S-Matrix of Liouville theory

George Jorjadze

FU of Tbilisi & RMI of TSU

Workshop "Frontiers of QCD"

Tbilisi - September 28, 2019

Contents

Liouville equation

$$\partial^2_{z\bar{z}}\,\varphi(z,\bar{z}) + \mu^2\, e^{2\varphi(z,\bar{z})} = 0$$

The general solution

$$\varphi(z,\bar{z}) = \frac{1}{2}\ \log\,\frac{A'(z)\,\bar{A}'(\bar{z})}{[1\,+\mu^2\,A(z)\bar{A}(\bar{z})]^2}$$

Liouville 1853

$z = \tau + \sigma$ and $\bar{z} = \tau - \sigma$ are chiral (light-cone) coordinates

τ is time and σ is the spatial coordinate

$\mu^2 > 0$ is constant

$$(\partial^2_{\tau\tau} - \partial^2_{\sigma\sigma})\varphi + 4\mu^2\, e^{2\varphi} = 0$$

Motivation

- Non-critical strings Polyakov 1981
- 2d CFT strings Curtright, Thorn 1982, Belavin, Polyakov, Zamolodchikov 1984
- 2d gravity Teitelboim 1983, Jackiw 1985, Polyakov 1988, Seiberg 1990
- Gauged WZW models O'Raifeartaigh,... 1989, Alekseev, Shatashvili, 1990
- D-brane dynamics Zamolodchikov, Zamolodchikov 1999, Teshner 2000
- AGT duality Alday, Gaiotto, Tachikawa 2009
- SYK model Maldacena, Stanford 2016, Jevicki 2017,...
- Schwarzian theory Stanford, Witten, 2017, ...
- High-dimensional LT Levy, Oz ... 2018-2019

String theory in the conformal gauge

$$\partial^2_{z\bar{z}} X^\mu(z,\bar{z}) = 0$$

Conformal and Poincare symmetries are compatible in $d = 26$

Liouville equation for one mode

$$\partial^2_{z\bar{z}} X_d(z,\bar{z}) + \mu^2\, e^{2X_d} = 0$$

Liouville theory as a model of 2d gravity

Conformal gauge

$$g_{\mu\nu} = e^{2\varphi} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Scalar curvature

$$R = 8\, e^{-2\varphi}\, \partial^2_{z\bar{z}}\varphi$$

Constant negative curvature $R = -8\mu^2$ is equivalent to

$$\partial^2_{z\bar{z}}\varphi + \mu^2\, e^{2\varphi} = 0$$

2d gravity model

$$R_{\mu\nu} - \frac{1}{2}\,\mu^2\; g_{\mu\nu} = 0$$

Liouville theory as a 2d CFT

Conformal transformations

$$z \;\mapsto\; \zeta(z) \qquad\qquad \bar{z} \;\mapsto\; \bar{\zeta}(\bar{z})$$

$$\zeta'(z) > 0 \qquad\qquad \bar{\zeta}'(\bar{z}) > 0$$

Conformal symmetry

the space of solutions of the Liouville equation is invariant under

$$\varphi(z, \bar{z}) \quad\mapsto\quad \varphi(\zeta(z), \bar{\zeta}(\bar{z})) + \frac{1}{2}\,\log \zeta'(z)\bar{\zeta}'(\bar{z})$$

Primary fields $\quad V_\beta = e^{2\beta\varphi}$

$$V_\beta(z, \bar{z}) \mapsto \quad (\zeta'(z)\bar{\zeta}'(\bar{z}))^\beta\; V_\beta(\zeta(z), \bar{\zeta}(\bar{z}))$$

Energy-momentum tensor

Chirality

$$T = (\partial_z \varphi)^2 - \partial^2_{zz}\varphi \ , \qquad \bar{T} = (\partial_{\bar{z}} \varphi)^2 - \partial^2_{\bar{z}\bar{z}}\varphi$$

$$\partial_{\bar{z}} T = 0 = \partial_z \bar{T} \qquad \text{Poincare}$$

$$T(z) = T_{zz} \qquad \bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}$$

