}\hat{G}_C^{(\pm)} \quad \rightsquigarrow \quad G_{SC}^{(+)}(E,\mathbf{0},\mathbf{0}) = \frac{G_C(E;\mathbf{0},\mathbf{0})}{1 - C_0(E^*)G_C(E;\mathbf{0},\mathbf{0})}$$

In **finite volume** a descent in rotational symmetry takes place:

$\rightsquigarrow$ the $\ell = 0$ irrep of SO(3) is mapped into the A_1 of the **cubic group** $\mathcal{O}$

$$G_{SC}^{(+)\,L}(E,\mathbf{0},\mathbf{0}) = \frac{J_C^L(\mathbf{p})}{1 - C_0^L(E^*)J_C^L(\mathbf{p})}$$

where $G_{SC}^{(+)}(E,\mathbf{0},\mathbf{0}) \to G_{SC}^{(+)\,L}(E,\mathbf{0},\mathbf{0})$ and $G_C^{(+)}(E,\mathbf{0},\mathbf{0}) \to G_C^{(+)\,L}(E,\mathbf{0},\mathbf{0}) \equiv J_C^L(\mathbf{p})$. The quantization condition can be read off the denominator of $G_{SC}^{(+)\,L}(E,\mathbf{0},\mathbf{0})$

$$C_0^L(E^*) = \frac{1}{J_C^L(\mathbf{p})}$$

where $J_C^L(\mathbf{p})$ to all orders in α contains sums over non-rational functions

$\rightsquigarrow$ the Sommerfeld factor depends on the summed momenta!

Perturbative treatment of QED in finite volume

Without zero modes, in FV the momenta are $|\mathbf{p}| \geq 2\pi/L$, i.e. $\eta \sim \alpha ML$

$\rightsquigarrow$ if $ML \ll 1/\alpha$ then $\eta \ll 1$: large volume required

$\rightsquigarrow$ in LQCD $M \ll 1/L$: QED can be treated perturbatively

Consequences

The Sommerfeld factor in $J_C^L(\mathbf{p})$ can be expanded in power series of α

$$J_C^L(\mathbf{p}) = -\frac{M}{4\pi^2 L}\sum_{\mathbf{n}}^{\Lambda_n}\frac{1}{|\mathbf{n}|^2-\tilde{p}^2} + \frac{\alpha M}{16\pi^5}\sum_{\mathbf{n}}^{\Lambda_n}\sum_{\mathbf{m}\neq\mathbf{n}}\frac{1}{|\mathbf{n}|^2-\tilde{p}^2}\frac{1}{|\mathbf{m}|^2-\tilde{p}^2}\frac{1}{|\mathbf{n}-\mathbf{m}|^2} + \mathcal{O}(\alpha^2)$$

where $\tilde{p} = L|\mathbf{p}|/2\pi$ and $\Lambda_n = L\Lambda/2\pi \rightsquigarrow$ three-momentum cutoff

▶ **Trick**: Regulation of the divergent sums by means of the Cutoff- and Dimensionally Regularized version of $J_C(\mathbf{p})$ up to first order in α in ∞-volume:

$$\frac{1}{C_0^L(E^*)} - \mathfrak{Re}\, J_C^{\infty\{\mathrm{DR}\}}(\mathbf{p}) = J_0^L(\mathbf{p}) - \mathfrak{Re}\, J_C^{\infty\{\Lambda\}}(\mathbf{p})$$

$\rightsquigarrow C_0^L(E^*)$ is expressed in terms of a **Lüscher function**

The finite volume ERE for S-waves with QED

The sums appearing in the QC for the $\ell = 0$ strong coupling constant are rewritten as

$$\frac{1}{C_0^L(E^*)} = -\frac{M}{4\pi^2 L}\mathcal{S}(\tilde{p}) + \frac{\alpha M^2}{16\pi^5}\mathcal{S}_2(\tilde{p}) + \frac{\alpha M^2}{4\pi}\left[\frac{1}{3-d} - \frac{1}{2}\gamma_E + 1 - \log\frac{2\pi}{\mu L} - \log\sqrt{\pi}\right]$$

where $\mathcal{S}^C(x) = \mathcal{S}(x) + \frac{\alpha M}{4\pi^3}\mathcal{S}_2(x)$ is the Lüscher function
and $\mathcal{S}(x)$ & $\mathcal{S}_2(x)$ are finite summations (zero modes are absent!)

