Symmetries and their realization

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Short introduction: Symmetries

Some remarks on symmetries

- Symmetries are encountered in every day life

  $\hookrightarrow$ often related to beauty

- Generally, these are not perfect

  $\hookrightarrow$ breaking of symmetries

  $\hookrightarrow$ hidden symmetries

- Symmetries entail conservation laws

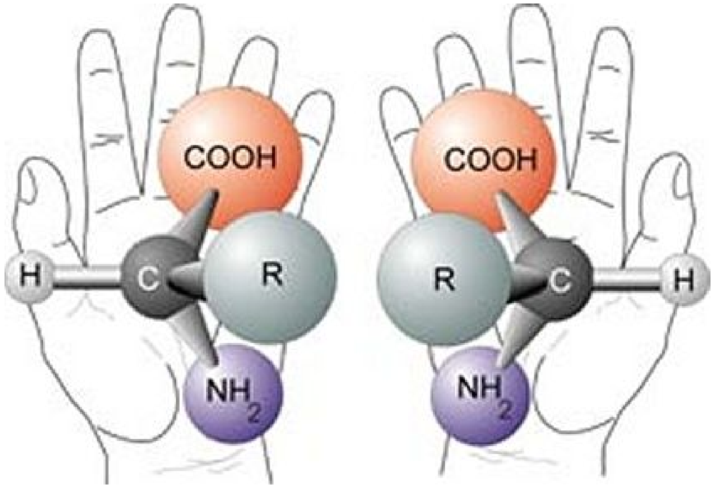
  $\hookrightarrow$ Rotational $O(3)$ symmetry

  $\hookrightarrow$ $[H, J] = 0$

- In the quantum world, more is possible

  $\hookrightarrow$ spontaneous symmetry breaking

  $\hookrightarrow$ anomalous symmetry breaking



First remarks on symmetry breaking

- Explicit symmetry breaking

$$\mathcal{L}_0 = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 \,, \quad \text{invariant under } x \to -x$$

$$\mathcal{L} = \mathcal{L}_0 + \epsilon x \,, \quad |\epsilon| \ll 1$$

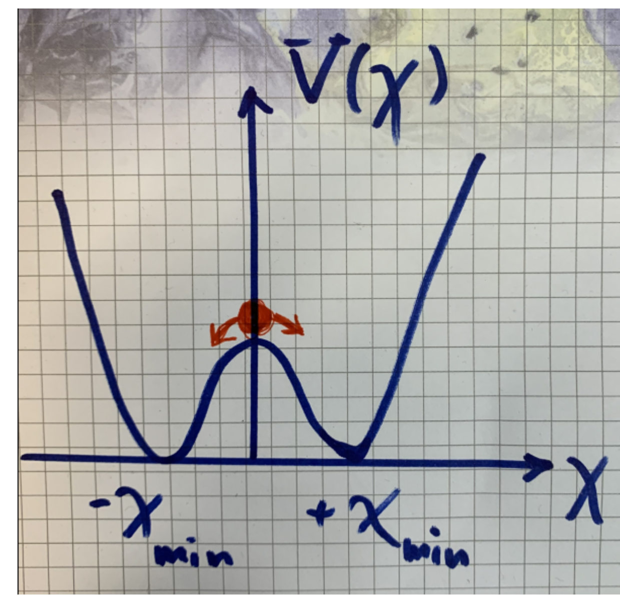
$\to$ approximate symmetry

$\to$ perturbation theory

- Spontaneous symmetry breaking
  (ground state has less symmetry than $\mathcal{L}$)

$$V(\chi) = a\chi^2 + b\chi^4 \,, \text{ with } V(-\chi) = V(\chi)$$

$$\chi_{\rm min} = \pm\sqrt{-a/2b}$$



- Anomalous symmetry breaking

$\hookrightarrow$ upon quantization, no classical analogue

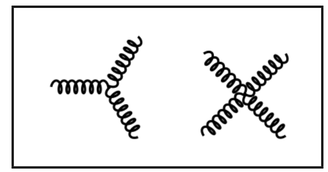
QCD – basic facts

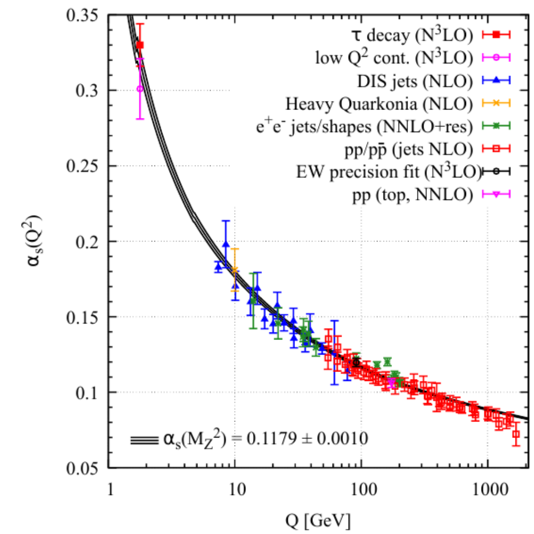
- $$\mathcal{L}_{\rm QCD} = -\frac{1}{4}\, G^a_{\mu\nu} G^{\mu\nu,a} + \sum_f \bar{q}_f (i\not{D} - \mathcal{M}) q_f + \ldots$$

$$D_\mu = \partial_\mu - igA^a_\mu \lambda^a/2$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g[A^b_\mu, A^c_\nu]$$

$$f = (u, d, s, c, b, t)$$





. . . covered by Andreas

- running of $\alpha_s = \dfrac{g^2}{4\pi} \Rightarrow \Lambda_{\rm QCD} = 210 \pm 14\,$MeV $\quad (N_f = 5, \overline{MS}, \mu = 2\,GeV)$

- light (u,d,s) and heavy (c,b,t) quark flavors:

$$m_{\rm light} \ll \Lambda_{\rm QCD}$$

$$m_{\rm heavy} \gg \Lambda_{\rm QCD}$$

$m_u = 2.2^{+0.5}_{-0.4}$ MeV

$m_d = 4.7^{+0.5}_{-0.2}$ MeV

$m_s = 93^{+11}_{-5}$ MeV

$m_c = 1.27 \pm 0.02$ GeV

$m_b = 4.18^{+0.03}_{-0.02}$ GeV

$m_t = 172.8 \pm 0.3$ GeV

Symmetries of QCD

- $SU(3)_c$ gauge symmetry (local)

  – basic construction principle: $q(x) \to q^g(x) = U(x)q(x)\ , \quad U(x) \in SU(3)$

  $$A_\mu(x) \to A_\mu^g(x) = U(x)A_\mu(x)U(x)^\dagger - \frac{i}{g}\partial_\mu U(x)\cdot U^\dagger(x)$$

  $\hookrightarrow$ relates the fermion-gauge field coupling to the 3-gluons and 4-gluon couplings

- $SU(3)_L \times SU(3)_R$ chiral symmetry

  $\to$ spontaneously broken / Goldstone bosons

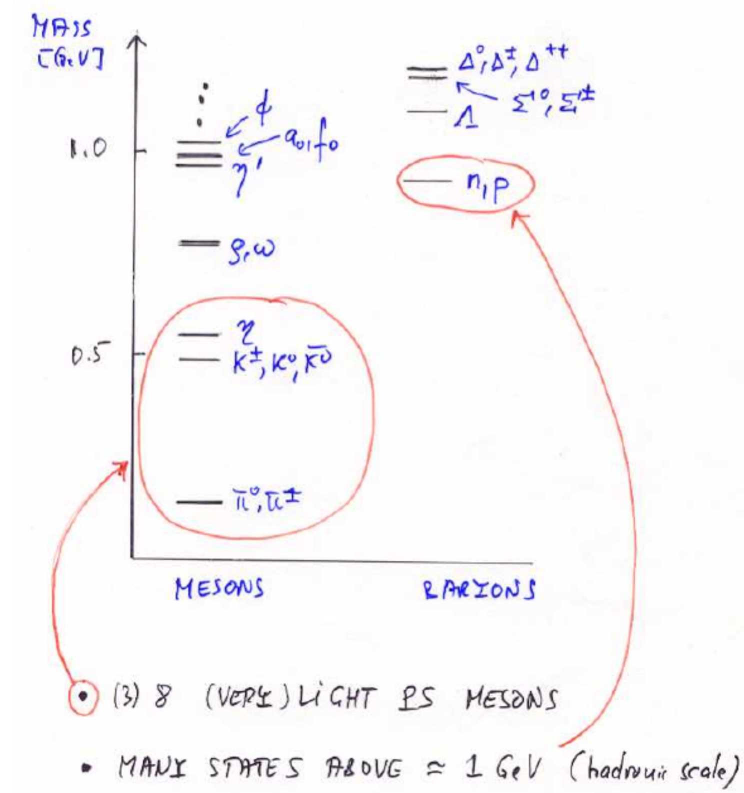
  $\to$ GBs clearly visible in the spectrum

- $U(1)_V \times U(1)_A$ "chiral" symmetry

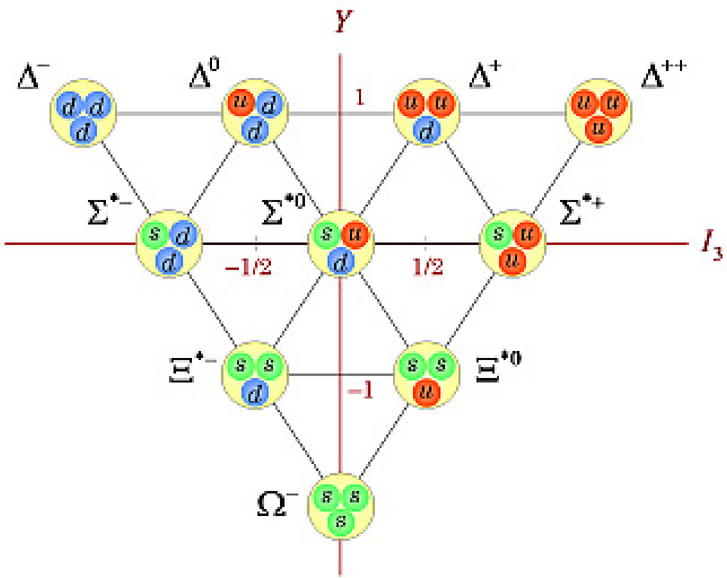
  $\to U(1)_V \sim \#\text{quarks} - \#\text{antiquarks}$

  $\to$ baryon number conservation

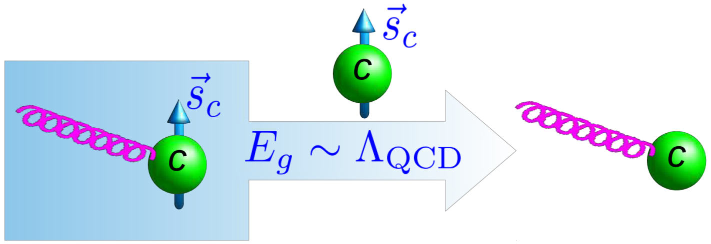
  $\to U(1)_A$: anomalously broken $\to$ mass of the η'



Symmetries of QCD II



- Light quark flavor symmetries
  - SU(2) isospin symmetry (u, d quarks)
  - originally introduced by Heisenberg (NN int.)
  - SU(3) flavor symmetry (u, d, s quarks)
  - Gell-Mann's and Zweig's eightfold way



ⓒFengkun Guo

- Heavy quark spin-flavor symmetries
  - SU(2) flavor symmetry (c, b quarks)
  - SU(2) spin symmetry (c, b quarks)

$\hookrightarrow$ these are all **approximate symmetries**

- combine these (also with chiral symmetry) in heavy-light systems
- light quarks: breakings controlled by $\Delta m_q/\Lambda_{\rm QCD}$
- heavy quarks: breakings controlled by $\Lambda_{\rm QCD}/m_Q$

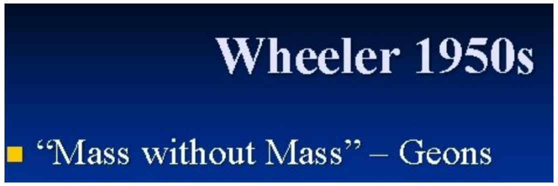
Symmetries of QCD III

- Dilatation symmetry: Classical, massless QCD is invariant under scale trafo's

$$\psi(x) \to \lambda^{3/2}\psi(\lambda x)\ ,\quad A_\mu(x) \to \lambda A_\mu(\lambda x)\ ,\quad \lambda \in \mathbb{R} \setminus \{0\}$$

$\to$ no massive states in QCD in this limit

$\to$ anomalously broken (trace anomaly)



- Discrete symmetries: P, C and T

$\to$ Parity transformation: $(x_0, \vec{x}) \to (x_0, -\vec{x})$
$\to$ Charge conjugation: charge $\to$ $-$charge
$\to$ Time (motion) reversal: $(x_0, \vec{x}) \to (-x_0, \vec{x})$

} fields $\psi(x), A_\mu(x)$ accordingly

$\hookrightarrow$ QCD is a Lorentz-invariant, micro-causal theory $\to$ CPT is conserved

Lüders, Wentzel, Pauli, ...