Energy density

$$\mathcal{E} = T + \bar{T} = \frac{1}{2}\ (\partial_\tau \varphi)^2 + \frac{1}{2}\ (\partial_\sigma \varphi)^2 + 2\mu^2\, e^{2\varphi} - \partial^2_{\sigma\sigma}\varphi$$

Momentum density

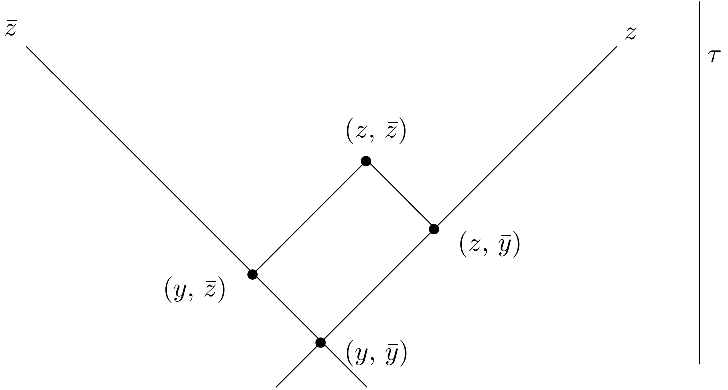
$$\mathcal{P} = T - \bar{T} = \partial_\tau \varphi\, \partial_\sigma \varphi - \partial^2_{\tau\sigma}\varphi$$

NET Poisson brackets

$V = e^{-\varphi}$

$$\{V(z,\bar{z}), V(y,\bar{y})\} = \frac{1}{4}\Big(\mathsf{sign}(z-y) + \mathsf{sign}(\bar{z}-\bar{y})\Big) \times \Big(V(z,\bar{z})V(y,\bar{y}) - 2V(z,\bar{y})V(y,\bar{z})\Big)$$

G.J, G. Weigt; 2001



Periodic boundary conditions

Periodic Liouville field $\varphi(\tau, \sigma + 2\pi) = \varphi(\tau, \sigma)$

$$\varphi(z, \bar{z}) = \frac{1}{2} \; \log \, \frac{A'(z) \, \bar{A}'(\bar{z})}{[1 \, + \mu^2 \, A(z)\bar{A}(\bar{z})]^2}$$

The class of parameterizing chiral fields:

$$A'(z) > 0, \quad \bar{A}'(\bar{z}) > 0$$

$$A(z + 2\pi) = e^{2\pi p} A(z) \qquad \bar{A}(\bar{z} + 2\pi) = e^{2\pi p} \bar{A}(\bar{z})$$

with $p > 0$

Free-field parameterization

Free-field

$$\varphi_{in}(\tau,\sigma) = \frac{1}{2}\log\ A'(z)\bar{A}'(\bar{z})$$

Chiral and mode decomposition

$$\varphi_{in}(z,\bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$$

$$\phi(z) = q + \frac{pz}{2} + i\sum_{n\neq 0}\frac{a_n}{n}\,e^{-inz} = \frac{p\zeta(z)}{2} + \frac{1}{2}\ \log\zeta'(z)$$

The equation $A'(z) = e^{2\phi(z)}$ is integrated to

$$A(z) = \int_{-\infty}^{z} dy\, e^{2\phi(y)}$$

Free-field parameterization of the Liouville field

$$e^{-\varphi(\tau,\sigma)} = e^{-\varphi_{in}(\tau,\sigma)}\left[1 + \mu^2 A(z)\bar{A}(\bar{z})\right]$$

Poisson brackets structure

'Improved' form

$$T(z) = \phi'^{\,2}(z) - \phi''(z) \qquad \bar{T}(\bar{z}) = \bar{\phi}'^{\,2}(\bar{z}) - \bar{\phi}''(\bar{z})$$

From the canonical brackets

$$\{\phi'(z), \phi(y)\} = \frac{1}{2}\,\delta(z-y)$$

follows

$$\{T(z), e^{2\beta\phi(y)}\} = \left(e^{2\beta\phi(y)}\right)'\delta(z-y) - \beta\, e^{2\beta\phi(y)}\delta'(z-y)$$

$$\{T(z), A(y)\} = A'(y)\delta(z-y)$$

2d conformal algebra

$$\{T(z), T(y)\} = T'(y)\,\delta(z-y) - 2T(y)\,\delta'(z-y) + \frac{1}{2}\,\delta'''(z-y)$$