$$\mathcal{S}_2(x) = \sum_{\mathbf{n}}^{\Lambda_n}\sum_{\mathbf{m}\neq\mathbf{n}} \frac{1}{|\mathbf{n}|^2 - x^2}\frac{1}{|\mathbf{m}|^2 - x^2}\frac{1}{|\mathbf{n}-\mathbf{m}|^2} - 4\pi^4 \log\Lambda_n \qquad \mathcal{S}(x) = \sum_{\mathbf{n}}^{\Lambda_n}\frac{1}{|\mathbf{n}|^2 - x^2} - 4\pi\Lambda_n$$

- The divergences in $1/C_0^L(E^*)$ can be removed by means of the $\overline{\mathrm{MS}}_{FV}$ scheme:

$$\frac{1}{C_0^L(E^*;\mu)} = -\frac{M}{4\pi^2 L}\mathcal{S}(\tilde{p}) + \frac{\alpha M^2}{16\pi^5}\mathcal{S}_2(\tilde{p}) - \frac{\alpha M^2}{4\pi}\left[\log\left(\frac{\pi}{\mu L}\right) - \gamma_E\right]$$

$\rightsquigarrow$ the following quantity has beeen subracted to $C_0^L(E^*)^{-1}$ and to the l.h.s. of the original ERE, modulo a multiplication factor of $4\pi/M^2$ PRD 90, 074511 (2014)

$$\Delta\frac{1}{C_0^L(E^*)}\Bigg|^{\overline{\mathrm{MS}}_{FV}} = \frac{\alpha M^2}{4\pi}\left[\frac{1}{3-d} - \frac{1}{2}\gamma_E + 1 + \log\frac{\sqrt{\pi}}{2}\right]$$

The finite volume ERE for S-waves with QED

The sums appearing in the QC for the $\ell = 0$ strong coupling constant are rewritten as

$$\frac{1}{C_0^L(E^*)} = -\frac{M}{4\pi^2 L}\mathcal{S}(\tilde{p}) + \frac{\alpha M^2}{16\pi^5}\mathcal{S}_2(\tilde{p}) + \frac{\alpha M^2}{4\pi}\left[\frac{1}{3-d} - \frac{1}{2}\gamma_E + 1 - \log\frac{2\pi}{\mu L} - \log\sqrt{\pi}\right]$$

where $\mathcal{S}^{\mathrm{C}}(x) = \mathcal{S}(x) + \frac{\alpha M}{4\pi^3}\mathcal{S}_2(x)$ is the Lüscher function
and $\mathcal{S}(x)$ & $\mathcal{S}_2(x)$ are finite summations (zero modes are absent!)

$$\mathcal{S}_2(x) = \sum_{\mathbf{n}}^{\Lambda_n}\sum_{\mathbf{m}\neq\mathbf{n}} \frac{1}{|\mathbf{n}|^2 - x^2}\frac{1}{|\mathbf{m}|^2 - x^2}\frac{1}{|\mathbf{n}-\mathbf{m}|^2} - 4\pi^4\log\Lambda_n \qquad \mathcal{S}(x) = \sum_{\mathbf{n}}^{\Lambda_n}\frac{1}{|\mathbf{n}|^2 - x^2} - 4\pi\Lambda_n$$

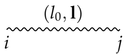
- The removal of the divergence via the $\overline{\mathrm{MS}}_{FV}$ scheme in both $C_0^L(E^*)^{-1}$ and the original $\ell = 0$ generalized FV ERE, allows for the sought rewriting of the latter:

$$\frac{1}{\pi L}\mathcal{S}(\tilde{p}) - \frac{\alpha M}{4\pi^3}\mathcal{S}_2(\tilde{p}) + \alpha M\left[\log\left(\frac{2\pi}{\alpha M L}\right) - \gamma_E\right] = \frac{1}{a_{\mathrm{C}}'^{(0)}} + \frac{1}{2}r_0'^{(0)}\mathbf{p}^2 + r_1'^{(0)}\mathbf{p}^4 + \ldots$$

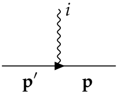
Improvements: By considering **transverse photon** contributions, further $\mathcal{O}(\alpha)$ terms can be included in $J_{\mathrm{C}}^L(\mathbf{p}) \rightsquigarrow$ photon exchanges occur also between the bubbles!