$\hookrightarrow$ QCD is separately invariant under P, C and T, if $\theta = 0$

$\to$ Andreas' lectures

Symmetries and their realization

Symmetry realization in QFT I

- Consider a Hamiltonian $\mathcal{H}$ invariant under some group $\mathcal{G}$

$$\rightarrow U\mathcal{H}U^\dagger = \mathcal{H} \ , \quad U \in \mathcal{G}$$

- U connects states that form an irrep of the group: $U|A\rangle = |B\rangle$

$\rightarrow$ this implies a multiplet structure: $E_A = \langle A|\mathcal{H}|A\rangle = \langle A|U^\dagger\mathcal{H}U|A\rangle$
$= \langle B|\mathcal{H}|B\rangle = E_B$

- This is the **Wigner-Weyl realization**: $\boxed{Q^a|0\rangle = 0}$ $\quad [U = \exp(i\epsilon_a Q^a)]$

$\rightarrow$ vacuum is annihilated by the symmetry charges

$\rightarrow$ embodied in Coleman's theorem

> Be Q^a a generator of a continous symmetry group $\mathcal{G}$ given as a space-time integral over the current density $J^a_\mu(\vec{x},t)$ and $Q^a|0\rangle = 0$. Then it follows that $\mathcal{H}$ remains invariant under transformatons of the fields according to $\mathcal{G}$ and the current is conserved, $\partial^\mu J^a_\mu = 0$.

Symmetry realization in QFT II

- **Nambu-Goldstone realization:** for some a let $\boxed{Q^a|0\rangle \neq 0}$

- Broken charge, consider volume V: $Q_V^a(t) = \int_V d^3x\, J_0^a(\vec{x}, t)$

  $\to$ since $\partial_\mu J_a^\mu = 0$, for any local operator A we have

  $$\to \int_V [\partial_\mu J_\mu^a, A] d^3x = 0 \quad \to \lim_{V\to\infty} \left[\partial_t Q_V^a(t), A\right] = 0$$

  or $$\lim_{V\to\infty} \left[Q_V^a(t), A\right] \equiv B^a \quad \text{with} \quad \frac{dB^a}{dt} = 0$$

  $\Rightarrow$ $\boxed{\langle 0|B^a|0\rangle \neq 0}$ signals the Nambu-Goldstone symmetry realization

- Fabri-Picasso theorem: If $Q^a|0\rangle \neq 0$ then $\lim_{V\to\infty} Q_V^a(t)$ does not exist

  $\hookrightarrow$ never need the broken charge, only its well-defined commutators $\surd$

Symmetry realization in QFT III

- Goldstone theorem:

In any local translationally invariant field theory with a conserved four-current, $\partial^\mu J_\mu^a = 0$, and a vacumm that is not annihilated by the charge $Q_V^a(t) = \int d^3x J_0^a(\vec{x}, t)$, i.e. $\langle 0| \left[Q_V^a(t), A\right] |0\rangle \neq 0$, there are neccessarily particles with zero mass, the so-called Goldstone bosons (in SUSY also fermions).

- Derivation in a nut-shell:

consider a broken generator $[Q, H] = 0$ but $Q|0\rangle \neq 0$

define $|\psi\rangle \equiv Q|0\rangle$

$$\to H|\psi\rangle = HQ|0\rangle = QH|0\rangle = 0$$

$\to$ not only G.S. $|0\rangle$ has $E = 0$

There exist massless excitations $|n\rangle$, non-interacting as $E_n, p_n \to 0$

- Important property of Goldstone bosons: $\langle 0|J_0^a(0)|n\rangle \neq 0$

Proof of Goldstone's theorem

- Consider the vev of B^a:

$$\lim_{V\to\infty}\sum_n \left[\langle 0|Q_V^a(t)|n\rangle\langle n|A|0\rangle - (Q_V^a \leftrightarrow A)\right] = \langle 0|B^a(\vec{x})|0\rangle$$

$$= \lim_{V\to\infty}\sum_n \left[\langle 0|\int_V d^3x J_0^a(x)|n\rangle\langle n|A|0\rangle - (J_0^a \leftrightarrow A)\right]$$

$$= \lim_{V\to\infty}\sum_n \left[\langle 0|\int_V d^3x J_0^a(0)|n\rangle\langle n|A|0\rangle e^{-ip_n x} - (J_0^a \leftrightarrow A)e^{+ip_n x}\right]$$

$$= \sum_n (2\pi)^3\delta^{(3)}(\vec{p}_n)\left[\langle 0|J_0^a(0)|n\rangle\langle n|A|0\rangle e^{-iE_n t} - \langle 0|A|n\rangle\langle n|J_0^a(0)|0\rangle e^{+iE_n t}\right]$$

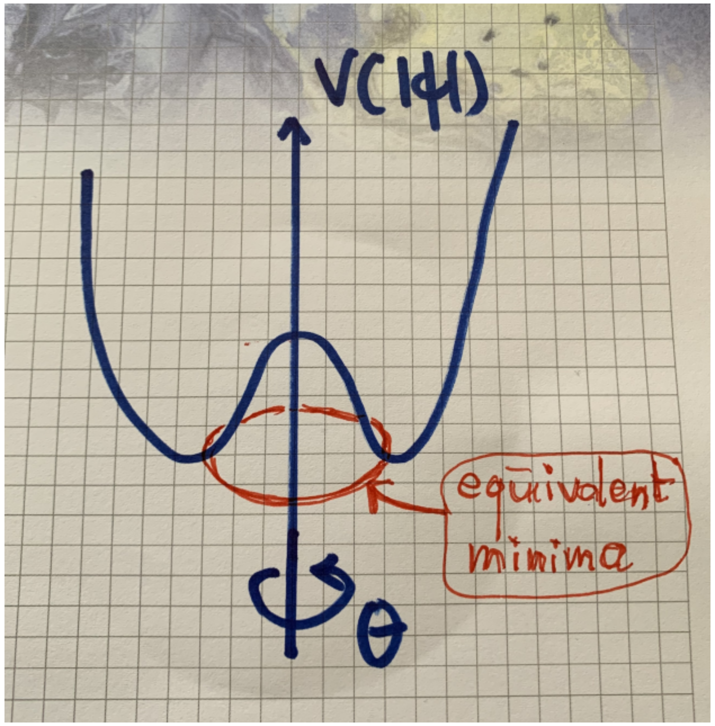
Now $\frac{d}{dt}(\text{l.h.s.}) \sim E_n$ but $\frac{d}{dt}(\text{r.h.s.}) \sim \dot{B}^a = 0$

$\Rightarrow$ $\exists$ states for which $E_n\delta^{(3)}(\vec{p}_n)$ vanishes, i.e. $\boxed{\vec{p}_n = 0\,,\quad E_n = 0\,,\quad M_n^2 = p^2 = 0}$

- These massless states have the same quantum #s as J_0^a (scalar or pseudoscalar)
- Important property: $\langle 0|J_0^a(0)|n\rangle \neq 0$ (neccessary & sufficient condition for SSB)
- Theorem holds independently of perturbation theory
- Requires a minimum # of dimensions: $d > 2$ (continous symmetry), $d > 1$ (discrete symmetry)

Coleman, Mermin, Wagner

Spontaneous symmetry breaking



- Consider a scalar field theory

$$\mathcal{L} = |\partial_\mu \phi|^2 - V(|\phi|) \ , \ \phi \text{ complex}$$

- Set: $\phi = \frac{1}{\sqrt{2}} \rho e^{i\theta}$

- Rotational symmetry: $\theta \to \theta + \alpha$

- Select one value of $\theta \Rightarrow$ SSB: $\rho(x) = \rho_0 \ , \ \ \theta = 0$

- Expand around this minimum: $\rho = \rho_0 + \chi$

$$\hookrightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu \chi)^2 + \frac{1}{2}\rho_0^2(\partial_\mu \theta)^2 - V(\rho_0/2) - \frac{1}{2}\chi^2 \, V''(\rho_0/2) + \cdots$$

$$\hookrightarrow M_\chi^2 = V''(\rho_0/2) \neq 0 \ , \quad M_\theta^2 = 0$$

$\hookrightarrow$ massless excitations in the θ-direction = Goldstone boson mode

Anomalous symmetry breaking

- What is an anomaly?

Anomaly = classical symmetry broken upon quantization

- Consider $\mathcal{L}(\psi, \bar{\psi}, \ldots)$ with a symmetry

$$\psi \mapsto \psi' = e^{iS}\psi \;\; \Rightarrow \;\; \mathcal{L}(\psi', \bar{\psi}', \ldots) = \mathcal{L}(\psi, \bar{\psi}, \ldots)$$

- Quantum effects via the path integral:

$$\mathcal{Z} = \int [d\psi][d\bar{\psi}] \exp\left\{ i \int d^4x \mathcal{L}(\psi, \bar{\psi}, \ldots) \right\}$$

e^{iS} : $\quad [d\psi][d\bar{\psi}] \mapsto [d\psi'][d\bar{\psi}']\mathcal{J}$ $\quad \leftarrow \mathcal{J}$ is the Jacobian

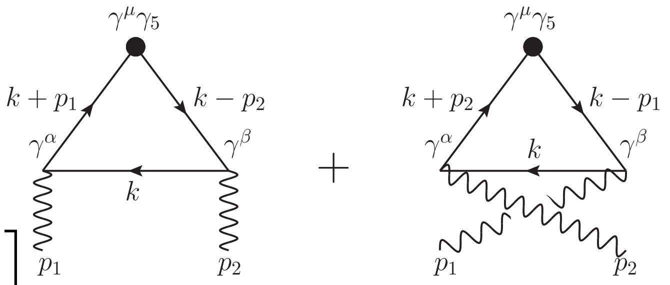
$\mathcal{J} \neq 1 \Leftrightarrow$ **anomaly**

Triangle anomaly

- First observed in $\pi^0 \to 2\gamma$

Sutherland, Adler, Bell, Jackiw

- QED three-point (VAA) function (perturbative calc.)



$$T^{\mu\alpha\beta}(p_1,p_2) = e^2\int\frac{d^4k}{(2\pi)^4 i}\,\mathrm{tr}\left[\gamma^\alpha\frac{1}{m_e-\not{k}}\gamma^\beta\frac{1}{m_e-\not{k}+\not{p}_2}\gamma^\mu\gamma_5\frac{1}{m_e-\not{k}-\not{p}_1}\right] + [\alpha\leftrightarrow\beta, p_1\leftrightarrow p_2]$$

- If we shift $k \to k - p_1$ or $k \to k + p_2$ in the integrand parts, then:

$$(p_1+p_2)_\mu T^{\mu\alpha\beta}(p_1,p_2) = -e^2\int\frac{d^4k}{(2\pi)^4 i}\,\mathrm{tr}\left[\gamma^\alpha\frac{1}{m_e-\not{k}}\gamma^\beta\frac{1}{m_e-\not{k}+\not{p}_2}2m_e\gamma_5\frac{1}{m_e-\not{k}-\not{p}_1}\right] + [\alpha\leftrightarrow\beta, p_1\leftrightarrow p_2]$$

- This corresponds to the naive identity:

$$\partial_\mu A^\mu(x) = \partial_\mu\left[\bar\psi(x)\gamma^\mu\gamma_5\psi(x)\right] = 2im_e\bar\psi(x)\gamma_5\psi(x)$$

$\hookrightarrow$ as on the classical level (no π^0 decay)

Triangle anomaly - a second look

- The integrals are divergent, use a Pauli-Villars type regularization:

$$T^{\mu\alpha\beta}_{\text{reg}}(p_1,p_2) = T^{\mu\alpha\beta}(p_1,p_2) - T^{\mu\alpha\beta}_{\Lambda}(p_1,p_2)$$

$$\Rightarrow (p_1+p_2)_\mu T^{\mu\alpha\beta}_{\text{reg}}(p_1,p_2)$$

$$= \Bigg(-e^2 \int \frac{d^4k}{(2\pi)^4 i}\,\text{tr}\left[\gamma^\alpha \frac{1}{m_e - \not{k}}\,\gamma^\beta \frac{1}{m_e - \not{k} + \not{p}_2}\,2m_e\gamma_5 \frac{1}{m_e - \not{k} - \not{p}_1}\right]$$

$$+e^2 \int \frac{d^4k}{(2\pi)^4 i}\,\text{tr}\left[\gamma^\alpha \frac{1}{\Lambda - \not{k}}\,\gamma^\beta \frac{1}{\Lambda - \not{k} + \not{p}_2}\,2\Lambda\gamma_5 \frac{1}{\Lambda - \not{k} - \not{p}_1}\right]\Bigg)$$

$$+[\alpha \leftrightarrow \beta, p_1 \leftrightarrow p_2]$$

- Using Feynman parametrization and some algebra leads to:

$$\lim_{\Lambda\to\infty}(p_1+p_2)_\mu T^{\mu\alpha\beta}_{\Lambda}(p_1,p_2) = -\frac{ie^2}{2\pi^2}\,\varepsilon^{\alpha\beta\lambda\rho}p_{1\lambda}p_{2\rho}$$

$$\hookrightarrow \quad \boxed{\partial_\mu A^\mu(x) = 2im_e\bar\psi(x)\gamma_5\psi(x) - \underbrace{\frac{e^2}{16\pi^2}\,\varepsilon_{\mu\nu\alpha\beta}\mathcal{F}^{\mu\nu}(x)\mathcal{F}^{\alpha\beta}(x)}_{\text{anomaly}}}$$