Asymptotic fields

The time asymptotic behavior

$$e^{-\varphi_{in}(\tau,\sigma)} \sim e^{-p\tau} \qquad \mu^2 e^{-\phi(\tau,\sigma)} A(z)\bar{A}(\bar{z}) \sim e^{p\tau}$$

Since $p > 0$

$$e^{-\varphi_{in}(\tau,\sigma)} \to 0 \qquad \tau \to \infty$$

$$\mu^2 e^{-\varphi_{in}(\tau,\sigma)} A(z)\bar{A}(\bar{z}) \to 0 \qquad \tau \to -\infty$$

Liouville field in terms of the asymptotic fields

$$e^{-\varphi(\tau,\sigma)} = e^{-\varphi_{in}(\tau,\sigma)} + e^{-\varphi_{out}(\tau,\sigma)}$$

The out-field in terms of the in-field

$$e^{-\varphi_{out}(\tau,\sigma)} = \mu^2 \, e^{-\varphi_{in}(\tau,\sigma)} \, A(z)\bar{A}(\bar{z})$$

Chiral structure of the map

From the chiral structure of the free-fields follows

$$\phi_{out}(z) = \phi_{in}(z) - \log \mu A_{in}(z) \qquad \bar{\phi}_{out}(\bar{z}) = \bar{\phi}_{in}(\bar{z}) - \log \mu \bar{A}_{in}(\bar{z})$$

The map preserves the stress tensor

$$\phi'^2_{out}(z) - \phi''_{out}(z) = \phi'^2_{in}(z) - \phi''_{in}(z)$$

The S-matrix has a chiral structure

It is conformally invariant: $[S, T(z)] = 0$

Canonical map from the *in* field to the *out* field

Canonical transformation at a fixed τ

$$\varphi_{out}(\sigma) = \varphi_{in}(\sigma) - \log \mu^2 A_{in}(\sigma)\, \bar{A}_{in}(-\sigma)$$

$$\pi_{out}(\sigma) = \pi_{in}(\sigma) - \frac{A'_{in}(\sigma)}{A_{in}(\sigma)} - \frac{\bar{A}'_{in}(-\sigma)}{\bar{A}_{in}(-\sigma)}$$

The generating functional

$$\delta G(\varphi_{out}, \varphi_{in}) = \int_0^{2\pi} d\sigma [\pi_{in}(\sigma)\delta\varphi_{in}(\sigma) - \pi_{out}(\sigma)\delta\varphi_{out}(\sigma)]$$

Semi-classical S-matrix

$$\langle \varphi_{out} \,|\, \varphi_{in} \rangle = e^{-\frac{i}{\hbar}\, G(\varphi_{out}, \varphi_{in})}$$

The semi-classical S-matrix

The solution for the generating functional

$$G = \int_0^{2\pi} d\sigma \left[\lambda(\sigma) + \frac{\varphi_-{}'(\sigma)}{2} \log \frac{\lambda(\sigma) - \varphi_-{}'(\sigma)}{\lambda(\sigma) + \varphi_-{}'(\sigma)} \right]$$

$$\lambda(\sigma) = \sqrt{\varphi'^2_-(\sigma) + 4\mu^2 e^{\varphi_+(z)}}$$

$$\varphi_\pm(\sigma) = \varphi_{in}(\sigma) \pm \varphi_{out}(\sigma)$$

No explicit chiral structure is observed.

