On Some Universal Features of Simple Lie Algebras

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Vogel's plane

Vogel's plane is the projective plane P^2 (CP^2, RP^2) with homogeneous coordinates α, β, γ,

$$(\alpha, \beta, \gamma) \sim (\lambda\alpha, \lambda\beta, \lambda\gamma), \lambda \neq 0$$

factorized over group S_3 of permutations of α, β, γ:

$$P^2/S_3$$

It appears in Vogel's (1995) study of finite Vassiliev's knots invariants .

Vogel's parameterization.

Let $2t$ be an eigenvalue of second Casimir operator on adjoint representation: $C_2(ad) = 2t$. Then for all simple (super)algebras:

$$S^2 ad = \underline{\mathbf{1}} + Y_2(\alpha) + Y_2(\beta) + Y_2(\gamma), \ \alpha + \beta + \gamma = t$$
$$C_2(Y_2(\alpha)) = 4t - 2\alpha$$
$$C_2(Y_2(\beta)) = 4t - 2\beta$$
$$C_2(Y_2(\gamma)) = 4t - 2\gamma$$

Table: Vogel's parameters for simple Lie algebras

Algebra/Parameters	α	β	γ	t	Line
$\mathfrak{sl}_N$	-2	2	N	N	$\alpha + \beta = 0$
$\mathfrak{so}_N$	-2	4	$N-4$	$N-2$	$2\alpha + \beta = 0$
$\mathfrak{sp}_N$	-2	1	$N/2+2$	$N/2+1$	$\alpha + 2\beta = 0$
$Exc(n)$	-2	$2n+4$	$n+4$	$3n+6$	$\gamma = 2(\alpha+\beta)$

For the exceptional line $n = -2/3, 0, 1, 2, 4, 8$ for $\mathfrak{g}_2, \mathfrak{so}_8, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, respectively.

Parameters α, β, γ belong to Vogel's plane P^2/S_3. This table is on agreement with $SO(N) \leftrightarrow Sp(N)$ under $N \leftrightarrow -N$ duality (King 1971, Cvitanovich 1981, RM 1981).

Dimension formulae

Dimension of simple Lie algebras in universal form:

$$dim\, \mathfrak{g} \quad = \quad \frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma}, \quad t=\alpha+\beta+\gamma$$

Further dimension formulae (Vogel 1995-2011, Landsberg, Manivel, 2002,2006)

$$dim\, Y_2(\alpha) \quad = \quad -\frac{(3\,\alpha-2\,t)\,(\beta-2\,t)\,(\gamma-2\,t)\,t\,(\beta+t)\,(\gamma+t)}{\alpha^2\,\beta\,\gamma\,(\alpha-\beta)\,(\alpha-\gamma)}$$

$$dim\, Y_3(\alpha) \quad = \quad \frac{(5\,\alpha-2\,t)\,(\alpha-2\,t)\,(\beta-2\,t)\,(\gamma-2\,t)\,t\,(\beta-\alpha+t)\,(\beta+t)\,(\gamma+t)}{\alpha^3\,\beta\,\gamma\,(\alpha-\beta)\,(\alpha-\gamma)\,(2\alpha-\beta)\,(2\alpha-\gamma)}$$

(For $Y_i(\beta)$, $Y_i(\gamma), i=2,3$ permute α,β,γ)

Quantum dimension of the adjoint representation.

Universal form of character of the adjoint representation at line $x\rho$, i.e. quantum dimension of adjoint (Westbury 2004; Veselov, RM 2012):

$$\chi_{ad}(x\rho) = r + \sum_{\mu \in R} e^{x(\mu,\rho)} \equiv f(x)$$

$$f(x) = \frac{\sinh(x\frac{\alpha-2t}{4})}{\sinh(x\frac{\alpha}{4})} \frac{\sinh(x\frac{\beta-2t}{4})}{\sinh(x\frac{\beta}{4})} \frac{\sinh(x\frac{\gamma-2t}{4})}{\sinh(x\frac{\gamma}{4})}$$

X_2 representation

$$\wedge^2 \mathfrak{g} = \mathfrak{g} + X_2$$

The representation X_2 is irreducible w.r.t. the semidirect product of simple Lie algebra on the automorphism of corresponding Dynkin diagram and has the highest weight in terms of fundamental ones given in the table below. We assume the numeration of nodes of Dynkin diagram as in Mathematica program LieART.