Transverse photon corrections

The terms depending on the vector potential $\mathbf{A}$ in $-1/2(\mathbf{E}^2 - \mathbf{B}^2)$ and in the $\propto \Psi^\dagger \mathbf{D}^2/2M\Psi$ term of $\mathcal{L}^{\text{QED}}$ originate the **transverse photon** propagator and vertex



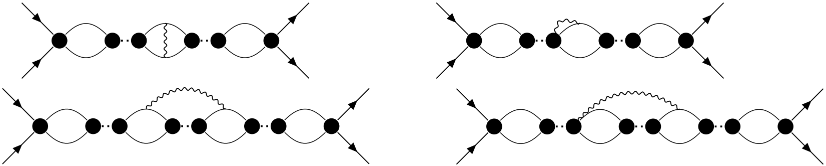
$$\frac{\delta_{ij} - \frac{l_i l_j}{\mathbf{l}^2+\lambda^2}}{l_0^2 - \mathbf{l}^2 - \lambda^2 + i\epsilon}$$



$$-e\frac{p_i + p_i'}{2M}$$

If these photons are included into the total Lagrangian, four new classes of diagrams sum up to the scattering amplitude T_{SC} to order $\mathcal{O}(\alpha)$

$\rightsquigarrow$ photons are exhanged between the bubbles!



Remarks

- In comparison with the bubbles with one Coulomb photon insertions, the new diagrams are suppressed in the IR momentum region

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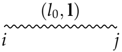

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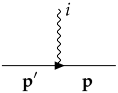

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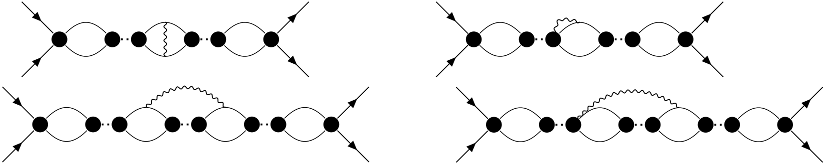

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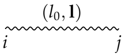

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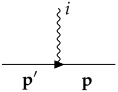

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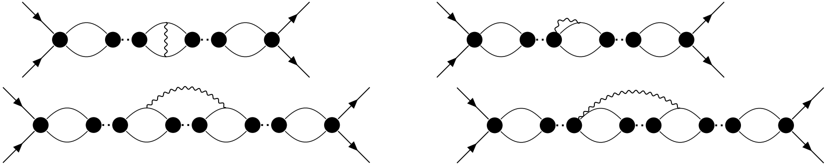

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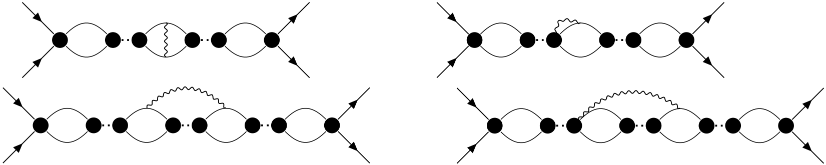

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Taking these new $\mathcal{O}(\alpha)$ diagrams into account, the generalized ERE in FV becomes

$$\frac{1}{\pi L}\mathcal{S}(\tilde{p}) - \frac{\alpha M}{4\pi^3}\mathcal{S}_2(\tilde{p}) + \frac{\alpha M a_{C(0)}{}^2 r_0^{(0)}}{\pi^3 L^3}\mathcal{I}[\mathcal{S}(\tilde{p})]^2 + \alpha M\left[\log\left(\frac{2\pi}{\alpha M L}\right) - \gamma_E\right] = \frac{1}{a'_C{}^{(0)}} + \frac{1}{2} r'_0{}^{(0)}\mathbf{p}^2 + r'_1{}^{(0)}\mathbf{p}^4 + \dots$$

FVECs for S-wave scattering states with QED

We search for the FVEC to the non-interacting g.s. A_1^+ with total energy $E^* = 2M^L$

Strategy

- Low-momentum approx. $\rightsquigarrow$ for $|\mathbf{n}| \neq 0$ we expand the arguments of the sums in the Lüscher function $\mathcal{S}^C(\tilde{p})$ as function of $\tilde{p}/|\mathbf{n}|$

$$\frac{1}{|\mathbf{n}|^2 - \tilde{p}^2} = \frac{1}{|\mathbf{n}|^2} \frac{1}{1 - \frac{\tilde{p}^2}{|\mathbf{n}|^2}} = \frac{1}{|\mathbf{n}|^2} + \frac{\tilde{p}^2}{|\mathbf{n}|^4} + \frac{\tilde{p}^4}{|\mathbf{n}|^6} + \mathcal{O}\left(\frac{\tilde{p}^6}{|\mathbf{n}|^8}\right)$$