The chiral canonical form

The chiral canonical 1-form

$$p\delta q + \frac{i}{2}\sum_{n>0}\frac{1}{n}(a_n^*\delta a_n - a_n\delta a_n^*)$$

The chiral-fields mode-expansion

$$\phi_{in}(z) = q + \frac{pz}{2} + i\sum_{n\neq 0}\frac{a_n}{n}\, e^{-inz}$$

$$\phi_{out}(z) = \tilde{q} + \frac{\tilde{p}z}{2} + i\sum_{n\neq 0}\frac{b_n}{n}\, e^{-inz}$$

Since the map is canonical

$$-(q+\tilde{q})\delta p + 2i\sum_{n>0}\frac{1}{n}(a_n^*\delta a_n + b_n\delta b_n^*) = \delta F(p, b^*, a)$$

Generating function

The generating function $F(p, b^*, a)$ satisfies the equations

$$\frac{\partial F}{\partial p} = -(q + \tilde{q}) \qquad \frac{\partial F}{\partial a_m} = \frac{2i}{m}\, a_m^* \qquad \frac{\partial F}{\partial b_m^*} = \frac{2i}{m}\, b_m$$

with $\quad m > 0$.

Here the right hand sides are treated as functions of (b^*, p, a), with $b^* := (b_1^*, b_2^*, \dots)$ and $a := (a_1, a_2, \dots)$.

Quantum mechanical interpretations of $F(b^*, p, a)$?

The p-dependent ($p > 0$) vacuum state for the chiral in-field

$$\hat{p}|p, 0\rangle = p|p, 0\rangle \qquad \hat{a}_m|p, 0\rangle = 0 \quad \text{for} \quad m > 0$$

The out-field 'bra' vectors $\langle b^*, \tilde{p}|$ are given similarly.

Holomorphic semi-classical S-matrix

The coherent states are constructed by

$$|p, a\rangle = \exp\left(\frac{2}{\hbar}\sum_{m>0}\frac{1}{m}\,a_m\,\hat{a}_m^\dagger\right)|p, 0\rangle$$

From the canonical commutators follow

$$\hat{a}_m\,|p, a\rangle = a_m\,|p, a\rangle \qquad \hat{a}_m^\dagger\,|p, a\rangle = \frac{\hbar m}{2}\,\frac{\partial}{\partial a_m}\,|p, a\rangle$$

$$\langle b^*, \tilde{p}|\,\hat{b}_m^\dagger = b_m^*\,\langle b^*, \tilde{p}| \qquad \langle b^*, \tilde{p}|\,\hat{b}_m = \frac{\hbar m}{2}\,\frac{\partial}{\partial b_m^*}\,\langle b^*, \tilde{p}|$$

To analyze the matrix elements $\langle b^*, p'|p, a\rangle$, we insert canonical operators

$$\langle b^*, \tilde{p}|\hat{\tilde{p}} + \hat{p}|p, a\rangle = (\tilde{p} + p)\langle b^*, \tilde{p}|p, a\rangle$$

Holomorphic semi-classical S-matrix

Since $\hat{\tilde{p}} = -\hat{p}$, the left hand side of the first equation vanishes and

$$\langle b^*, \tilde{p} | p, a \rangle = \mathcal{S}(b^*, p, a)\, \delta(p + \tilde{p})$$

Assuming that the classical relations are valid on the quantum level

$$i\hbar \frac{\partial \mathcal{S}}{\partial p} = \frac{\partial F}{\partial p} \mathcal{S} \qquad \hbar \frac{\partial \mathcal{S}}{\partial a_m} = -i \frac{\partial F}{\partial a_m} \mathcal{S} \qquad \hbar \frac{\partial \mathcal{S}}{\partial b_m^*} = -i \frac{\partial F}{\partial b_m^*} \mathcal{S}$$

These equations lead to

$$\mathcal{S}(b^*, p, a) = e^{-\frac{i}{\hbar} F(b^*, p, a)} \ .$$

Setting here $b^* = a^*$, we conclude that $e^{-\frac{i}{\hbar} F(a^*, p, a)}$ describes the normal symbol of the S-matrix semi-classically.

Virasoro generators

The mode-expansions of the stress-tensor

$$T(x) = \sum_{n\in\mathbb{Z}} L_n \, e^{-inx}$$

provides L_n's in the Fourier modes of the in-field

$$L_0 = \frac{1}{4}\, p^2 + \sum_{k\neq 0} a_{-k}\, a_k \qquad L_n = (p + i\, n) a_n + \sum_{k,l\neq 0} a_k\, a_l\, \delta_{k+l,n} \quad (n \neq 0)$$

and the out-field mode expansion is obtained by the replacements $p \mapsto -p$, $a_n \mapsto b_n$.

