TBILISI STATE UNIVERSITY

SPLIT-OCTONIONS AND GEOMETRY OF ITS AUTOMORPHISM GROUP G_2^{NC}

Alexandre Gurchumelia

(PhD student)

Supervisor: Assoc. Prof. Merab Gogberashvili

School and Workshop “Frontiers of QCD” (2019/09/24 - Tbilisi, Georgia)

Supported by a joint grant of Volkswagen Foundation and SRNSF (Ref. 93 562 & #04/48)

Cayley-Dickson doubling procedure

- Given algebra $\mathbb{A}$, we can double it

$$(a_1, a_2) \in \mathbb{A} \times \mathbb{A}, \qquad a_1, a_2 \in \mathbb{A}$$

- Multiplication rule

$$(a_1, a_2)(b_1, b_2) = (a_1 b_1 + \delta b_2 a_2^*, a_1^* b_2 + b_1 a_2)$$

- Involution

$$(a_1, a_2)^* = (a_1^*, -a_2)$$

- Norm of $a = (a_1, a_2)$

$$|a|^2 = a^* a = (|a_1|^2 + |a_2|^2, 0)$$

- Complex numbers: $i = (0,1)$

$$(a_1, a_2) = a_1 + i a_2$$

- Setting $\delta = 1$ and iterating

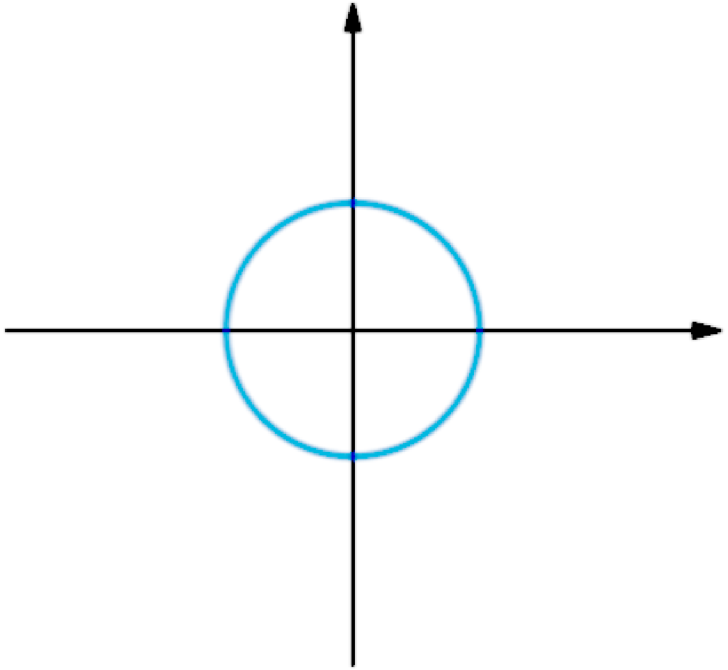
$$\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O} \to \mathbb{S} \to \cdots$$

Algebra	Dimension	Ordered	Multiplication properties				Nontriv. zero divisors
			Commutative	Associative	Alternative	Power-assoc.	
Real numbers	1	Yes	Yes	Yes	Yes	Yes	No
Complex num.	2	No	Yes	Yes	Yes	Yes	No
Quaternions	4	No	No	Yes	Yes	Yes	No
Octonions	8	No	No	No	Yes	Yes	No
Sedenions	16	No	No	No	No	Yes	Yes
	> 16						