	λ_{ad}	λ_{X_2}
G_2	ω_2	$3\omega_1$
F_4	ω_1	ω_2
E_6	ω_6	ω_3
E_7	ω_1	ω_2
E_8	ω_7	ω_6
A_1	2ω	0
$A_i,\ i > 1$	$\omega_1 + \omega_i$	$(2\omega_1 + \omega_{i-1}) \oplus (\omega_2 + 2\omega_i)$
B_2	$2\omega_2$	$\omega_1 + 2\omega_2$
B_3	ω_2	$\omega_1 + 2\omega_3$
$B_i,\ i > 3$	ω_2	$\omega_1 + \omega_3$
C_i	$2\omega_1$	$2\omega_1 + \omega_2$
D_4	ω_2	$\omega_1 + \omega_3 + \omega_4$
$D_i,\ i > 4$	ω_2	$\omega_1 + \omega_3$

The universal quantum dimension of X_2 is (Deligne 2013)

$$D_Q^{X_2} = \frac{\sinh\left(\frac{x}{4}(2t-\alpha)\right)\sinh\left(\frac{x}{4}(2t-\beta)\right)\sinh\left(\frac{x}{4}(2t-\gamma)\right)}{\sinh\left(\frac{\alpha x}{4}\right)\sinh\left(\frac{\beta x}{4}\right)\sinh\left(\frac{\gamma x}{4}\right)} \times$$

$$\frac{\sinh\left(\frac{x}{4}(t+\alpha)\right)\sinh\left(\frac{x}{4}(t+\beta)\right)\sinh\left(\frac{x}{4}(t+\gamma)\right)}{\sinh\left(\frac{\alpha x}{2}\right)\sinh\left(\frac{\beta x}{2}\right)\sinh\left(\frac{\gamma x}{2}\right)} \times$$

$$\frac{\sinh\left(\frac{x}{2}(t-\alpha)\right)\sinh\left(\frac{x}{2}(t-\beta)\right)\sinh\left(\frac{x}{2}(t-\gamma)\right)}{\sinh\left(\frac{x}{4}(t-\alpha)\right)\sinh\left(\frac{x}{4}(t-b)\right)\sinh\left(\frac{x}{4}(t-\gamma)\right)}$$

Quantum dimension of the Cartan product $X_2^k ad^n$

Quantum dimension of irrep with highest weight $k\lambda_{X_2} + n\lambda_{ad}$

$$X(x,k,n,\alpha,\beta,\gamma) = L_{31} \cdot L_{32} \cdot L_{21s1} \cdot L_{21s2} \cdot L_{21s3} \cdot L_{10s1} \cdot L_{10s2} \cdot L_{10s3} \cdot L_{11s1} \cdot L_{11s2} \cdot L_{11s3} \cdot L_{01} \cdot L_{c2}$$

$$L_{31} = \sinh\left[\frac{x}{4} : \frac{-2(\beta+\gamma)+\alpha(-4+3k+n)}{2(2\alpha+\beta+\gamma)} = \frac{\sinh[\frac{x}{4}(-2(\beta+\gamma)+\alpha(-4+3k+n))]}{\sinh[\frac{x}{4}(2(2\alpha+\beta+\gamma))}\right.$$

$$L_{32} = \sinh\left[\frac{x}{4} : \frac{-2(\beta+\gamma)+\alpha(-3+3k+2n)}{3\alpha+2(\beta+\gamma)}\right.$$

$$L_{21s1} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{2k+n} \frac{-2(\beta+\gamma)+\alpha(-5+i)}{-2\beta+\alpha(i-2)}\right.$$

$$L_{21s2} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{2k+n} \frac{\beta+2\gamma-\alpha(-3+i)}{\beta+\gamma-\alpha(i-2)}\right.$$