Triangle anomaly - more remarks

- Requires regularization, same results with dim. reg. or point-split techniques

- In more detaill: Add a polynomial in p_1, p_2 to $T^{\mu\alpha\beta}(p_1, p_2)$, use Lorentz-invariance, invariance under parity and Bose symmetry:

$$T^{\mu\alpha\beta}(p_1,p_2) \to T^{\mu\alpha\beta}(p_1,p_2) + a\varepsilon^{\mu\alpha\beta\nu}(p_1 - p_2)_\nu$$

$$\hookrightarrow (p_1+p_2)_\mu T^{\mu\alpha\beta}(p_1,p_2) \to (p_1+p_2)_\mu T^{\mu\alpha\beta}(p_1,p_2) - 2a\varepsilon^{\mu\nu\alpha\beta} p_{1\mu} p_{2\nu}$$

- Choose $a = -\dfrac{ie^2}{4\pi^2}$ $\to$ anomalous term disappears

- But what happens to the Ward identities for the vector current?

$$p_{1\alpha} T^{\mu\alpha\beta}(p_1,p_2) = p_{2\beta} T^{\mu\alpha\beta}(p_1,p_2) = 0$$

since $p_{1\alpha}\varepsilon^{\mu\alpha\beta\nu}(p_1 - p_2)_\nu = -\varepsilon^{\mu\alpha\beta\nu} p_{1\alpha} p_{2\nu} \neq 0 \quad \Rightarrow \quad a = 0$

- **No choice of** a that allows for both naive Ward identities to hold

$\hookrightarrow$ Choose a so that the conserved vector current is anomaly free (QED, QCD)

$\hookrightarrow$ Requires anomaly cancellation in the Standard model

Fujikawa determinant

Fujikawa, PRD **21** (1980) 2848, PRD **22** (1980) 1499 [E]

- Path integral formulation: Non-invariance of the fermionic measure = anomaly

$$Z = \int d\psi d\bar\psi \,\exp\left\{-\int d^4x \bar\psi(x)\,\gamma_\mu D_\mu \psi(x)\right\}, \quad \underbrace{D_\mu = \partial_\mu + G_\mu}_{\text{Dirac operator}}$$

- Expand the massless Dirac operator in eigenfunctions

$$\not{D}\psi_\lambda = i\lambda\psi_\lambda \;,\quad \bar\psi_\lambda\,(\overleftarrow{\not{D}}) = i\lambda\bar\psi_\lambda \;,\quad \int d^4x\,\bar\psi_\lambda(x)\psi_{\lambda'}(x) = \delta_{\lambda\lambda'}$$

$$\psi(x) = \sum_\lambda a_\lambda\psi_\lambda(x) \;,\quad \bar\psi(x) = \sum_\lambda \bar\psi_\lambda(x)\bar a_\lambda \;\to\; \boxed{d\psi d\bar\psi = \prod_\lambda da_\lambda d\bar a_\lambda}$$

- Local singlet axial transformations:

$$a_\lambda \mapsto \int d^4x\,\bar\psi_\lambda(x)(1 + i\beta^0(x)\gamma^5)\psi(x) = \sum_{\lambda'}(\delta_{\lambda\lambda'} + C_{\lambda\lambda'})a_{\lambda'}$$

$$C_{\lambda\lambda'} = i\int d^4x\,\beta^0(x)\,\bar\psi_\lambda(x)\gamma^5\psi_{\lambda'}(x)$$

- Jacobian: $d\psi d\bar\psi \mapsto J^{-2} d\psi d\bar\psi$

$$\hookrightarrow J = \det(\mathbb{1} + C) = \exp(\mathrm{Tr}(\ln(\mathbb{1} + C))) = \exp\left\{\sum_\lambda C_{\lambda\lambda} + O(\beta^2)\right\}$$

Fujikawa determinant continued

- Evaluation of the Jacobian: $\ln J = i \int d^4x \beta^0(x) \underbrace{\sum_\lambda \bar\psi_\lambda(x)\gamma^5\psi_\lambda(x)}_{\doteq S(x)}$

- Regulate the divergent sum over EV:

$$\sum_\lambda \bar\psi_\lambda(x)\gamma^5\psi_\lambda(x) \to \lim_{M\to\infty} \sum_\lambda e^{-\lambda^2/M^2} \bar\psi_\lambda(x)\gamma^5\psi_\lambda(x) = \lim_{M\to\infty} S_M(x)$$

$$S_M(x) = \sum_\lambda \bar\psi_\lambda(x)\gamma^5 e^{\not{D}^2/M^2}\psi_\lambda(x) = \left\langle x \middle| \mathrm{tr}\left(\gamma^5 e^{\not{D}^2/M^2}\right) \middle| x \right\rangle, \quad \not{D}^2 = D^2 - \tfrac{i}{2}\,\sigma_{\mu\nu}F_{\mu\nu}$$

tr = trace over Dirac, color and flavor indices

- After some algebra: $\lim_{M\to\infty} S_M(x) = \frac{N_f}{32\pi^2}\,\varepsilon_{\mu\nu\alpha\beta}\mathrm{tr}_c(F_{\mu\nu}F_{\alpha\beta})$

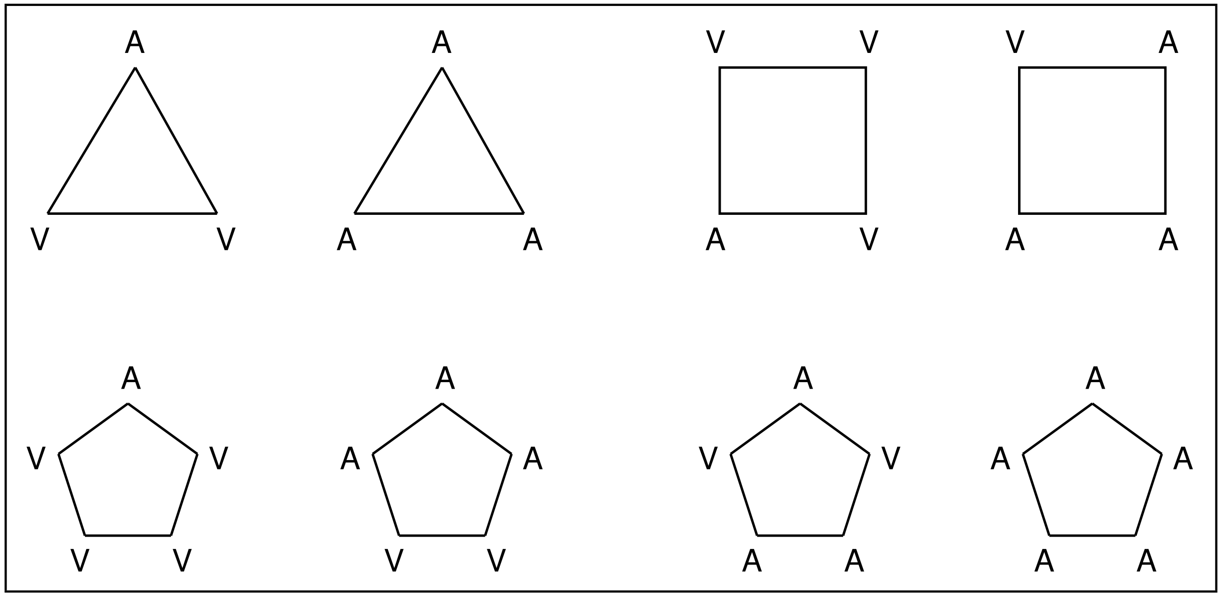
$$\hookrightarrow J = \exp\left\{ i\int d^4x \beta^0(x)\underbrace{\frac{N_f}{32\pi^2}\,\varepsilon_{\mu\nu\alpha\beta}\mathrm{tr}_c(F_{\mu\nu}F_{\alpha\beta})}_{\text{anomaly}} + O(\beta^2)\right\}$$

$\hookrightarrow \partial_\mu A^0_\mu(x) = \frac{N_f}{16\pi^2}\,\varepsilon_{\mu\nu\alpha\beta}\mathrm{tr}_c(F_{\mu\nu}F_{\alpha\beta}) \to$ known result w/o perturbation th'y!

More on anomalies

- Anomalies appear rather often, a generic feature of most QFTs

- In the general non-abelian case, the following diagrams can lead to anomalies



- Anomaly matching relates anomalous W.I. between fundamental & effective theories

't Hooft (1980)

- In the language of the EFT of QCD, anomalies are represented in terms of the Wess-Zumino–Witten effective action + Witten-Veneziano formula + ...

Wess, Zumino (1971), Witten (1979, 1983), Veneziano (1979

Chiral symmetry in QCD

Introduction to chiral symmetry

- Massless fermions exhibit **chiral symmetry**:

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi$$

- left/right-decomposition:

$$\psi = \frac{1}{2}(1-\gamma_5)\psi + \frac{1}{2}(1+\gamma_5)\psi = P_L\psi + P_R\psi = \psi_L + \psi_R$$

- projectors:

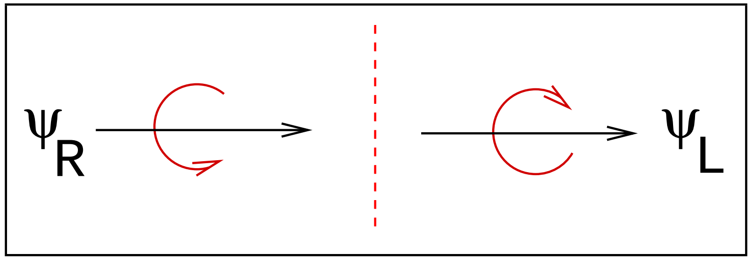
$$P_L^2 = P_L,\ P_R^2 = P_R,\ P_L \cdot P_R = 0,\ P_L + P_R = \mathbb{1}$$

- helicity eigenstates:

$$\frac{1}{2}\hat{h}\psi_{L,R} = \pm\frac{1}{2}\psi_{L,R} \quad \hat{h} = \vec{\sigma}\cdot\vec{p}/|\vec{p}\,|$$

- L/R fields do **not** interact $\to$ conserved L/R currents

$$\mathcal{L} = i\bar{\psi}_L\gamma_\mu\partial^\mu\psi_L + i\bar{\psi}_R\gamma_\mu\partial^\mu\psi_R$$



- mass terms break chiral symmetry: $\bar{\psi}\mathcal{M}\psi = \bar{\psi}_R\mathcal{M}\psi_L + \bar{\psi}_L\mathcal{M}\psi_R$

Chiral symmetry of QCD

- Three flavor QCD:

$$\boxed{\mathcal{L}_{\rm QCD} = \mathcal{L}^0_{\rm QCD} - \bar{q}\mathcal{M}q}\,,\ q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}\,,\quad \mathcal{M} = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

- $\mathcal{L}^0_{\rm QCD}$ is invariant under **chiral** $SU(3)_L \times SU(3)_R$ (split off U(1)'s)

$$\mathcal{L}^0_{\rm QCD}(G_{\mu\nu}, q', D_\mu q') = \mathcal{L}^0_{\rm QCD}(G_{\mu\nu}, q, D_\mu q)$$

$$q' = RP_R q + LP_L q = Rq_R + Lq_L \quad R, L \in SU(3)_{R,L}$$

- conserved L/R-handed [vector/axial-vector] Noether currents:

$$J^{\mu,a}_{L,R} = \bar{q}_{L,R}\gamma^\mu \frac{\lambda^a}{2} q_{L,R}\,, \quad a = 1, \ldots, 8$$

$$\partial_\mu J^{\mu,a}_{L,R} = 0 \quad [\text{or}\ \ V^\mu = J^\mu_L + J^\mu_R\,,\ \ A^\mu = J^\mu_L - J^\mu_R]$$

- Is this symmety reflected in the vacuum structure/hadron spectrum?

The fate of QCD's chiral symmetry

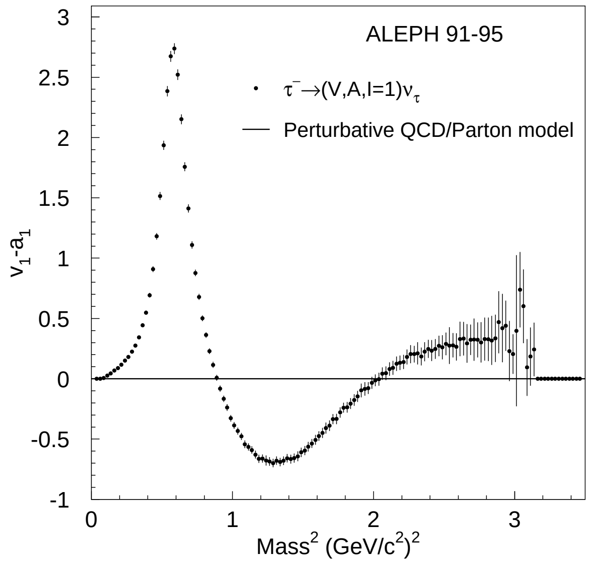
- The chiral symmetry is not "visible", it is "hidden" (spontaneously broken)

  - no parity doublets

  $\hookrightarrow M_\rho \neq M_{a_1}, M_N \neq M_{S_{11}}, \ldots$

  - $\langle 0|AA|0\rangle \neq \langle 0|VV|0\rangle$

  $\hookrightarrow \int \frac{ds}{s}[\rho_V(s) - \rho_A(s)] = F_\pi^2$

  - scalar condensate $\bar{q}q = \bar{q}_L q_R + \bar{q}_R q_L$ acquires v.e.v.