The equality between L_n's of the in and out-fields leads to the equations

$$|n|(|n| + ip)\frac{\partial F}{\partial a_n} = 2i(|n| - ip)a_{-n} - i \sum_{k,l\neq 0} [\epsilon(k) - \epsilon(l)]\, |l|\, a_k \frac{\partial F}{\partial a_{-l}} \delta_{k+l+n}$$

$$+ \sum_{k,l\neq 0} [\epsilon(k) + \epsilon(l)] \left(a_k\, a_l + \frac{|k|\,|l|}{4} \frac{\partial F}{\partial a_{-k}} \frac{\partial F}{\partial a_{-l}} \right) \delta_{k+l+n}$$

Solutions for the generating function

Equation for $n = 0$

$$\sum_{k \neq 0} k \; a_k \frac{\partial F}{\partial a_k} = 0$$

A monomial $a_{n_1} \cdots a_{n_\nu}$ is a solution of this linear equation if $n_1 + \cdots + n_\nu = 0$.

The function F then is represented in the form

$$F = F^{(0)} + \sum_{\nu \geq 2} F^{(\nu)} , \quad \text{with} \quad F^{(\nu)} = \sum_{n_1 \ldots n_\nu} f^{(\nu)}_{n_1 \ldots n_\nu}(p) \, \delta_{n_1 + \cdots + n_\nu} \, a_{n_1} \cdots a_{n_\nu}$$

where the expansion coefficients $f^{(\nu)}_{n_1 \ldots n_\nu}$ are symmetric under the permutation of the indices.

Solutions for the generating function

$$f^{(2)}_{n\,-n} = f^{(2)}_{-n\,n} = \frac{i}{|n|}\left(\frac{|n| - ip}{|n| + ip}\right)$$

$$f^{(3)}_{n_1\,n_2\,n_3} = -\frac{4i\,p}{3}\,\frac{\epsilon(n_1)\,\epsilon(n_2)\,\epsilon(n_3)}{(|n_1| + ip)(|n_2| + ip)(|n_3| + ip)}$$

$$f^{(4)}_{n_1\,n_2\,n_3\,n_4} = \frac{2\,p}{3}\prod_{\alpha=1}^{4}\left(\frac{\epsilon(n_\alpha)}{|n_\alpha| + ip}\right)\,U_{n_1\,n_2,\,n_3}$$

$$U_{n_1\,n_2\,n_3} = \left(1 - \frac{ip}{|n_1 + n_2| + ip} - \frac{ip}{|n_2 + n_3| + ip} - \frac{ip}{|n_3 + n_1| + ip}\right)$$

Solutions for the generating function

Starting from $\nu = 4$, the following recursive relations hold

$$|n|\left(|n|+ip\right)\frac{\partial F^{(\nu+1)}}{\partial a_n} =$$

$$\sum_{k,l\neq 0}\Bigg[\left([\epsilon(k)+\epsilon(l)]\frac{|k|}{2}\frac{\partial F^{(2)}}{\partial a_{-k}} - i\,[\epsilon(k)-\epsilon(l)]\,a_k\right)|l|\,\frac{\partial F^{(\nu)}}{\partial a_{-l}}$$

$$+\,[\epsilon(k)+\epsilon(l)]\,\frac{|k|\,|l|}{4}\left(\sum_{j=3}^{\nu-1}\frac{\partial F^{(j)}}{\partial a_{-k}}\,\frac{\partial F^{(\nu+2-j)}}{\partial a_{-l}}\right)\Bigg]\delta_{k+l+n}$$

$$f^{(\nu)}_{n_1\ldots n_\nu} = \frac{(2i)^\nu}{\nu!}\,p\prod_{j=1}^{\nu}\left(\frac{\epsilon(n_j)}{|n_j|+ip}\right)U_{n_1\ldots n_{\nu-1}}\ ,$$