Quantum dimension of Cartan product $X_2^k ad^n$

$$L_{31} = \sinh\left[\frac{x}{4} : \frac{-2(\beta+\gamma)+\alpha(-4+3k+n)}{2(2\alpha+\beta+\gamma)}\right.$$

$$L_{32} = \sinh\left[\frac{x}{4} : \frac{-2(\beta+\gamma)+\alpha(-3+3k+2n)}{3\alpha+2(\beta+\gamma)}\right.$$

$$L_{21s1} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{2k+n} \frac{-2(\beta+\gamma)+\alpha(-5+i)}{-2\beta+\alpha(i-2)}\right.$$

$$L_{21s2} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{2k+n} \frac{\beta+2\gamma-\alpha(-3+i)}{\beta+\gamma-\alpha(i-2)}\right.$$

$$L_{21s3} = \sinh\left[\frac{x}{4} : \frac{2\beta+\gamma+\alpha(3-2k-n)}{3\alpha+2\beta+\gamma}\right.$$

$$L_{10s1} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{k} \frac{2\gamma-\alpha(i-3)}{-\alpha i}\right.$$

$$L_{10s2} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{k} \frac{\beta+\gamma-\alpha(i-3)}{\beta-\alpha(i-2)}\right.$$

$$L_{10s3} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{k} \frac{-2\beta+\alpha(i-3)}{\gamma-\alpha(i-2)}\right.$$

$$L_{11s1} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{k+n} \frac{2\beta+\gamma-\alpha(i-4)}{\alpha(i+2)}\right.$$

$$L_{11s2} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{k+n} \frac{\beta+\gamma-\alpha(i-2)}{\beta-\alpha(i-1)}\right.$$

$$L_{11s3} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{k+n} \frac{-2\beta+\alpha(i-2)}{\gamma+\alpha(1-i)}\right.$$

$$L_{01} = \sinh\left[\frac{x}{4} : \frac{\alpha(1+n)}{\alpha}\right.$$

$$L_{c2} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{k} \frac{\gamma+2\beta-\alpha(i+k+n-4))}{\alpha(i+k+n-2)-2\gamma}\right.$$

Proposition.
The function $X(x, k, n, \alpha, \beta, \gamma)$ on the points from Vogel's table equals to the quantum dimensions of representations of simple Lie algebras given in the tables below.
((k,0): M.Avetisyan, R.Mkrtchyan, 2018 arXiv1812.07914, (k,n): M.Avetisyan, R.Mkrtchyan, 2019, arXiv:1909.02076)

Table: $X(x, k, n, \alpha, \beta, \gamma)$ for classical algebras

k, n	$0, n$	$1, n$	$k, n \ (k > 1)$
A_1	$n\lambda_{ad}$	0	0
$A_i, i \geq 2$	$n\lambda_{ad}$	$\lambda_{X_2} + n\lambda_{ad}$	$k\lambda_{X_2} + n\lambda_{ad}$
B_2	$n\lambda_{ad}$	$\lambda_{X_2} + n\lambda_{ad}$	0
$B_i, \ i > 2$	$n\lambda_{ad}$	$\lambda_{X_2} + n\lambda_{ad}$	$k\lambda_{X_2} + n\lambda_{ad}$
$C_i, \ i > 2$	$n\lambda_{ad}$	$\lambda_{X_2} + n\lambda_{ad}$	0
$D_i, \ i > 3$	$n\lambda_{ad}$	$\lambda_{X_2} + n\lambda_{ad}$	$k\lambda_{X_2} + n\lambda_{ad}$

Table: $X(x, k, n, \alpha, \beta, \gamma)$ for the exceptional algebras

k, n	k, n
L	$k\lambda_{X_2} + n\lambda_{ad}$

L is any of exceptional simple Lie algebras.

Permutations of the parameters α, β, γ

Table: $X(x, k, n, \beta, \alpha, \gamma)$ for exceptional algebras

k, n	1,0	1,1	1,2	1,3	1,4	1,5	2,0	2,1	2,2
G_2	$3\omega_1$	$\omega_1 + \omega_2$	0	0	0	0	0	0	0
F_4	ω_2	$\omega_3 + \omega_4$	$\omega_1 + \omega_4$	0	0	0	$3\omega_4$	0	0
E_6	ω_3	E:$(\omega_1 + \omega_2)$ $\oplus(\omega_4 + \omega_5)$	ω_3	0	-1	0	E:$3\omega_1 \oplus 3\omega_5$	$-\omega_3$	$-\omega_6$
E_7	ω_2	$\omega_6 + \omega_7$	0	E:$2\omega_6$	0	0	0	E:$-\omega_6 - \omega_7$	$-\omega_5$
E_8	ω_6	ω_8	$-\omega_8$	$-\omega_6$	0	1	0	0	ω_6