  $\hookrightarrow$ another order parameter of SSB

  - Vafa-Witten theorem [NPB 234 (1984) 173]

  $\hookrightarrow$ If $\theta = 0$, vector symmetries can not be spontaneously broken

  - (almost) massless pseudoscalar bosons

The fate of QCD's chiral symmetry II

- Wigner mode $Q_5^a|0\rangle = Q^a|0\rangle = 0 \; (a = 1,\ldots,8)$?

- parity doublets: $dQ_5^a/dt = 0 \to [H, Q_5^a] = 0$

  single particle state: $H|\psi_p\rangle = E_p|\psi_p\rangle$

  axial rotation: $H(\mathrm{e}^{iQ_5^a}|\psi_p\rangle) = \mathrm{e}^{iQ_5^a} H|\psi_p\rangle = \underbrace{E_p(\mathrm{e}^{iQ_5^a}|\psi_p\rangle)}$

  same mass but opposite parity

- VV and AA spectral functions (without pion pole):

$$\begin{array}{c}
\langle 0|VV|0\rangle = \langle 0|(L+R)(L+R)|0\rangle = \langle 0|L^2+R^2+2LR|0\rangle = \langle 0|L^2+R^2|0\rangle \\
|| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad || \\
\langle 0|AA|0\rangle = \langle 0|(L-R)(L-R)|0\rangle = \langle 0|L^2+R^2-2LR|0\rangle = \langle 0|L^2+R^2|0\rangle
\end{array}$$

since L and R are orthogonal

The fate of QCD's chiral symmetry III

- Chiral symmetry is realized in the Nambu-Goldstone mode

  - pions indeed couple to the vacuum

  $\hookrightarrow \langle 0|A^j_\mu(0)|\pi^k(p)\rangle = iF_\pi p_\mu \delta^{jk}\,, \quad F_\pi \simeq 92\,$MeV

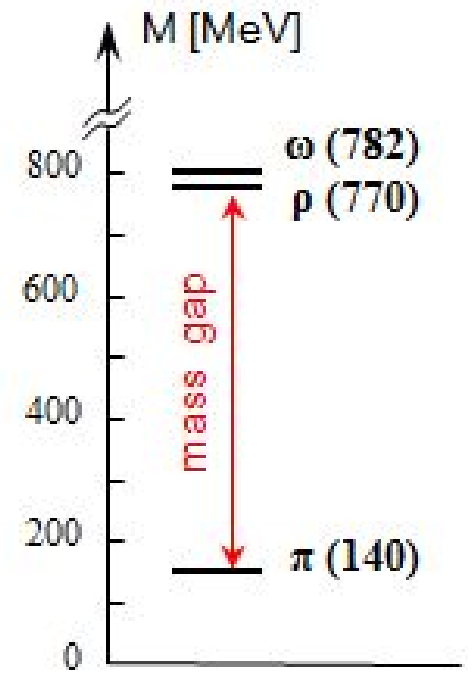
  - weakly interacting massless pseudoscalar excitations

  $\hookrightarrow$ this can be tested experimentally

  - approximate symmetry (small quark masses)

  $\hookrightarrow \; M^2_{\pi^\pm} \sim (m_u + m_d)$

  $\hookrightarrow \pi, K, \eta$ as Pseudo-Goldstone Bosons



  - An appropriately tailored effective field theory can be set up

  $\hookrightarrow$ Chiral Perturbation Theory

  $\hookrightarrow$ perturbative expansion in p/Λ and M_π/Λ including loops

Weinberg, Gasser, Leutwyler

Chiral perturbation theory in a nutshell

- Low-energy EFT of QCD: $\mathcal{L}_{\rm QCD}(q,\bar q,G) \to \mathcal{L}_{\rm eff}(U,s,p,v,a)$

  $\hookrightarrow$ same Ward identities = exact mapping

Leutwyler, Weinberg

- Chiral effective Lagrangian:

  $$\mathcal{L}_{\rm eff} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots$$

  $\hookrightarrow$ systematic expansion in small momenta and quark (pion) masses

  $$D = 2 + \sum_d{}' N_d\,(d-2) + 2N_L$$

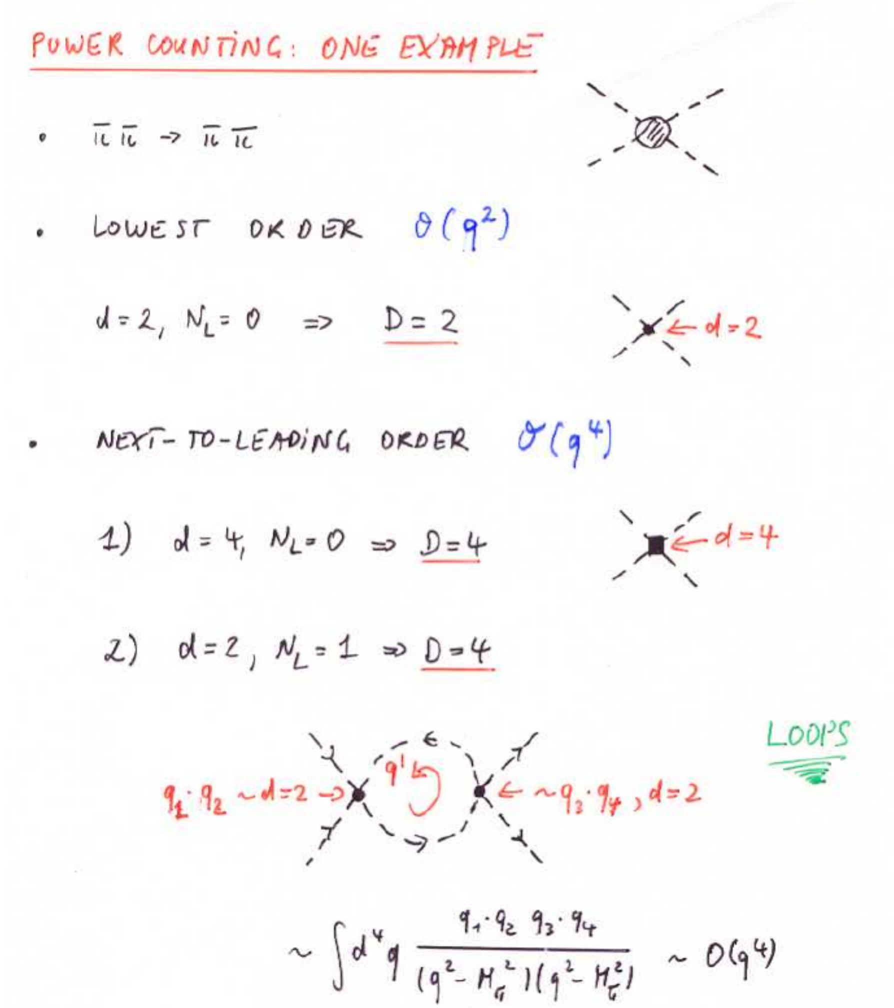
  $\hookrightarrow$ leading order: trees

  next-to-leading order:
  trees and one-loop graphs , ...

  $\hookrightarrow$ order-by-order renormalization

  $\hookrightarrow$ some processes to two loops

  $\hookrightarrow$ very successful framework



Elastic pion-pion scattering

- Purest process in two-flavor chiral dynamics (really light quarks)

- Scattering amplitude at threshold: two numbers (a_0, a_2)

- Very precise prediction: match 2-loop representation to Roy equation solution

Roy + 2-loop: $a_0 = 0.220 \pm 0.005$

Colangelo, Gasser, Leutwyler, Phys. Lett . B488 (2000) 261

- Same precision for a_2, but corrections very small ...

- Experiment: Kaon decays ($K_{\ell 4}, K \to 3\pi$) and the lifetime of pionium

$$a_0 = 0.2210 \pm 0.0047_{\rm stat} \pm 0.0040_{\rm sys}$$

$$a_2 = -0.0429 \pm 0.0044_{\rm stat} \pm 0.0028_{\rm sys}$$

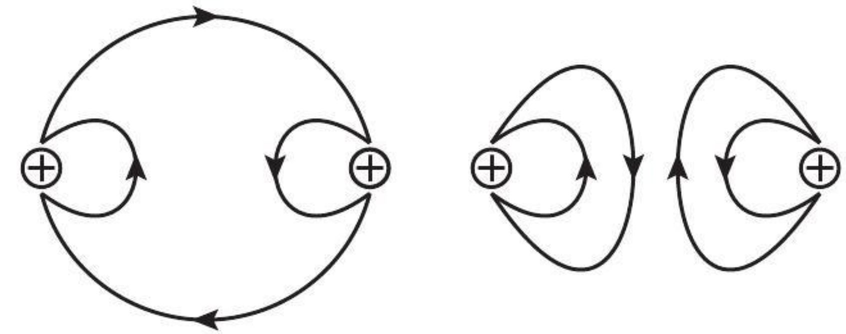
Batley et al. [NA48/2 Coll.] EPJ C70 (2010) 635

$$|a_0 - a_2| = 0.2533^{+0.0107}_{-0.0137}$$

Adeva et al. [DIRAC Coll.] Phys. Lett. B704 (2011) 24

Elastic pion-pion scattering – lattice a_0

- Only a few lattice determinations of a_0
  - $\hookrightarrow$ disconnected diagrams difficult
  - $\hookrightarrow$ quantum numbers of the vacuum
  - $\hookrightarrow$ qualitative insight from large-N_C



Guo, Liu, UGM, Wang, Phys. Rev. D88 (2013) 074506

- Available unquenched lattice results:

Author(s)	a_0	Fermion action	Pion mass range
Fu	0.214(4)(7)	asqtad staggered	240 - 430 MeV
Liu et al.	0.198(9)(6)	twisted mass	250 - 320 MeV

Fu, PRD87 (2013) 074501; Liu et al., PRD96 (2017) 054516

$\rightarrow$ use EFT of PQQCD to investigate these contributions

Acharya, Guo, UGM, Seng, Nucl. Phys. B922 (2017) 480

$\rightarrow$ more work needed!

Chiral anomaly and the Wess-Zumino-Witten term

- Redundant symmetries of the chiral effective Lagrangian, consider (massless) $\mathcal{L}^{(2)}$

$$\mathcal{L}^{(2)} = \frac{F_\pi^2}{4}\langle \partial_\mu U \partial^\mu U^\dagger \rangle \ , \quad U(x) = \exp\left(i\lambda^a \phi^a(x)/F_\pi\right) \ , \ U \in SU(3)$$

- Parity: $PU(\vec{x},t)P^{-1} = U^\dagger(-\vec{x},t)$ like QCD

- $\mathcal{L}^{(2)}$ has two extra symmetries, unlike QCD

  $U(\vec{x},t) \mapsto U(-\vec{x},t)\ ,\ U(\vec{x},t) \mapsto U^\dagger(\vec{x},t)$ conserves intrinsic parity (P_I)

- Intrinsic parity: $P_I = +1/-1$ for a true/pseudo-tensor of rank k

- Examples

$$\pi\pi \to \pi\pi \quad (-1)\cdot(-1) = (-1)\cdot(-1)\ \surd \qquad \pi^0 \to 2\gamma \quad (-1) = (+1)\cdot(+1)\ ?$$

$$\gamma\pi^+ \to \pi^+ \quad (+1)\cdot(-1) = (-1)\ \surd \qquad \gamma \to \pi^+\pi^-\pi^0 \quad (+1) = (-1)\cdot(-1)\cdot(-1)\ ?$$

$$\eta \to \pi^+\pi^-\pi^0 \quad (-1) = (-1)\cdot(-1)\cdot(-1)\ \surd \qquad K\bar{K} \to \pi^+\pi^-\pi^0 \quad (-1)^2 = (-1)^3\ ?$$

$\hookrightarrow$ $\mathcal{L}^{(2)}$ conserves P_I, i.e. the number of Goldstone bosons mod 2 [holds for all $\mathcal{L}^{(2n)}$]

but QCD does **not**!