Complex & split-complex numbers

$$i^2 = -1$$

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

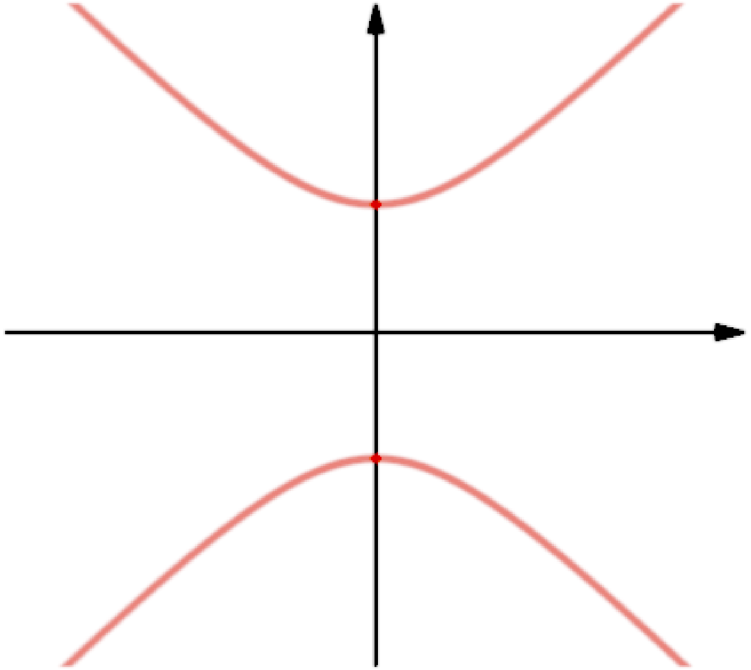


$$z = x + iy \in \mathbb{C}$$

$$|z|^2 = x^2 + y^2$$

$$j^2 = 1$$

$$e^{j\varphi} = \cosh\varphi + j\sinh\varphi$$



$$z = t + jx \in \text{split-complex}$$

$$|z|^2 = t^2 - x^2$$

Quaternions $\mathbb{H}$: $i^2 = j^2 = k^2 = ijk = -1$ Slide: 4

- Quaternionic algebra

$$q_1^2 = q_2^2 = q_3^2 = q_1 q_2 q_3 = -1$$

- Spatial rotations of $\vec{\mathbf{A}} = A_n q_n$

$$\vec{\mathbf{A}}' = \exp\left(\frac{1}{2}\theta_n q_n\right)\vec{\mathbf{A}}\exp\left(-\frac{1}{2}\theta_n q_n\right)$$

- Isomorphism with Clifford algebra

$$\mathbb{C} \otimes \mathbb{H} \simeq \mathbb{C}l(3)$$

$$iq_1 \leftrightarrow \sigma_1, \quad iq_2 \leftrightarrow \sigma_2, \quad iq_3 \leftrightarrow \sigma_3$$

- Grassmann numbers (nilpotent)

$$a = \frac{1}{2}(q_1 + iq_2), \quad a^\dagger = -\frac{1}{2}(q_1 - iq_2)$$

- Projection operator (idempotent)

$$aa^\dagger = \frac{1}{2}(1 - iq_3)$$

- Ladder structure

$$\{a, a\} = \{a^\dagger, a^\dagger\} = 0, \quad \{a, a^\dagger\} = 1$$

C. Furey 2018

Eur. Phys. J. C (2018) 78:375
https://doi.org/10.1140/epjc/s10052-018-5844-7

THE EUROPEAN PHYSICAL JOURNAL C

CrossMark

Slide: 5

Regular Article - Theoretical Physics

$SU(3)_C \times SU(2)_L \times U(1)_Y \ (\times U(1)_X)$ as a symmetry of division algebraic ladder operators

C. Furey[1,2,a]

[1] Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wi
[2]
B3 0HI

$$\overleftarrow{a_1} \equiv \frac{1}{2}\left(-\overleftarrow{e_5} + i\,\overleftarrow{e_4}\right), \quad \overleftarrow{a_2} \equiv \frac{1}{2}\left(-\overleftarrow{e_3} + i\,\overleftarrow{e_1}\right),$$

$$\overleftarrow{a_3} \equiv \frac{1}{2}\left(-\overleftarrow{e_6} + i\,\overleftarrow{e_2}\right), \quad \overleftarrow{a_1}^{\dagger} \equiv \frac{1}{2}\left(\overleftarrow{e_5} + i\,\overleftarrow{e_4}\right),$$

$$\overleftarrow{a_2}^{\dagger} \equiv \frac{1}{2}\left(\overleftarrow{e_3} + i\,\overleftarrow{e_1}\right), \quad \overleftarrow{a_3}^{\dagger} \equiv \frac{1}{2}\left(\overleftarrow{e_6} + i\,\overleftarrow{e_2}\right).$$