Table: $X(x, k, n, \gamma, \alpha, \beta)$ for the exceptional algebras

k, n	1,0	1,1	1,3	2,0	2,1
G_2	$3\omega_1$	$-3\omega_1$	1	$3\omega_1$	ω_2
F_4	ω_2	$-\omega_2$	1	ω_2	ω_1
E_6	ω_3	$-\omega_3$	1	ω_3	ω_6
E_7	ω_2	$-\omega_2$	1	ω_2	ω_1
E_8	ω_6	$-\omega_6$	1	ω_6	ω_7

Notation E: (and SL:, SO:) means restriction on exceptional (respectively special linear and orthosymplectic) line.
That is necessary for the points in Vogel's plane, where the function $X(x, k, n, \alpha, \beta, \gamma)$ is singular.
E.g., for the (k,n)=(1,3) case for E_7 is singular at the point $\alpha = -2, \beta = 8, \gamma = 12$. To study that singularity, consider deviation $\alpha = -2, \beta = 8 + p, \gamma = 12 + q, p, q \to 0$.
Then the singular factor of the function $X(x, k, n, \alpha, \beta, \gamma)$ is

$$\frac{p-q}{3p-2q}$$

Depending on how p, q tend to 0, answer may be arbitrary. However, if we restrict them on the exceptional line $\gamma = 2(\alpha + \beta)$, i.e. $q = 2p$, we obtain a unique answer for E_7.

Table: $X(x, k, n, \beta, \alpha, \gamma)$ for classical algebras. Data for A_i, C_i are valid for sufficiently large ranks i (depending on k, n)

k, n	$1, n$	$k, n\ ,\ k \geq 2$
A_i	$(\omega_1 + \omega_{1+n} + \omega_{i-1-n}) \oplus (\omega_i + \omega_{i-n} + \omega_{n+2})$	$(\omega_k + \omega_{k+n} + \omega_{i+1-2k-n}) \oplus (\omega_{2k+n} + \omega_{i+1-k} + \omega_{i+1}$
B_i	$\omega_1 + \omega_{2n+3}$	0
C_i	$\omega_1 + \omega_{n+1} + \omega_{n+2}$	$\omega_k + \omega_{k+n} + \omega_{2k+n}$
D_i	$\omega_1 + \omega_{2n+3}$	0

For the small (i.e. not "sufficiently large") values of the rank i the function $X(x, k, n, \beta, \alpha, \gamma)$ still gives (quantum) dimensions of irreducible representations of corresponding algebra. However, the picture is chaotic.

Table: $X(x, k, n, \gamma, \alpha, \beta)$ for the classical algebras

$k, n > 0$	$1, 2$
A_i	-1
L	0

L is any classical algebra, except A_i

According to this table, for the classical algebras, $X(x, k, n, \gamma, \alpha, \beta)$ is non-zero (besides previously known case $(k, n) = (k, 0)$) for $(k, n) = (1, 2)$, A_i, only. In that case $X(x, k, n, \gamma, \alpha, \beta) = -1$.

Universal eigenvalues of Casimir operator

Eigenvalues of second Casimir operator on the Cartan product of k X_2 representations and n adjoint ones can be presented in universal form (M.Avetisyan 2019, arXiv:1908.08794)

$$C_{k,n} = \alpha(3k - 3k^2 + n - n^2 - 3kn) + t(4k + 2n) \tag{1}$$

This coincides with eigenvalues, given in (Cohen, deMan 1996) for representations (Cartan products of) X_2, X_2^2, gX_2, g^2X_2 (with Casimir's eigenvalues $C_{1,0}, C_{2,0}, C_{1,1}, C_{1,2}$, respectively). Permutations of parameters give eigenvalues for corresponding representations.

The further aim is to establish which objects in the theory of simple Lie algebras and their applications can be presented in the universal form.
Thanks!