Chiral anomaly and the Wess-Zumino-Witten term II

- Break the redundenat symmetries (EoM) Witten (1983)

$$\frac{i}{2}F_\pi^2\partial^\mu L_\mu + \lambda\epsilon^{\mu\nu\alpha\beta}L_\mu L_\nu L_\alpha L_\beta = 0 \;,\; L_\mu = U^\dagger\partial_\mu U$$

- This can not be written as a 4-dimensional Lagrangian, so [some algebra]

$$\boxed{S_{WZW} = -\frac{in}{240\pi^2}\int_{S^5} d^5x\,\epsilon^{\mu\nu\alpha\beta\gamma}\langle L_\mu L_\nu L_\alpha L_\beta L_\gamma\rangle} \;,\quad S^5 = \partial M^5$$

- topological quantization to resolve path ambiguity in S^5
- electromagnetic gauging $\to n = N_C$ (c.f. $T(\pi^0 \to 2\gamma)$)
- many testable predictions, e.g.

$$T(\gamma \to \pi^+\pi^-\pi^0) = -\epsilon_{\mu\nu\alpha\beta}\epsilon^\mu k^\nu p_-^\alpha p_+^\beta\,F(s,t,u)$$

$$\hookrightarrow F(0,0,0) = \frac{eN_c}{12\pi^2 F_\pi^3} = 9.7\ \text{GeV}^{-3}$$

Light quark flavor symmetries

Isospin symmetry

- Nucleon-nucleon interactions are approximately invariant under

$$N \mapsto N' = U\,N\,, \quad U \in SU(2)\,, \quad N = \begin{pmatrix} p \\ n \end{pmatrix}$$

Heisenberg (1932)

- For $m_u = m_d$, QCD is invariant under SU(2) isospin transformations

$$q \mapsto q' = U\,q\,, \quad q = \begin{pmatrix} u \\ d \end{pmatrix} \quad U = \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix} \quad |a|^2 + |b|^2 = 1$$

- Rewrite the QCD quark mass term

$$\mathcal{H}_{\rm QCD}^{\rm mass} = m_u\,\bar{u}u + m_d\,\bar{d}d = \underbrace{\frac{1}{2}(m_u + m_d)(\bar{u}u + \bar{d}d)}_{I=0} + \underbrace{\frac{1}{2}(m_u - m_d)(\bar{u}u - \bar{d}d)}_{I=1}$$

- Sources of isospin breaking: $m_u \neq m_d$ and electromagnetism

$\hookrightarrow$ strong breaking expected to be small: $\boxed{\dfrac{m_d}{m_u} \simeq 2 \;\; \text{but} \;\; \dfrac{m_d - m_u}{\Lambda_{\rm QCD}} \ll 1}$

$\hookrightarrow$ em breaking small $\sim \alpha \sim 1/137$

The proton-neutron mass difference

- Two contributions of similar size to combine to $m_n - m_p = 1.3$ MeV

- Mass difference given by: $m_p - m_n = \underbrace{4c_5 B(m_d - m_u)}_{\text{strong}} - \underbrace{F_\pi^2 e^2 f_2}_{\text{em}}$

- EM contribution form the Cottingham sum rule

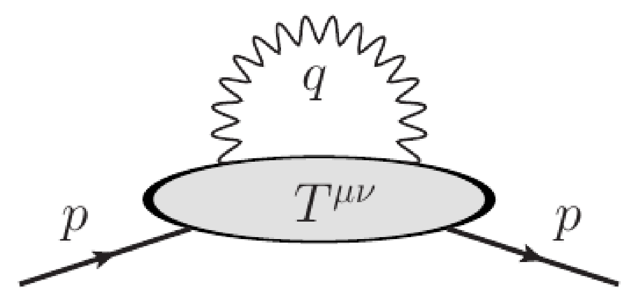
Cottingham (1968)

$$\delta m_N^{\text{em}} \sim e^2 \int d^4q D(q^2) g_{\mu\nu} \left(T_p^{\mu\nu}(p,q) - T_n^{\mu\nu}(p,q)\right)$$

$$= \quad 0.58 \pm 0.16 \text{ MeV}$$

Gasser, Leutwyler, Rusetsky (2021)

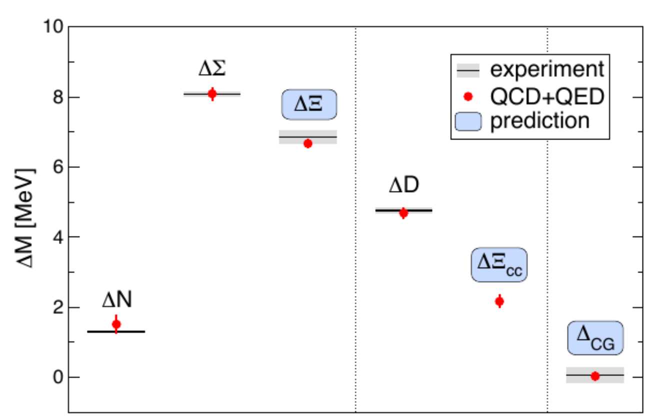
$$\hookrightarrow \delta m_N^{\text{strong}} = -1.87 \mp 0.15 \text{ MeV}$$

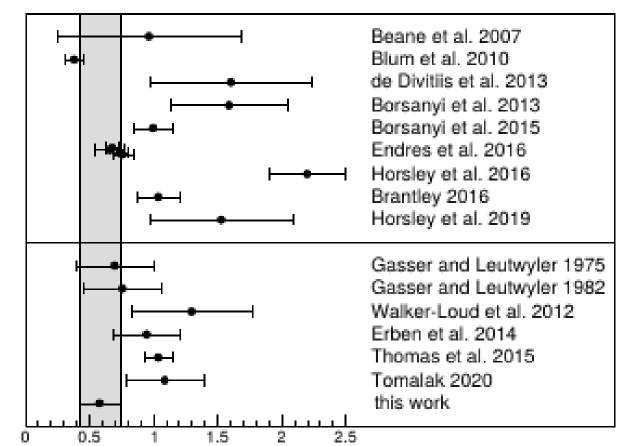


- Other determinations of δm_N^{em} using the Cottingham sum rule or lattice QCD (also $\delta m_N^{\text{strong}}$)

Gasser, Leutwyler, Rusetsky (2020)

Borsanyi et al. (2015)





SU(3) flavor symmetry

- Introduced to bring order into the hadron zoo, the (constituent) quark model

  Gell-Mann, Zweig (1964)

- Flavor SU(3) in QCD refers to the light up, down and strange (current) quarks
- For $m_u = m_d = m_s$, QCD is invariant under SU(3) flavor transformations

  [the unbroken SU(3)]

$$q \mapsto q' = U\, q\,, \quad q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad U \in SU(3) \quad \text{[Gell-Mann matrices]}$$

- Sources of SU(3) flavor breaking: $m_s \neq m_d, m_u$ and electromagnetism

  $\hookrightarrow$ strong breaking expected to be sizeable: $\boxed{m_s \gg m_d, m_u \ \text{ and } \ \frac{m_s}{\Lambda_{\rm QCD}} \simeq \frac{1}{2}}$

  $\hookrightarrow$ em breaking small $\sim \alpha \sim 1/137$

- Still, some rather precise predictions (Gell-Mann–Okubo mass formula):

$$\frac{1}{4}\,(m_N + m_\Xi) = \frac{3}{4}\,(m_\Lambda + m_\Sigma) \quad \text{i.e.} \quad 1128.5 \text{ MeV} \simeq 1135.3 \text{ MeV}$$

Calculation of hadron masses

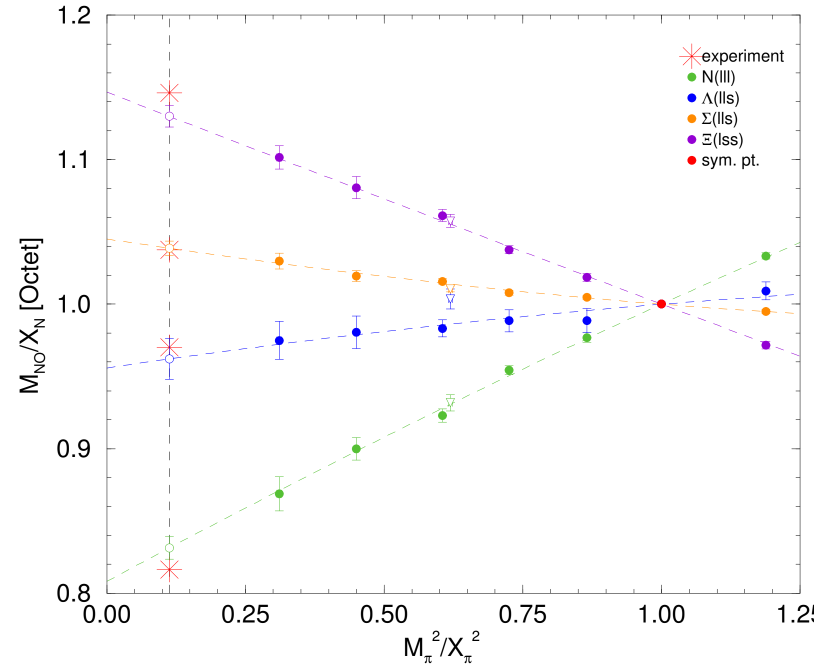
- SU(3) limit $m_s = m_d = m_u$ is

  $\hookrightarrow$ starting point in lattice QCD calculations

  $\hookrightarrow$ increase m_s & decrease m_d, m_u

  $\hookrightarrow$ fan-plots & rather accurate calculations

  Bietenholz et al., Phys. Rev. D **84** (2011) 054509



- SU(3) limit $m_s = m_d = m_u$ is

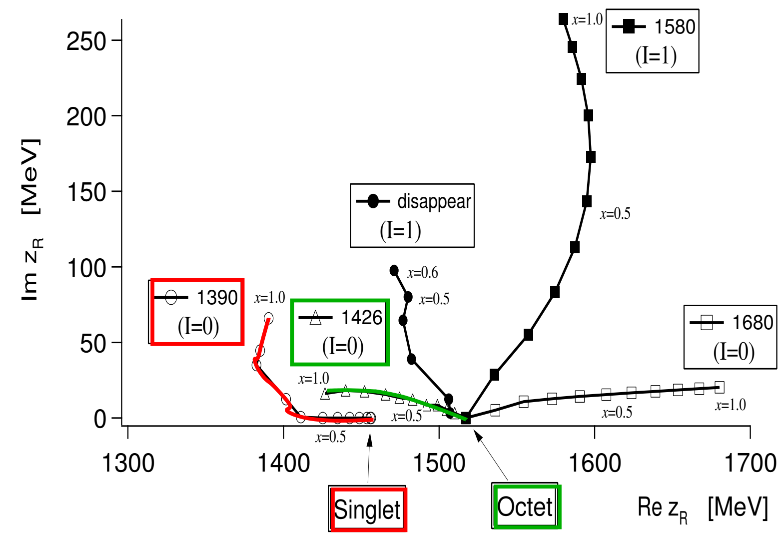
  $\hookrightarrow$ starting point in unitarized CHPT calculations

  $\hookrightarrow$ $x = 0 \to 1$ from the sym. pt. to the phys. world

  $\hookrightarrow$ two-pole structure of the $\Lambda(1405)$ emerges

  Oller, UGM, Phys. Lett. **500** (2001) 263

  Jido, Oller, Oset, Ramos, UGM, Nucl. Phys. **725** (2003) 181



Dimensional transmutation, the trace anomaly & all that

Scale invariance and its breaking

- Massless classical QCD is scale-invariant, no massive particles emerge

  $\hookrightarrow$ this scale invariance is obviously broken (anomaly), e.g. $m_p \neq 0$

  $\hookrightarrow$ dimensional transmutation generates a scale $\Lambda_{\rm QCD}$, so $m_p \sim \Lambda_{\rm QCD}$

  $\hookrightarrow$ how are these phenomena linked?

- Classical QCD, rewritten in a symmetrized form:

$$\mathcal{L}_{\rm QCD} = \frac{1}{2g^2}\,{\rm tr}_c(F_{\mu\nu}F^{\mu\nu}) + \bar\psi\big(\tfrac{i}{2}\,\gamma^\mu \overset{\leftrightarrow}{D}_\mu - \mathcal{M}\big)\psi$$

- Conserved and gauge-invariant *energy-momentum tensor*:

$$\bar\theta_{\mu\nu} = \frac{i}{2}\,\bar\psi\gamma_\mu\overset{\leftrightarrow}{D}_\nu\psi + 2{\rm tr}_c\big(F_{\mu\lambda}F_\nu^\lambda - \tfrac{1}{4}\,g_{\mu\nu}F_{\lambda\rho}F^{\lambda\rho}\big)\;, \quad \bar\theta^\mu_\mu = \bar\psi\mathcal{M}\psi$$

- M.E. of $\bar\theta_{\mu\nu}$ in some hadron $|k\rangle$:

  $\langle k|\bar\theta_{\mu\nu}(0)|k\rangle = a k_\mu k_\nu + b g_{\mu\nu} \to a = 2,\, b = 0$ from momentum operator

  $\hookrightarrow \langle k|\bar\theta^\mu_\mu(0)|k\rangle = 2M^2 \to$ all hadrons are massless as $\mathcal{M} \to 0$???

Scale transformations in QCD

- Scale transformations: $\psi(x) \mapsto \lambda^{3/2}\psi(\lambda x)\,,\;\; G_\mu(x) \mapsto \lambda G_\mu(\lambda x)\,,\;\; \lambda \in \mathbb{R}\backslash\{0\}$

- Simplified derivation of the trace anomaly, gluons described by an external field

$$Z(G_\mu) = \int d\psi d\bar\psi \,\exp\left\{ i \int d^4x \mathcal{L}_{\rm QCD} \right\}$$

- Inf. scale trafo's: $\psi(x) \mapsto \left(1 - \frac{\epsilon(x)}{2}\right)\psi(x),\;\; \bar\psi(x) \mapsto \left(1 - \frac{\epsilon(x)}{2}\right)\bar\psi(x)$

$$\hookrightarrow Z(G_\mu) = \int d\psi d\bar\psi \, J^{-2} \,\exp\left\{ i \int d^4x \left(\mathcal{L}_{\rm QCD} - \epsilon(\bar\theta^\mu_\mu(x) - \bar\psi(x)\mathcal{M}\psi(x))\right) \right\}$$

- Ward identity ot first order in $\epsilon(x)$:

$$\int d\psi d\bar\psi \left(i \int d^4x \,\epsilon(x)(\bar\theta^\mu_\mu(x) - \bar\psi(x)\mathcal{M}\psi(x)) + 2\ln J \right) \exp\left\{ i \int d^4x \,\mathcal{L}_{\rm QCD} \right\} = 0$$

  - if $J = 1$, same as on the classical level
  - the anomaly will emerge from the femionic measure, let's calculate it!