$$S^u = \mathcal{V} v_c$$
$$+\bar{\mathcal{D}}^{\mathrm{r}} a_1^{\dagger} v_c + \bar{\mathcal{D}}^{\mathrm{g}} a_2^{\dagger} v_c + \bar{\mathcal{D}}^{\mathrm{b}} a_3^{\dagger} v_c$$
$$+\mathcal{U}^{\mathrm{r}} a_3^{\dagger} a_2^{\dagger} v_c + \mathcal{U}^{\mathrm{g}} a_1^{\dagger} a_3^{\dagger} v_c + \mathcal{U}^{\mathrm{b}} a_2^{\dagger} a_1^{\dagger} v_c$$
$$+\mathcal{E}^{+} a_3^{\dagger} a_2^{\dagger} a_1^{\dagger} v_c,$$

Abstract We demonstrate a model which captures certain attractive features of $SU(5)$ theory, while providing a possible escape from proton decay. In this paper we show how ladder operators arise from the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. From the $SU(n)$ symmetry of these ladder operators, we then demonstrate a model which has much structural similarity to Georgi and Glashow's $SU(5)$ grand unified theory. However, in this case, the transitions leading to proton decay are expected to be blocked, given that they coincide with presumably forbidden transformations which would incorrectly mix distinct algebraic actions. As a result, we find that we are left with $G_{sm} = SU(3)_C \times SU(2)_L \times U(1)_Y/\mathbb{Z}_6$. Finally, we point out that if $U(n)$ ladder symmetries are used in place of $SU(n)$, it may then be possible to find this same $G_{sm} = SU(3)_C \times SU(2)_L \times U(1)_Y/\mathbb{Z}_6$, together with an extra $U(1)_X$ symmetry, related to $B-L$.

- Multiplication table

1	J_1	J_2	J_3	I	j_1	j_2	j_3
J_1	1	j_3	$-j_2$	j_1	I	$-J_3$	J_2
J_2	$-j_3$	1	j_1	j_2	J_3	I	$-J_1$
J_3	j_2	$-j_1$	1	j_3	$-J_2$	J_1	I
I	$-j_1$	$-j_2$	$-j_3$	1	$-J_1$	$-J_2$	$-J_3$
j_1	$-I$	$-J_3$	J_2	J_1	-1	j_3	$-j_2$
j_2	J_3	$-I$	$-J_1$	J_2	$-j_3$	-1	j_1
j_3	$-J_2$	J_1	$-I$	J_3	j_2	$-j_1$	-1

- Split-octonion & its conjugte

$$s = \omega + \lambda_n J_n + tI + x_n j_n$$

$$\bar{s} = \omega - \lambda_n J_n - tI - x_n j_n$$

- Norm

$$\bar{s}s = \omega^2 - \lambda_m \lambda^m - t^2 + x_n x^n$$

- Other involutions (conjugtions)

$$\mathrm{conj}_{I,j}(s) = J_3(J_2(J_1 s J_1)J_2)J_3$$
$$\mathrm{conj}_{J,j}(s) = s^{\dagger_I} = IsI$$
$$\mathrm{conj}_{J,I}(s) = I(J_3(J_2(J_1 s J_1)J_2)J_3)I$$

Spin & chirality with split-octonions

- Eluding non-associativity

$$ABCf \equiv A\big(B(Cf)\big)$$

- Ladder operators

$$\alpha = \frac{1}{2}(J_1 - j_2), \quad \alpha^\dagger = \frac{1}{2}(J_1 + j_2)$$

- Commutation relations

$$\{\alpha, \alpha\} = \{\alpha^\dagger, \alpha^\dagger\} = 0, \quad \{\alpha, \alpha^\dagger\} = 1$$

- "Complex" numbers, A, B, C, D

$$x = x_0 + j_3 x_1$$

- Idempotents

$$v_R = \alpha\alpha^\dagger = \frac{1}{2}(1 - J_3)$$
$$v_L = \alpha^\dagger\alpha = \frac{1}{2}(1 + J_3)$$