As a result:

$$\mathcal{S}(\tilde{p}) = -\frac{1}{\tilde{p}^2} + \mathcal{I} + \mathcal{J}\tilde{p}^2 + \mathcal{K}\tilde{p}^4 + \mathcal{L}\tilde{p}^6 + \ldots$$

$$\mathcal{S}_2(\tilde{p}) = -\frac{2}{\tilde{p}^2}\mathcal{J} + \mathcal{R} - 2\mathcal{K} + 2\tilde{p}^2(\mathcal{R}_{24} - \mathcal{L}) + \tilde{p}^4(\mathcal{R}_{44} + 2\mathcal{R}_{26}) + \ldots$$

where $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{R}_{st}$ are 3D Riemann sums:

$$\mathcal{I} = \sum_{\mathbf{n}\neq 0}^{\Lambda_n} \frac{1}{|\mathbf{n}|^2} - 4\pi\Lambda_n = -8.9136 \qquad \mathcal{J} = \sum_{\mathbf{n}\neq 0}^{\infty} \frac{1}{|\mathbf{n}|^4} = 16.5323 \qquad \mathcal{K} = \sum_{\mathbf{n}\neq 0}^{\infty} \frac{1}{|\mathbf{n}|^6} = 8.4019$$

$$\mathcal{R} = \sum_{\mathbf{n}\neq 0}^{\Lambda_n} \sum_{\mathbf{m}\neq 0,\mathbf{n}}^{\infty} \frac{1}{|\mathbf{n}|^2|\mathbf{m}|^2} \frac{1}{|\mathbf{m}-\mathbf{n}|^2} - 4\pi \log \Lambda_n \qquad \mathcal{L} = \sum_{\mathbf{n}\neq 0}^{\infty} \frac{1}{|\mathbf{n}|^8} = 6.9458 \qquad \mathcal{R}_{st} = \sum_{\mathbf{n}\neq 0}^{\infty} \sum_{\mathbf{m}\neq 0,\mathbf{n}}^{\infty} \frac{1}{|\mathbf{n}|^s|\mathbf{m}|^t} \frac{1}{|\mathbf{m}-\mathbf{n}|^2}$$

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- Solution of the approximate FV ERE in terms of $\tilde{p} \equiv \tilde{p}(a_C'^{(0)}, r_0'^{(0)}, M; L)$
  $\rightsquigarrow$ up to order $\tilde{p}^0$ the polynomial is biquadratic
  $\rightsquigarrow$ improvements in $\tilde{p}^{2m}$: iterative approach

$$-\frac{1}{a_C'^{(0)}} + \frac{2\pi^2}{L^2} r_0'^{(0)}\tilde{p}^2 + \frac{16\pi^4}{L^4} r_1'^{(0)}\tilde{p}^4 = \frac{\alpha\, a_C'^{(0)2} r_0'^{(0)} M}{\pi^3 L^3}\frac{1}{\tilde{p}^4} + \left[-\frac{1}{\pi L} + \frac{\alpha M}{4\pi^4}2\mathcal{J} - \frac{\alpha\, a_C'^{(0)2} r_0'^{(0)}}{\pi^3 L^3}2\mathcal{I}^2\right]\frac{1}{\tilde{p}^2}$$
$$+\frac{\mathcal{I}}{\pi L} - \frac{\alpha M}{4\pi^4}(\tilde{\mathcal{R}} - 2\mathcal{K}) + \left[\frac{\mathcal{J}}{\pi L} - \frac{\alpha M}{4\pi^4}2(\mathcal{R}_{24} - \mathcal{L}) + \frac{\alpha\, a_C'^{(0)2} r_0'^{(0)}}{\pi^3 L^3}2\mathcal{I}(\mathcal{J}\mathcal{I} - \mathcal{K})\right]\tilde{p}^2$$
$$+\left[\frac{\mathcal{K}}{\pi L} - \frac{\alpha M}{4\pi^4}2(\mathcal{R}_{26} + \mathcal{R}_{44}) + \frac{\alpha\, a_C'^{(0)2} r_0'^{(0)}}{\pi^3 L^3}\mathcal{I}(\mathcal{J}^2 - 2\mathcal{L} + 2\mathcal{K}\mathcal{I})\right]\tilde{p}^4 + \ldots$$

where $\tilde{\mathcal{R}} = \mathcal{R} - 4\pi^4\left[\log\left(\frac{4\pi}{\alpha M L}\right) - \gamma_E\right]$

small quantities: $\alpha,\ 1/L,\ a_C'^{(0)}/L,\ r_0'^{(0)}/L$ and $r_1'^{(0)}/L \ll 1$