Trace anomaly

- Use the Fujikawa method:

$$\ln J = \ln(\det(e^{-\epsilon/2})) = -\lim_{M\to\infty}\left\{\int d^4x\, \frac{\epsilon(x)}{2}\, \left\langle x \middle| \mathrm{tr}\left(e^{(i\not{D})^2/M^2}\right) \middle| x \right\rangle\right\}$$

- Use $[D_\mu, D_\nu] = F_{\mu\nu}$ and let $M \to \infty$:

$$\ln J = i \int d^4x\, \frac{\epsilon(x)}{2}\, \frac{N_f}{24\pi^2}\, \mathrm{tr}_c(F_{\mu\nu}(x)F^{\mu\nu}(x))$$

$\Rightarrow$ the trace of the energy-momentum tensor is:

$$\bar{\theta}^\mu_\mu = \bar{\psi}\mathcal{M}\psi - \frac{N_f}{24\pi^2}\, \mathrm{tr}_c(F_{\mu\nu}(x)F^{\mu\nu}(x)) \neq 0 \text{ as } \mathcal{M} \to 0$$

- Full calculation: Collins, Crewther, Chanowitz, Ellis, Nielsen

$$\bar{\theta}^\mu_\mu(x) = (1+\gamma_m(g_r))[\bar{\psi}(x)\mathcal{M}_r\psi(x)]_r - \frac{\beta(g_r)}{g_r^3}\, [\mathrm{tr}_c(F_{\mu\nu}(x)F^{\mu\nu}(x))]_r$$

Anatomy of the nucleon mass

$$m_N = \langle N(p)|\theta^\mu_\mu|N(p)\rangle$$
$$= \langle N(p)|\underbrace{\frac{\beta_{\rm QCD}}{2g}G^a_{\mu\nu}G^{\mu\nu}_a}_{\rm field\ \ energy} + \underbrace{m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s}_{\rm Higgs}|N(p)\rangle$$

- Dissect the various contributions:

$\star\ \langle N(p)|m_u\bar{u}u + m_d\bar{d}d|N(p)\rangle = 40\ldots70\ {\rm MeV}\ \doteq \sigma_{\pi N}$

from the analysis of the pion-nucleon sigma term & lattice QCD (before 2015)

Gasser, Leutwyler, Sainio; Borasoy & M., Büttiker & M., Pavan et al., Alarcon et al. . . .

$\star\ \langle N(p)|m_s\bar{s}s|N(p)\rangle = 20\ldots60\ {\rm MeV}$ from lattice

⇒ bulk of the nucleon mass is generated by the gluon fields / field energy

⇒ this is a central result of QCD

⇒ requires better Roy-Steiner analysis of πN and lattice data

↪ discuss this w/o all details

σ-term basics

- Scalar form factor of the nucleon (isospin limit $\hat{m} = (m_u + m_d)/2$):

$$\sigma_{\pi N}(t) = \langle N(p')|\hat{m}(\bar{u}u + \bar{d}d)|N(p)\rangle \ , \ t = (p' - p)^2$$

- Cheng-Dashen Low-Energy Theorem (LET): Cheng, Dashen (1971)

$$\bar{D}^+(\nu = 0, t = 2M_\pi^2) = \sigma(2M_\pi^2) + \Delta_R$$

$\left[\nu = \frac{s-u}{4m_N}\right]$

- $\bar{D}^+$ – isospin-even, Born-term subtracted pion-nucleon scattering amplitude

$$\bar{D}^+(0, 2M_\pi^2) = A^+(m_N^2, 2M_\pi^2) - \frac{g_{\pi N}^2}{m_N}$$

$\hookrightarrow$ best determined from πN data using dispersion relations (unphysical region)

- reminder Δ_R, calculated in CHPT to $\mathcal{O}(p^4)$, no chiral logs

$$\Delta_R \lesssim 2\,\text{MeV}$$

Bernard, Kaiser, UGM (1996)

σ-term basics continued

- Standard decomposition of the σ-term: $\sigma_{\pi N} = \sigma_{\pi N}(0)$

$$\boxed{\sigma_{\pi N} = \Sigma_d + \Delta_D - \Delta_\sigma - \Delta_R}$$

$$\Sigma_d = F_\pi^2(d_{00}^+ + 2M_\pi^2 d_{01}^+) \quad \to \text{full RS analysis}$$

$$\left.\begin{aligned} \Delta_D &= \bar{D}^+(0, 2M_\pi^2) - \Sigma_d \\ \Delta_\sigma &= \sigma(2M_\pi^2) - \sigma_{\pi N} \end{aligned}\right\} \to \text{RS t-channel analysis}$$

- d_{00}^+, d_{01}^+ – subthreshold expansion coefficients (around $\nu = t = 0$)

- Strong $\pi\pi$ rescattering in Δ_D and Δ_σ, the difference is small!

Gasser, Leutwyler, Sainio (1991)

- Most precise analysis of the scalar form factor of the nucleon:

Hoferichter, Ditsche, Kubis, UGM (2012)

$$\Delta_D - \Delta_\sigma = (-1.8 \pm 0.2)\,\text{MeV}$$

Roy-Steiner equations in a nutshell

• Roy-Steiner (RS) equations = hyperbolic dispersion relations (HDRs):

$$\boxed{(s-a)(u-a)=b}\,, \qquad a,b\in\mathbb{R} \qquad [b=b(s,t,a)]$$

Steiner (1968), Roy(1971), Hite, Steiner (1973)

• why HDRs?

$\hookrightarrow$ combine all *physical regions*
very important for a reliable continuation to the subthreshold region

Stahov (1999)

$\hookrightarrow$ especially powerful for the determination of the σ-term

Koch (1982)

$\hookrightarrow$ $s \leftrightarrow u$ crossing is explicit

$\hookrightarrow$ absorptive parts are only needed in regions where
the corresponding partial expansions converge

$\hookrightarrow$ judicious choice of a allows to increase the range of convergence

Results for the σ-term

- Basic formula: $\sigma_{\pi N} = F_\pi^2(d_{00}^+ + 2M_\pi^2 d_{01}^+) + \Delta_D - \Delta_\sigma - \Delta_R$

- Subthreshold parameters output of the RS equations:

$$d_{00}^+ = -1.36(3) M_\pi^{-1} \qquad \text{[KH: } -1.46(10) M_\pi^{-1}\text{]}$$

$$d_{01}^+ = 1.16(3) M_\pi^{-3} \qquad \text{[KH: } 1.14(2)\ M_\pi^{-3}\text{]}$$

- $\Delta_D - \Delta_\sigma = (1.8 \pm 0.2)$ MeV — Hoferichter, Ditsche, Kubis, UGM (2012)

- $\Delta_R \lesssim 2$ MeV — Bernard, Kaiser, UGM (1996)

- Isospin breaking in the CD theorem shifts $\sigma_{\pi N}$ by $+3.0$ MeV

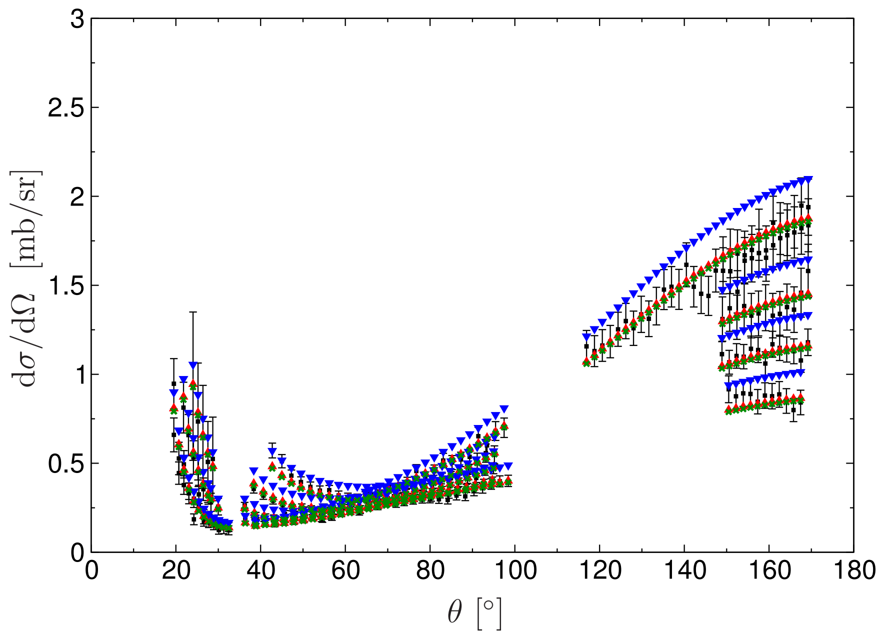
$\Rightarrow$

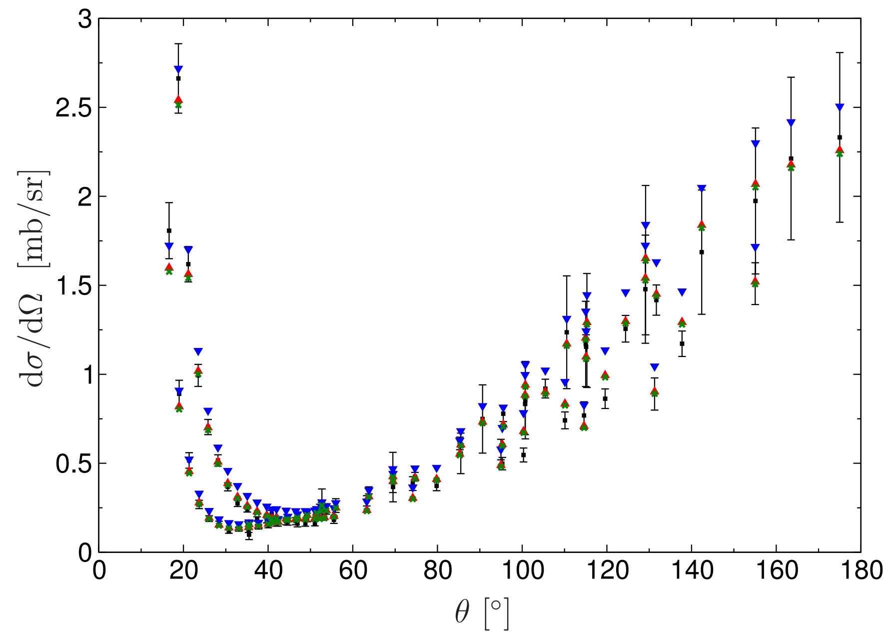
$$\sigma_{\pi N} = (59.1 \pm 1.9_{\rm RS} \pm 3.0_{\rm LET})\ \text{MeV} = (59.1 \pm 3.5)\ \text{MeV}$$

[NB: recover $\sigma_{\pi N} = 45$ MeV if KH80 scattering lengths are used]

Sanity check – no hadronic atom input

- Fit to the pion-nucleon data base (GWU), electromagnetic corrections to the data à la Tromberg et al and treat normalizations of the data as fit parameters
- Fit to low-energy data based on the RS representation that are dominated from the scattering lengths (up to $T_\pi^{\rm max} = 33 - 55$ MeV)





black: data with readjusted norm

green triangles: new RS solution

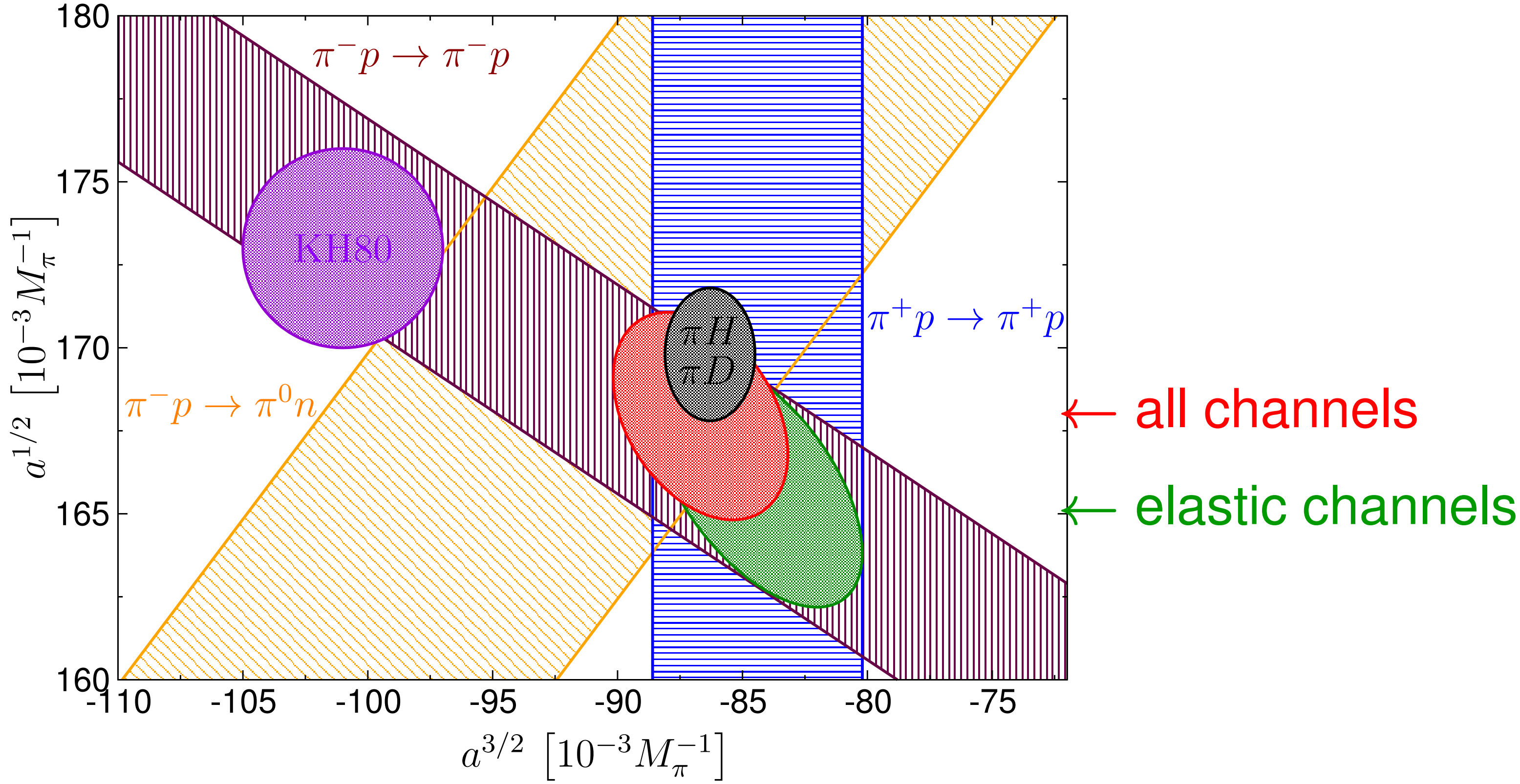
red triangles: RS solution with hadronic atom input