- Finding minimal left ideal (spinor)

$$\Psi = \left(\alpha|A + \alpha^\dagger|B + \alpha\alpha^\dagger|C + \alpha^\dagger\alpha|D\right)v$$

- Left and right Weyl spinors

$$\Psi_R = \alpha^\dagger v_R \psi_R^\uparrow + v_R \psi_R^\downarrow$$
$$\Psi_L = v_L \psi_L^\uparrow + \alpha v_L \psi_L^\downarrow$$

Dirac theory with with split-octonions

- Dirac spinor

$$\Psi_D = \Psi_R + \Psi_L$$
$$= \frac{1}{2}(\psi^{\uparrow}_{L0} + \psi^{\downarrow}_{R0}) + \frac{1}{2}(\psi^{\downarrow}_{L0} + \psi^{\uparrow}_{R0})J_1 + \frac{1}{2}(\psi^{\downarrow}_{L1} + \psi^{\uparrow}_{R1})J_2 + \frac{1}{2}(\psi^{\uparrow}_{L0} - \psi^{\downarrow}_{R0})J_3$$
$$- \frac{1}{2}(\psi^{\uparrow}_{L1} - \psi^{\downarrow}_{R1})I - \frac{1}{2}(\psi^{\downarrow}_{L1} - \psi^{\uparrow}_{R1})j_1 - \frac{1}{2}(\psi^{\downarrow}_{L0} - \psi^{\uparrow}_{R0})j_2 + \frac{1}{2}(\psi^{\uparrow}_{L1} + \psi^{\downarrow}_{R1})j_3$$

- Dirac equation in external 4-potential

$$j_3(\nabla\Psi) + \mathrm{conj}_{J,I}(\mathcal{A}\Psi) = m\Psi$$

$$\nabla = \frac{\partial}{\partial t} + J_m \frac{\partial}{\partial x_m}, \quad \mathcal{A} = A_0 + J_m A_m$$

$$\rho = \bar{\Psi}\Psi$$

- Another way of writing

$$J_3(\nabla\Psi) + I\,\mathrm{conj}_{I,J}(\mathcal{A}\Psi) = -m\Psi$$

$$\nabla = -I\frac{\partial}{\partial t} + j_m \frac{\partial}{\partial x_m}, \quad \mathcal{A} = -IA_0 + j_m A_m$$

$$\rho = -\bar{\Psi}\Psi = \overline{(\Psi I)}(\Psi I)$$

Exceptional Lie group G_2^{NC}

- Quadratic form

$$\bar{s}s = \omega^2 - \lambda_m \lambda^m - t^2 + x_n x^n$$

- Groups that preserve it

$$SO(4,4) \supset SO(4,3) \supset G_2^{\mathrm{NC}}$$

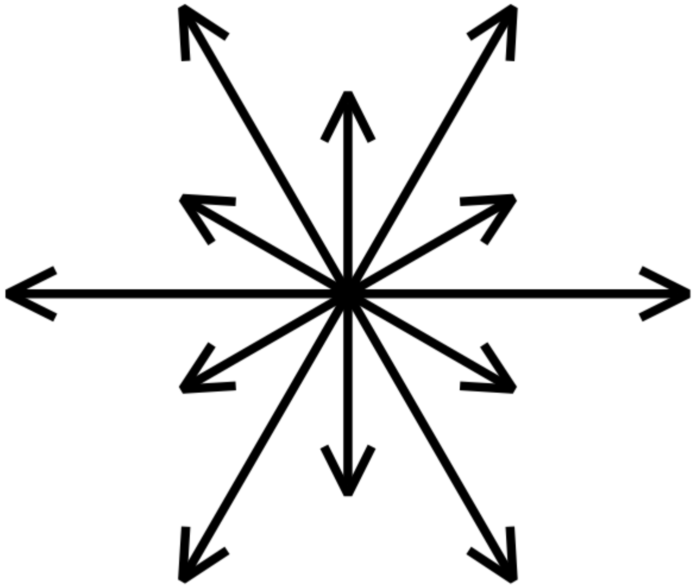
$$Y_{kk} = -z_k \frac{\partial}{\partial z^k} + y_k \frac{\partial}{\partial y^k} + \frac{1}{3} \sum_i \left(z_i \frac{\partial}{\partial z_i} - y_i \frac{\partial}{\partial y_i} \right) ,$$