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- Derivation of the FVECs $\rightsquigarrow \Delta E_S^{(0,A_1)} = \mathbf{p}^2/M$ where $\mathbf{p}$ solves the approx. ERE
  $\rightsquigarrow$ at order $\alpha\, a_C'^{(0)^3}/L^4$, $a_C'^{(0)^3}/L^5$ and $\alpha\, a_C'^{(0)^2} r_0'^{(0)}/L^4$ we have:

$$\Delta E_S^{(0,A_1)} = \frac{4\pi a_C^{(0)}}{ML^3}\left\{1 - \left(\frac{a_C^{(0)}}{\pi L}\right)\mathcal{I} + \left(\frac{a_C^{(0)}}{\pi L}\right)^2[\mathcal{I}-\mathcal{J}] + \ldots\right\} - \frac{2\alpha\, a_C^{(0)}}{L^2\pi^2}\left\{\mathcal{J} + \left(\frac{a_C^{(0)}}{\pi L}\right)[\mathcal{K} - \mathcal{I}\mathcal{J} - \tilde{\mathcal{R}}/2] + \frac{2a_C'^{(0)} r_0'^{(0)}\pi^2}{L^2}\mathcal{I} + \left(\frac{a_C^{(0)}}{\pi L}\right)^2\left[\tilde{\mathcal{R}}\mathcal{I} + \mathcal{I}\mathcal{J} - 2\mathcal{J}^2 - 2\mathcal{I}\mathcal{K} + \mathcal{L} - \mathcal{R}_{24}\right] + \ldots\right\}$$

FVECs for $\ell = 0$ bound states with QED

We search for two-body bound eigenstates with momentum $\mathbf{p} \equiv i\kappa$ with QED.

- Taking the large $\tilde{\kappa}$ limit of $\mathcal{S}^{C}(i\tilde{k})$ ($\equiv$ **deep binding limit**) we find:

$$\sum_{\mathbf{n}}^{\Lambda_n} \frac{1}{|\mathbf{n}|^2 + \tilde{\kappa}^2} \approx 4\pi\Lambda_n - 2\pi^2\tilde{k}$$

$$\sum_{\mathbf{n}}^{\Lambda_n} \sum_{\mathbf{m}\neq\mathbf{n}} \frac{1}{|\mathbf{n}|^2 + \tilde{\kappa}^2} \frac{1}{|\mathbf{m}|^2 + \tilde{\kappa}^2} \frac{1}{|\mathbf{n}-\mathbf{m}|^2} \approx 4\pi^4[\log\Lambda_n - \log(2\tilde{\kappa})] + \frac{\pi^2}{\tilde{\kappa}}\mathcal{I}$$

Plugging the approx. expr. of the Lüscher function into the ERE, the latter becomes

$$-\frac{1}{a_C^{(0)}} - \frac{1}{2} r_0^{(0)} \kappa^2 = -\kappa - \alpha M \left[\gamma_E + \log\left(\frac{\alpha M}{4\kappa}\right)\right] - \frac{\alpha M}{2\pi\kappa L}\mathcal{I}(1 - \kappa r_0)$$

- Performing a perturbative expansion of $\kappa = \kappa_0 + \kappa_1 + \ldots$ where κ_n is $\mathcal{O}(\alpha^n)$ and expressing κ_1 in terms of (κ_0, α) the energy of the lowest A_1^+ state in FV is found

$$E_B^{(0,A_1)}(L) = \frac{\kappa^2}{M} \approx \frac{\kappa_0^2}{M} + 2\frac{\kappa_0}{M}\kappa_1 = \frac{\kappa_0^2}{2\mu} - \frac{2\alpha\kappa_0}{1 - \kappa_0 r_0}\left[\gamma_E + \log\left(\frac{\alpha\mu}{2\kappa_0}\right)\right] - \frac{\alpha}{\pi L}\mathcal{I}$$

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from which the **LO FVECs** is inferred (remember: $\mathcal{I} < 0$) **PRD 90, 074511 (2014)**

$$\Delta E_B^{(0,A_1)} \equiv E_B^{(0,A_1)}(\infty) - E_B^{(0,A_1)}(L) = \frac{\alpha}{\pi L}\mathcal{I} + \mathcal{O}(\alpha^2)$$

Remark

As in the case without QED, FV corrections for the $\ell = 0$ state are negative

Conclusion

The preliminary calculations for the derivation of the correction formulae to two-body $\ell = 1$ bound and free energy eigenstates on a cubic lattice with PBC in the approach followed in PR D 90 665, 074511 (2014) have been presented.
In particular

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Thanks for your attention!

Grazie per l'attenzione!