blue triangles: KH80

Sanity check – no hadronic atom input cont'd

- Scattering lengths in $[10^{-3}/M_\pi]$ and σ-term:

$$a^{1/2} = -86.7(3.5)\ ,\ \ a^{3/2} = 167.9(3.2)\ ,\quad \boxed{\sigma_{\pi N} = 58(5)\ \text{MeV}}$$



- consistent picture!

[details in Ruiz De Elvira, Hoferichter, Kubis, UGM, J. Phys. G **45** (2018) 024001]

Results for the proton mass

- Precise determination of the pion-nucleon σ-term:

$$\sigma_{\pi N} = (59.1 \pm 3.5)\ \text{MeV}$$

- Consistency check w/o pionic atom data:

$$\sigma_{\pi N} = (58 \pm 5)\ \text{MeV}$$

- Not consistent w/ lattice determinations (presumably excited state contaminations)

Gupta et al. (2021)

- Strange σ-term: $\sigma_s = \langle N | m_s \bar{s} s | N \rangle$ (more safe in LQCD)

$\hookrightarrow$ FLAG average: $\underbrace{\sigma_s = 52.9(7.0)\ \text{MeV}}_{N_f=2+1}$ or $\underbrace{\sigma_s = 41.0(8.4)\ \text{MeV}}_{N_f=2+1+1}$

Consequences for the proton mass:
About 100 MeV from the Higgs, the rest is gluon field energy

Large-N_C QCD and the mass of the η'

Large-N_C QCD

t 'Hooft , Nucl. Phys. **B72** (1974) 461

- QCD is difficult to solve, often only numerical approaches allow *ab initio* calc's
- There is no small parameter except for very high-energy processes $\alpha_S \to 0$ (still confinement)
- Idea: Consider the limit $N_C \to \infty$ with $g^2 N_C = \text{constant}$

  $\to$ This gives a smooth limit as $N_C \to \infty$
- To leading order in $1/N_c$, only planar diagrams contribute

  $\to$ Mesons are stable with masses $\mathcal{O}(N_C^0)$ and widths $\mathcal{O}(1/N_C)$

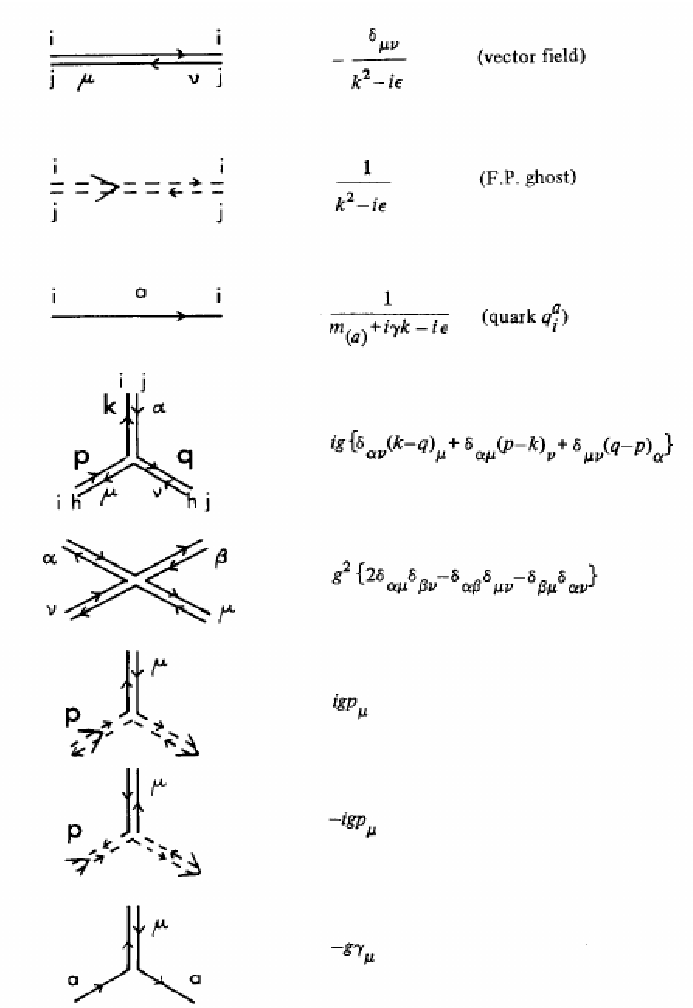
  $\to$ Meson interactions are weak, of order $\mathcal{O}(N_C^0)$

  $\to$ The OZI rule is exact in this limit

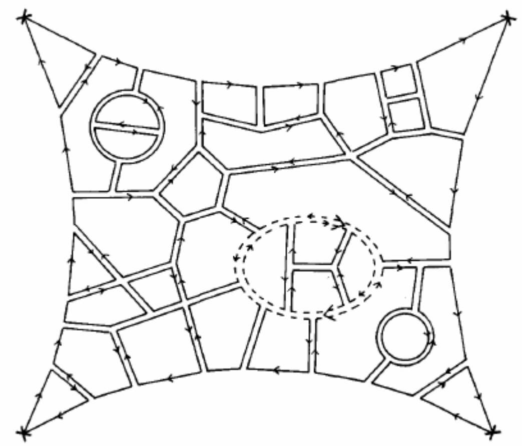
A few details on large-N_C

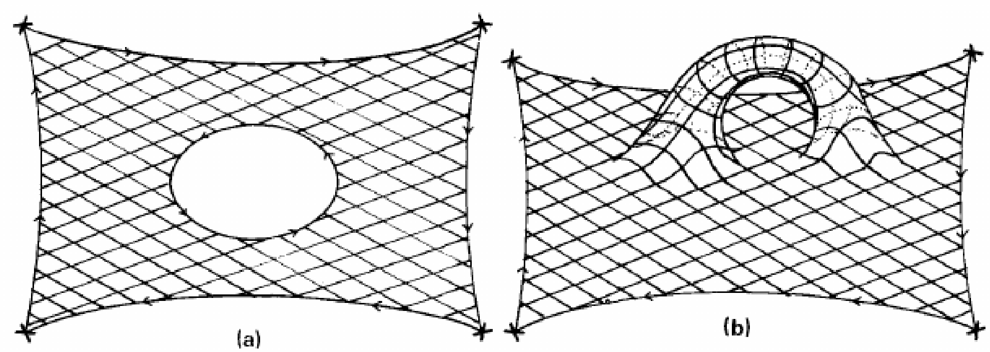
t 'Hooft, Nucl. Phys. **B72** (1974) 461

- Double line notation



- Planar diagrams dominate





suppressed by $1/N_C$ by $1/N_C^2$

A sample calculation

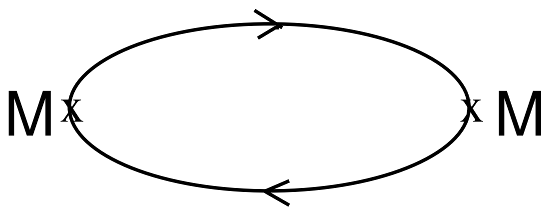
- Color singlet contribution from a $Q\bar{Q}$ pair

$$|Q^\alpha \bar{Q}^\beta\rangle_{\text{color singlet}} \sim \frac{1}{\sqrt{N_C}} b_i^\alpha d_i^{\beta\,\dagger}|0\rangle$$

α, β = flavor labels

$i = 1, \ldots N_C$ = color label

$1/\sqrt{N_c}$ = w.f. must be normalized to one

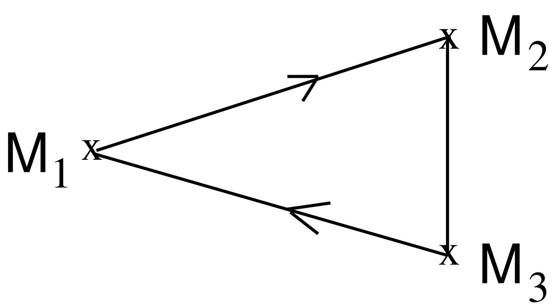


$\Rightarrow$ Propagator = quark loop $\times$ w.f. normalizations = $N_C \left(N_C^{-1/2}\right)^2 = \mathcal{O}(1)$

- Meson deacy widths:

3 meson w.f. normalizations $\times$ one quark loop

$$= \left(N_C^{-1/2}\right)^3 N_C = N_C^{-1/2}$$



$\Rightarrow$ Meson become stable, i.e. $\Gamma/M \to 0$, as $N_C \to \infty$

Spontaneous symmetry breaking in large-N_C

- Now we have $U(3)_L \times U(3)_R \to U(3)_V$

  $\hookrightarrow$ There is a ninth Goldstone boson, the η'

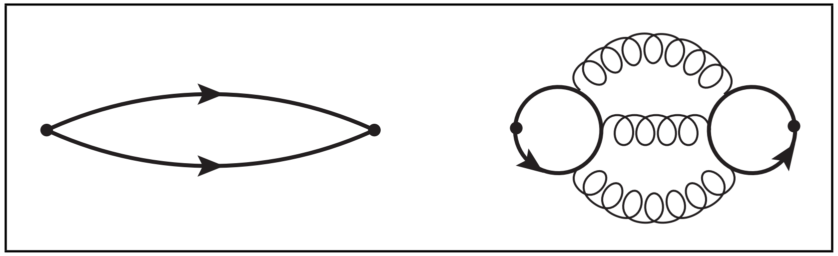
- However, at finite N_C, the residual $U(1)_A$ symmetry is anomalous

- Two-point function of the isosinglet axial current:

  $\hookrightarrow$ disconnected diagrams vanish as $N_C \to \infty$

  $\hookrightarrow$ related to the vacuum structure

  $\longrightarrow$ Andreas' lecture



- Explicit calculations give the Witten-Veneziano formula: Witten (1979), Veneziano (1979)

$$M_{\eta'}^2 = \frac{2N_f}{F_\pi^2}\chi_{\rm top} = \frac{const.}{N_c}$$

$\hookrightarrow$ in terms of the topological suszeptibility of the QCD vacuum

$\hookrightarrow$ obviously $M_{\eta'}^2 \to 0$ when $N_C \to \infty$

Heavy quark symmetries

Heavy quark Lagrangian

Caswell, Lepage, Voloshin, Shifman, Isgur, Wise, Neubert, Mannel, ...

- The heavy quark Lagrangian at leading order takes the form:

$$\boxed{\mathcal{L}_{\rm LO} = \bar{Q}\,(iv\cdot D)\,Q + \mathcal{O}(1/m_Q)}\,,\quad Q = \begin{pmatrix} c \\ b \end{pmatrix}\,,\quad D_\mu = \partial_\mu - igG^a_\mu T^a$$

$\hookrightarrow$ SU(2) flavor symmetry: $Q \mapsto Q' = UQ\,,\quad U \in SU(2)$

$\hookrightarrow$ SU(2) spin symmetry: $S^3_Q|P\rangle = -\frac{1}{2}|V\rangle\,,\quad S^3_Q|V\rangle = -\frac{1}{2}|P\rangle$

- HQ symmetry breaking $\sim \Lambda_{\rm QCD}/m_Q$ [better for b quarks]

$$\mathcal{L}_{\rm NLO} = \underbrace{\frac{1}{2m_Q}\bar{Q}\,(iD_\perp)^2\,Q}_{\text{breaks SU(2) flavor}}\underbrace{-c_F\frac{g}{4m_Q}\bar{Q}\,\sigma_{\alpha\beta}G^{\alpha\beta}\,Q}_{\text{breaks SU(2) spin \& flavor}}$$

$$D^\mu_\perp = D^\mu - v^\mu v\cdot D$$

$$[D^\mu, D^\nu] = gG^{\mu\nu}$$

- non-relativistic kinetc energy and chromo-magnetic term

- must be “dressed” to form physical states: $Q\bar{q}$, Qqq, etc.