$$Y_{k0} = -2t \frac{\partial}{\partial z^k} + y_k \frac{\partial}{\partial t} + \frac{1}{2} \sum_{i,j} \epsilon_{ijk} \left(z^i \frac{\partial}{\partial y_j} - z^j \frac{\partial}{\partial y_i} \right) ,$$

$$Y_{0k} = -2t \frac{\partial}{\partial y^k} + z_k \frac{\partial}{\partial t} + \frac{1}{2} \sum_{i,j} \epsilon_{ijk} \left(y^i \frac{\partial}{\partial z_j} - y^j \frac{\partial}{\partial z_i} \right) ,$$

$$Y_{ij} = -z_j \frac{\partial}{\partial z^i} + y_i \frac{\partial}{\partial y^j} . \qquad (i,j,k = 1,2,3)$$

- G_2 root system



$$Y_{11} + Y_{22} + Y_{33} = 0 ,$$

G_2^{NC} generators (other basis)

Gogberashvili, Gurchumelia (2019)

$$\Theta_k = -2\left(x_k\frac{\partial}{\partial t} + t\frac{\partial}{\partial x^k}\right) - \sum_{i,j}\epsilon_{ijk}\left(\lambda^i\frac{\partial}{\partial x_j} - x^j\frac{\partial}{\partial \lambda_i}\right) ,$$

$$B_k = -2\left(\lambda_k\frac{\partial}{\partial t} + t\frac{\partial}{\partial \lambda^k}\right) - \sum_{i,j}\epsilon_{ijk}\left(\lambda^i\frac{\partial}{\partial \lambda_j} - x^j\frac{\partial}{\partial x_i}\right) ,$$

$$\Gamma_k = \sum_{i,j}|\epsilon_{ijk}|\left(x^i\frac{\partial}{\partial \lambda_j} + \lambda^j\frac{\partial}{\partial x_i}\right) ,$$

$$(i, j, k = 1, 2, 3)$$

$$R_k = \sum_{i,j}\epsilon_{ijk}\left(\lambda^i\frac{\partial}{\partial \lambda_j} + x^i\frac{\partial}{\partial x_j}\right) ,$$

$$\Phi_k = \left(x_k\frac{\partial}{\partial \lambda^k} + \lambda_k\frac{\partial}{\partial x^k}\right) - \frac{1}{3}\sum_i\left(x_i\frac{\partial}{\partial \lambda_i} + \lambda_i\frac{\partial}{\partial x_i}\right) .$$

Infinitesimal G_2^{NC} transformations

$$\lambda'_k = \lambda_k + \sum_{i,j} \epsilon_{ijk} \left(\beta^i - \rho^i\right) \lambda^j - 2\beta_k t - \sum_{i,j} \left(\epsilon_{ijk}\theta^i + |\epsilon_{ijk}|\, \gamma^i\right) x^j - \left(\varphi_k - \frac{1}{3}\sum_i \varphi_i\right) x_k \ ,$$

$$t' = t + 2\sum_i \left(\beta_i \lambda^i + \theta_i x^i\right) \ ,$$

$$x'_k = x_k - \sum_{i,j} \epsilon_{ijk} \left(\beta^i + \rho^i\right) x^j + 2\theta_k t + \sum_{i,j} \left(\epsilon_{ijk}\theta^i - |\epsilon_{ijk}|\, \gamma^i\right) \lambda^j - \left(\varphi_k - \frac{1}{3}\sum_i \varphi_i\right) \lambda_k \ .$$

Finite transformation examples in G_2^{NC}

Non-commutativity bound $[x_i, x_j] < 10^{-8}\mathrm{GeV}^2$

Chaichian, Sheikh-Jabbari, Tureanu (2001)