Particle model

Classical equation

$$\ddot{\varphi}(\tau) + 4\mu^2\, e^{2\varphi(\tau)} = 0$$

General solution

$$e^{-\varphi(\tau)} = e^{-q-p\tau} + \frac{\mu^2}{p^2}\, e^{q+p\tau}$$

Canonical transformation

$$p_{out} = -p_{in} = -p \qquad q_{out} = -q + 2\log(p/\mu)$$

Generating function

$$-(q_{in} + q_{out})\delta p = \delta F(p) \qquad F(p) = -2p[\log(p/\mu) - 1]$$

The reflection amplitude

$$R(p) = e^{\frac{2ip}{\hbar}\, \log(\hbar/\mu)}\, \frac{\Gamma(ip/\hbar)}{\Gamma(-ip/\hbar)} \sim e^{-\frac{i}{\hbar}\, F(p)} \qquad \hbar \mapsto 0$$

Quantization

Canonical quantization $$[\phi(z), \phi'(y)] = -\frac{i\hbar}{2}\,\delta(z-y)$$

The Hilbert space $L^2(R_+) \otimes \mathcal{F}$

$\mathcal{F}$ is the Fock space

The vacuum state $|p;0\rangle \quad (p>0)$

Normal ordered operators

$$T(z) =: \phi'^{\,2}(z) : -\eta\phi''(z) \qquad e^{2\beta\phi(z)} =: e^{2\beta\phi(z)} :$$

The condition

$$[T(z), e^{2\phi(y)}] = \left(e^{2\phi(y)}\right)'\delta(z-y) - e^{2\phi(y)}\delta'(z-y)$$

defines $$\eta = 1 + \hbar \qquad (\hbar = b^2)$$

Quantum Virasoro generators

quantum Virasoro generators are defined as follows

$$\hat{L}_m = (\hat{p} + im\eta)\hat{a}_m + \sum_{j,j'\geq 1} \hat{a}_j \, \hat{a}_{j'} \, \delta_{j+j',m} + 2\sum_{j>0} \hat{a}_j^\dagger \, \hat{a}_{m+j}$$

$$\hat{L}_0 = \frac{1}{4}(\hat{p}^2 + \eta^2) + 2\sum_{j\geq 1} \hat{a}_j^\dagger \, \hat{a}_j$$

$$\hat{L}_{-m} = (\hat{p} - im\eta)\hat{a}_m^\dagger + \sum_{j,j'\geq 1} \hat{a}_j^\dagger \, \hat{a}_{j'}^\dagger \, \delta_{j+j',m} + 2\sum_{j\geq 1} \hat{a}_{n+j}^\dagger \, \hat{a}_j$$

Vertex operators and the reflection amplitude

Conformal properties and locality fixes the vertex operator

$$V(\tau,\sigma) = e^{-\varphi_{in}(\tau,\sigma)} + \mu_b^2 \; : e^{-\varphi_{in}(\tau,\sigma)} A(z)\bar{A}(\bar{z}) :$$

S-matrix

$$e^{-\varphi_{in}(\tau,\sigma)} \mathcal{S} \; = \mu_b^2 \, \mathcal{S} \, e^{-\varphi_{in}(\tau,\sigma)} A(z)\bar{A}(\bar{z})$$

The vacuum sector

$$\mathcal{S} \, |p;0\rangle = R(p) \, |-p;0\rangle$$

Equation for $R(p)$

$$R(p) = \frac{\mu_b^2 \, \Gamma(-ip - b^2) \, \Gamma(ip)}{\Gamma(1 + ip + b^2) \, \Gamma(1 - ip)} \, R(p - ib^2)$$

Solution

$$R(p) = - \left(\mu_b^2 \, \Gamma^2(b^2)\right)^{-\frac{ip}{b^2}} \; \frac{\Gamma\left(ip/b^2\right)}{\Gamma\left(-ip/b^2\right)} \, \frac{\Gamma(ip)}{\Gamma(-ip)}$$

The structure of S-matrix

S-matrix

$$\mathcal{S} = \mathcal{P}\, R(p)\, \mathcal{S}_p(a_n)\, \mathcal{S}_p(\bar{a}_n)$$