Heavy quark effective Lagrangian

- consider the quark field as a static, heavy source $\to$ four-velocity v_μ:

Georgi (1990)

$$\boxed{p_\mu = m_Q v_\mu + \ell_\mu}$$

with $$v^2 = 1, \quad p^2 = m_Q^2, \quad v \cdot \ell \ll m_Q$$

- velocity-projection: $\Psi(x) = \exp(-i m_Q v \cdot x)\,[Q(x) + \mathcal{Q}(x)]$

with $$\not{v} Q = Q, \; \not{v} \mathcal{Q} = -\mathcal{Q}$$ [“large/small” components]

- Q- and $\mathcal{Q}$-components decouple, separated by large mass gap $2m_Q$:

$$\mathcal{L}_Q^{(1)} = \bar{\Psi}\left(i\not{D} - m_Q + \ldots\right)\Psi$$

$$\to \boxed{\mathcal{L}_Q^{(1)} = \bar{Q}\,(iv \cdot D + \ldots)\,Q + \mathcal{O}\left(\frac{1}{m_Q}\right)}$$

Heavy hadron masses

- Hadron mass in HQET is $m_H - m_Q$
- Leading order: all heavy hadrons containing Q have the same mass

$$\frac{1}{2}\langle H^{(Q)}|\mathcal{H}_Q|H^{(Q)}\rangle \equiv \bar{\Lambda}$$

$\hookrightarrow$ $\bar{\Lambda}$ for B, B^*, D, D^* , $\bar{\Lambda}_\Lambda$ for Λ_b, Λ_c , $\bar{\Lambda}_\Sigma$ for $\Sigma_b, \Sigma_b^*, \Sigma_c, \Sigma_c^*$ states

$\hookrightarrow$ in case of SU(3) breaking: $\bar{\Lambda}_{u,d}$ and $\bar{\Lambda}_s$

- Next-to-leading order: two non-perturbative constants

$$\langle H^{(Q)}|\bar{Q}D_\perp^2\, Q|H^{(Q)}\rangle = -2\lambda_1$$

$$\langle H^{(Q)}|\bar{Q}a(\mu)g\sigma_{\alpha\beta}G^{\alpha\beta}\, Q|H^{(Q)}\rangle = 16(\vec{S}_Q\cdot\vec{S}_\ell)\lambda_2(m_Q)$$

$\hookrightarrow$ note renormalzation of the chromo-magnetic operator

$\hookrightarrow$ $\lambda_{1,2}$ are the same for all states in a spin-flavor multiplet

Heavy hadron masses continued

- after a bit of algebra:

$$m_B = m_b + \bar\Lambda - \frac{\lambda_1}{2m_b} - \frac{3\lambda_2(m_b)}{2m_b}\,, \qquad m_D = m_c + \bar\Lambda - \frac{\lambda_1}{2m_c} - \frac{3\lambda_2(m_c)}{2m_c}$$

$$m_{B^*} = m_b + \bar\Lambda - \frac{\lambda_1}{2m_b} + \frac{\lambda_2(m_b)}{2m_b}\,, \qquad m_{D^*} = m_c + \bar\Lambda - \frac{\lambda_1}{2m_c} + \frac{\lambda_2(m_c)}{2m_c}$$

$$m_{\Lambda_b} = m_b + \bar\Lambda_\Lambda - \frac{\lambda_{\Lambda,1}}{2m_b}\,, \qquad m_{\Lambda_c} = m_c + \bar\Lambda_\Lambda - \frac{\lambda_{\Lambda,1}}{2m_c}$$

$$m_{\Sigma_b} = m_b + \bar\Lambda_\Sigma - \frac{\lambda_{\Sigma,1}}{2m_b} - \frac{2\lambda_{\Sigma,2}(m_b)}{2m_b}\,, \qquad m_{\Sigma_c} = m_c + \bar\Lambda_\Sigma - \frac{\lambda_{\Sigma,1}}{2m_c} - \frac{2\lambda_{\Sigma,2}(m_c)}{2m_c}$$

$$m_{\Sigma_b^*} = m_b + \bar\Lambda_\Sigma - \frac{\lambda_{\Sigma,1}}{2m_b} + \frac{\lambda_{\Sigma,2}(m_b)}{2m_b}\,, \qquad m_{\Sigma_c^*} = m_c + \bar\Lambda_\Sigma - \frac{\lambda_{\Sigma,1}}{2m_c} + \frac{\lambda_{\Sigma,2}(m_c)}{2m_c}$$

- so that (ignore m_Q dependence of λ_2):

$$0.49\ \text{GeV}^2 \simeq m_{B^*}^2 - m_B^2 \simeq 4\lambda_2 \simeq m_{D^*}^2 - m_D^2 \simeq 0.55\ \text{GeV}^2$$

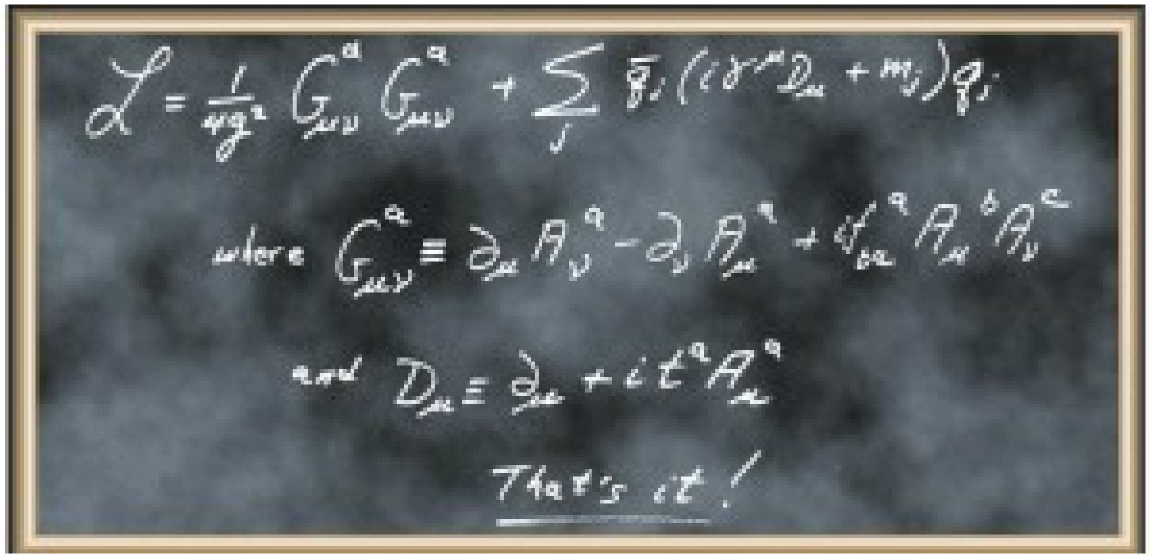
$$(90 \pm 3)\ \text{GeV} = m_{B_s} - m_{B_d} = \bar\Lambda_s - \bar\Lambda_{u,d} = m_{D_s} - m_{D_d} = (99 \pm 1)\ \text{GeV}$$

$$(345 \pm 9)\ \text{GeV} = m_{\Lambda_b} - m_B = \bar\Lambda_\Lambda - \bar\Lambda_{u,d} = m_{\Lambda_c} - m_D = (416 \pm 1)\ \text{GeV}$$

Summary

Summary

- QCD is a remarkably easy to write down theory



@ Frank Wilczek

- ... but has this rich cornucopia of symmetries and their realizations
  with many fascinating physical consequences & phenomena

- ... and this is not even all! $\rightarrow$ Andreas' lectures

SPARES

Pion-nucleon scattering

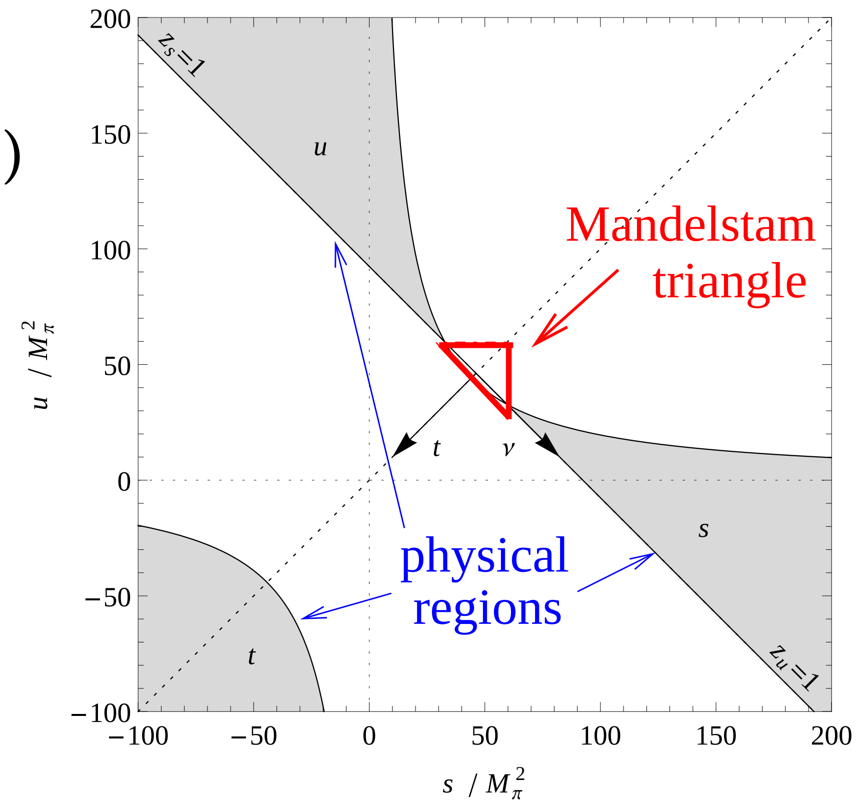
- s-channel: $\pi(q) + N(p) \;\; \to \pi(q') \; + \; N(p')$

  t-channel: $\pi(q) + \pi(-q') \to \bar{N}(-p) + N(p')$

- Mandelstam variables:

$$s = (p+q)^2, t = (p-p')^2, u = (p-q')^2$$

$$s + t + u = 2m_N^2 + 2M_\pi^2, \;\; s = W^2$$



- Isospin structure:

$$T^{ba}(s,t) = \delta^{ba} \, T^+(s,t) + i\epsilon_{abc}\tau^c \, T^-(s,t)$$

- Lorentz structure:

$$8\pi\sqrt{s}T^I(s,t) = \bar{u}(p')\left\{A^I(s,t) + \tfrac{1}{2}(\not{q} + \not{q}')B^I(s,t)\right\}u(p)\,, \quad I = +,-$$

$$I = 1/2, 3/2$$

- Crossing:

$$A^\pm(\nu,t) = \pm A^\pm(-\nu,t)\,, \;\; B^\pm(\nu,t) = \mp B^\pm(-\nu,t)\,, \quad \nu = \frac{s-u}{4m_N}$$

Pion-nucleon scattering continued

- Partial wave projection:

$$X_\ell^I(s) = \int_{-1}^{+1} dz_s P_\ell(z_s) X^I(s,t)\Big|_{t=-2q^2(1-z_s)}\,, \quad X \in \{A, B\}$$

$\Rightarrow$ partial wave expansion (total isospin I, ang. mom. ℓ, $j = \ell \pm 1/2$):

$$f_{\ell\pm}^I(W) = \frac{1}{16\pi W}$$

$$\times\Big\{(E+m)\big[A_\ell^I(s) + (W-m)B_\ell^I(s)\big] + (E-m)\big[-A_{\ell\pm1}^I(s) + (W+m)B_{\ell\pm1}^I(s)\big]\Big\}$$

- MacDowell symmetry: $f_{\ell+}^I(W) = -f_{(\ell+1)-}^I(-W) \quad \forall\, l \geq 0$ MacDowell (1959)

- Low-energy region: only S- and P-waves are relevant $\boxed{f_{0+}^{\pm},\, f_{1+}^{\pm},\, f_{1-}^{\pm}}$

$\Rightarrow$ low-energy amplitude can eventually be matched to chiral perturbation theory

Subthreshold expansion

- For the σ-term extraction, the πN amplitude $D = A + \nu B$ is most useful:

$$\bar{D}^+(\nu, t) = D^+(\nu, t) - \frac{g_{\pi N}^2}{m_N} - \nu g_{\pi N}^2 \left(\frac{1}{m_N^2 - s} - \frac{1}{m_N^2 - u} \right)$$

⋆ subtraction of pseudovector Born terms $\to \bar{D}$

- Subthreshold expansion: expand around $\nu = t = 0$:

$$\boxed{X(\nu, t) = \sum_{m,n} x_{mn}\, \nu^{2m}\, t^n} \quad , \quad X \in \left\{ \bar{A}^+, \frac{\bar{A}^-}{\nu}, \frac{\bar{B}^+}{\nu}, \bar{B}^-, \bar{D}^+, \frac{\bar{D}^-}{\nu} \right\}$$

⋆ $\boxed{x_{mn}}$ are the **subthreshold parameters** $\to$ can be calculated via sum rules

⋆ inside the Mandelstam triangle, scattering amplitudes are real polynomials