- Rotation

$$\lambda_1' = \lambda_1 \ , \qquad \lambda_2' = \lambda_2 \cos\rho_1 + \lambda_3 \sin\rho_1 \ , \qquad \lambda_3' = \lambda_3 \cos\rho_1 - \lambda_2 \sin\rho_1 \ ,$$

$$t' = t \ ,$$

$$x_1' = x_1 \ , \qquad x_2' = x_2 \cos\rho_1 + x_3 \sin\rho_1 \ , \qquad x_3' = x_3 \cos\rho_1 - x_2 \sin\rho_1 \ .$$

- This boost imitates translation, but is non-commutative

$$\lambda_1' = \lambda_1 \ , \qquad \lambda_2' = \lambda_2 \cosh\gamma_1 - x_3 \sinh\gamma_1 \ , \qquad \lambda_3' = \lambda_3 \cosh\gamma_1 - x_2 \sinh\gamma_1 \ ,$$

$$t' = t \ ,$$

$$x_1' = x_1 \ , \qquad x_2' = x_2 \cosh\gamma_1 - \lambda_3 \sinh\gamma_1 \ , \qquad x_3' = x_3 \cosh\gamma_1 - \lambda_2 \sinh\gamma_1 \ .$$

Casimir operator of G_2^{NC}

Gogberashvili, Gurchumelia (2019)

$$C_2 = \sum_k \left(\frac{1}{3}\Theta_k^2 - \frac{1}{3}B_k^2 + \Gamma_k^2 - R_k^2 + 2\Phi_k^2 \right) =$$

$$= 6\left[t\frac{\partial}{\partial t} + \sum_k \left(\lambda_k \frac{\partial}{\partial \lambda_k} + x_k \frac{\partial}{\partial x_k} \right) \right] + x^2 \frac{\partial^2}{\partial t^2} + \sum_k t^2 \frac{\partial^2}{\partial x_k^2} + 2t \sum_k x_k \frac{\partial^2}{\partial t \partial x_k} +$$

$$+ \sum_{i,j,k} |\epsilon_{ijk}| \left(x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} - x_i^2 \frac{\partial^2}{\partial x_j^2} \right) - \lambda^2 \left(\frac{\partial^2}{\partial t^2} - \sum_k \frac{\partial^2}{\partial x_k^2} \right) - \sum_k \left(t^2 - x^2 \right) \frac{\partial^2}{\partial \lambda_k^2} +$$

$$+ 2t \sum_k \lambda_k \frac{\partial^2}{\partial t \partial \lambda_k} + \sum_{i,j} \frac{2}{3} \lambda_i x_j \frac{\partial^2}{\partial \lambda_i \partial x_j} + \sum_{i,j,k} |\epsilon_{ijk}| \left(\lambda_i \lambda_j \frac{\partial}{\partial \lambda_i \partial \lambda_j} - \lambda_i^2 \frac{\partial}{\partial \lambda_j^2} \right) .$$

- Limiting extra dimensions $\lambda = \mathrm{const}$

$C_2 \rightarrow C_{\mathrm{Lorentz}} - \lambda^2 C_{\mathrm{KG}}$

Summery

- We constructed spin ladder structure and Dirac equation with split-octonions
- Geometrical application of split-octonions was also considered
- Representation of rotations by split-octonions gives G_2^{NC}
- Effective space-time is 7D (4 time, 3 space)
- G_2^{NC} imitates Poincaré transformations
- Translations are non-commutative due to extra time-like dimensions
- Casimir of G_2^{NC} reduces to Casimirs of Lorentz and Poincaré in the limit

References

- **Gogberashvili, M., & Gurchumelia, A. (2019). Geometry of the non-compact G (2). *Journal of Geometry and Physics*.**
- **Gogberashvili, M., & Sakhelashvili, O. (2015). Geometrical applications of split octonions. *Advances in Mathematical Physics, 2015*.**
- Chaichian, M., Sheikh-Jabbari, M. M., & Tureanu, A. (2001). Hydrogen atom spectrum and the Lamb shift in noncommutative QED. *Physical Review Letters*, *86*(13), 2716.
- Furey, C. (2018). $SU(3)_C \times SU(2)_L \times U(1)_Y (\times U(1)_X)$ as a symmetry of division algebraic ladder operators. *The European Physical Journal C*, *78*(5), 375.