$\mathcal{P}$ is the parity operator

$$\mathcal{P}\, p\, \mathcal{P} = -p \qquad \mathcal{P}\, q\, \mathcal{P} = -q$$

$\mathcal{S}_p$ does not depend on q

$$\mathcal{S}_p\, |p;0\rangle = |p;0\rangle$$

The conformal symmetry

$$[\mathcal{S}, T(z)] = 0$$

provides the operator relations

$$\mathcal{S}_p L_n(p) = L_n(-p)\, \mathcal{S}_p$$

Equation for the S-matrix

Projecting the operator equations between the coherent states

$$\langle a^*|\,\hat{\mathcal{S}}\,\hat{L}_n(\hat{p})\,|p,a\rangle = \langle a^*|\,\hat{L}_n(-\hat{p})\,\hat{\mathcal{S}}\,|p,a\rangle$$

Representing $\mathcal{S}$ in the form

$$\mathcal{S} = R(p)\,e^{-\frac{i}{\hbar}F_q}$$

one finds that $R(p)$ is canceled in the equations and F_q satisfies

$$\frac{|n|}{2}(|n|\eta + ip)\frac{\partial F_q}{\partial a_n} = i(|n|\eta - ip)a_{-n} - \frac{i}{2}\sum_{k,l\neq 0}\left[\epsilon(k)-\epsilon(l)\right]|l|a_k\frac{\partial F_q}{\partial a_{-l}}\delta_{k+l+n}$$

$$+\frac{1}{2}\sum_{k,l\neq 0}\left[\epsilon(k)+\epsilon(l)\right]\left[a_k\,a_l + \frac{|k||l|}{4}\left(\frac{\partial F_q}{\partial a_{-k}}\frac{\partial F_q}{\partial a_{-l}} + i\hbar\frac{\partial^2 F_q}{\partial a_{-k}\partial a_{-l}}\right)\right]\delta_{k+l+n}$$

Transition amplitudes

$$f^{(q,2)}_{-1\,1} = i\,\frac{1+\hbar-ip}{1+\hbar+ip}\ ,$$

$$f^{(q,3)}_{-1\,-1\,2} = \frac{-4ip(1+\hbar)}{3(1+\hbar+ip)(1+2\hbar+ip)(2+\hbar+ip)} = -f^{(q,3)}_{-2\,1\,1}$$

$$f^{(q,4)}_{-1\,-1\,1\,1} = \frac{-4p(1+\hbar)}{3(1+\hbar+ip)^2(1+2\hbar+ip)(2+\hbar+ip)}$$

$$f^{(q,2)}_{-2\,2} = \frac{i(2+7\hbar+7\hbar^2+2\hbar^3)-(3+5\hbar+3\hbar^2)p-p^3}{2(1+\hbar+ip)(1+2\hbar+ip)(2+\hbar+ip)}$$

$$f^{(q,6)}_{-1\,-1\,-1\,1\,1\,1} = \frac{32p(1+\hbar)(1+\hbar-ip)}{15(1+\hbar+ip)^3(1+2\hbar+ip)(1+3\hbar+ip)(2+\hbar+ip)(3+\hbar+ip)}$$

Scattering and bound states

The conformal group orbit for a constant negative stress tensor $T = -\frac{\theta^2}{4}$

$$\phi(z) = \frac{i\theta\zeta(z)}{2} + \frac{1}{2}\log\zeta'(z)$$

The mode expansion

$$\phi'(z) = \frac{i\rho}{2} + \sum_{n\neq 0} a_n\, e^{-inz}$$

Canonical commutation relations

$$[a_m, a_n] = \hbar\, m\, \delta_{m+n,0}$$

However,

$$a^\dagger_{-n} \neq a_n \qquad\qquad a^\dagger_n = b_n$$

The Fock space with the vacuum state $|\theta_k, 0\rangle$

Summary and outlook

We have derived equations for the generating function of the canonical transformation between the in and out fields of Liouville theory in the holomorphic variables.

We have calculated semi-classical S-matrix in a compact form.

We have derived equations for the normal symbol of S-matrix.

A closed form of the amplitudes have been found for scattering of Liouville particles (up to 8).

- Boundary Liouville theory and bound states.
- High-dimensional Liouville theory
- String in the static gauge