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Split octonionic Dirac equation

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Introduction

- Division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ are associated with mass, electric charge, spin and color symmetries respectively.
- Octonions are rarely used and are understudied because of inconvenience of non-associativity
- Non-associativity can describe time irreversibility[Gogberashvili M 2002]
- Split-octonionic quadratic form is invariant under $SO(4,4)$ which has Lorentz group $SO(3,1)$ as a subgroup
- Split octonionic (pseudo-)analyticity condition contains Dirac and Maxwell equations
- Triality symmetry (which only exists in $\mathbb{R}^8$) of trilinear form on split octonions is an interaction Lagrangian of two fermions and a boson
- Since vectors and spinors are indistinguishable in this setting[Gurchumelia A, Gogberashvili M, 2021] unless some restriction is imposed this is an analog of supersymmetric theory. But this theory might not contain superpartners.

Framework: Cayley-Dickson constructions & Hurwitz algebras

Given an algebra $\mathbb{A}_n$ we can define $\mathbb{A}_{n+1}$ for $\mathbb{A}_{n+1} = \mathbb{A}_n \times \mathbb{A}_n$ where

$$
\begin{aligned}
(a,b) + (c,d) &= (a+c, b+d)\ , && \text{(with } a,b,c,d \in \mathbb{A}_n\text{)}\\
(a,b) \times (c,d) &= (ac - \gamma d^* b, da + bc^*)\ , && \text{(with } \gamma = \pm 1\text{)}\\
(a,b)^* &= (a^*, -b)\ .
\end{aligned}
$$

For example starting from $\mathbb{A}_0 = \mathbb{R}$ and $\gamma = 1$ we get a chain $\mathbb{R} \to \mathbb{C} \to \mathbb{H} \to \mathbb{O} \to \mathbb{S} \to \cdots$.

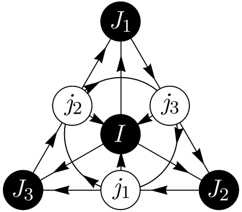
	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$	$\mathbb{S}$	$\cdots$	$\mathbb{A}_n$
dimension:	1	2	4	8	16	$\cdots$	2^n
order: $a < b$	✓	×	×	×	×	$\cdots$	×
commutative: $ab = ba$	✓	✓	×	×	×	$\cdots$	×
associative: $a\,(bc) = (ab)\,c$	✓	✓	✓	×	×	$\cdots$	×
division: $\forall\,(a \neq 0)\,\exists a^{-1}$ such that $aa^{-1} = 1$	✓	✓	✓	✓	×	$\cdots$	×

Split versions of the algebras $\mathbb{C}'$, $\mathbb{H}'$, $\mathbb{O}'$ are obtained by setting $\gamma = -1$ at least once in the chain

Framework: split octonions $\mathbb{O}'$

- As linear space $\mathbb{O}' \simeq \mathbb{R}^8$
- $\mathbb{O}'$ algebraic relations and the Fano plane:

$$j_m j_n = -\delta_{mn} + \sum_k \epsilon_{mnk} j_k \ , \qquad (\mathbb{H} \text{ subalgebra})$$
$$I^2 = 1 \ ,$$
$$J_m J_n = \delta_{mn} + \sum_k \epsilon_{mnk} j_k \ , \qquad (m,n,k = 1,2,3)$$
$$j_n I = J_n \ ,$$
$$J_m j_n = \delta_{mn} I - \sum_k \epsilon_{mnk} J_k \ .$$



- Real split octonion is $\mathrm{X} = x_0 + x_1 j_1 + x_2 j_2 + x_3 j_3 + x_4 I + x_5 J_1 + x_6 J_2 + x_7 J_3$ (for $x_0, x_1, \ldots \in \mathbb{R}$)
- Dot product: $\cdot : \mathbb{O}' \times \mathbb{O}' \to \mathbb{R}$ is a bilinear map

$$\mathrm{X} \cdot \mathrm{Y} = \frac{1}{2}\left(\overline{\mathrm{X}}\mathrm{Y} + \overline{\mathrm{Y}}\mathrm{X}\right) = x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4 - x_5 y_5 - x_6 y_6 - x_7 y_7$$

- Quadratic form $Q(\mathrm{X}) = \overline{\mathrm{X}}\mathrm{X} = \mathrm{X} \cdot \mathrm{X}$

- Given $f : \mathbb{O}' \to \mathbb{O}'$ its Wirtinger derivatives are

$$\frac{\overrightarrow{\partial}}{\partial x} f = \frac{1}{2}\left(\partial_0 + j_1\partial_1 + j_2\partial_2 + j_3\partial_3 + I\partial_4 + J_1\partial_5 + J_2\partial_6 + J_3\partial_7\right) f\,,$$

$$\frac{\overrightarrow{\partial}}{\partial \overline{x}} f = \frac{1}{2}\left(\partial_0 - j_1\partial_1 - j_2\partial_2 - j_3\partial_3 - I\partial_4 - J_1\partial_5 - J_2\partial_6 - J_3\partial_7\right) f\,.$$

  (arrow indicates action direction)
- x and $\overline{x}$ are not independent in $\mathbb{O}$ and $\mathbb{H}$ unlike in $\mathbb{C}$ therefore in general

$$\frac{\overrightarrow{\partial}}{\partial \overline{x}} f\left(x\right) \neq 0 \qquad \text{and} \qquad \frac{\overrightarrow{\partial}}{\partial x} f\left(\overline{x}\right) \neq 0\,.$$

- The expected rules only work for first order polynomials

$$\frac{\overrightarrow{\partial}}{\partial \overline{x}} x = \frac{\overrightarrow{\partial}}{\partial x} \overline{x} = 0, \qquad \frac{\overrightarrow{\partial}}{\partial x} x = \frac{\overrightarrow{\partial}}{\partial \overline{x}} \overline{x} = 1\,.$$

Previous results: triality symmetry

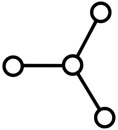
- Trilinear form $\mathcal{F} : \mathbb{O}' \times \mathbb{O}' \times \mathbb{O}' \to \mathbb{R}$ is defined as $\mathcal{F}(\Phi, \mathrm{X}, \Psi) = -\overline{\Phi} \cdot (\mathrm{X}\Psi)$
- The form is conserved under simultaneous $SO(4,4)$ and $Spin(4,4)$ which has the following representation in $\mathbb{O}'$

$$\begin{cases} \Phi \mapsto (\Phi u)\,\overline{\mathrm{T}}_{uv} \\ \mathrm{X} \mapsto \mathrm{T}_{uv}\,(u\mathrm{X}v)\,\mathrm{T}_{uv} \\ \Psi \mapsto \overline{\mathrm{T}}_{uv}\Psi \end{cases}$$

where u and v are the imaginary basis units of $\mathbb{O}'$ and

$$\mathrm{T}_{uv} = \begin{cases} u\cos\left(\frac{\vartheta}{2}\right) + v\sin\left(\frac{\vartheta}{2}\right), & u^2v^2 = -1 \\ u\cosh\left(\frac{\vartheta}{2}\right) + v\sinh\left(\frac{\vartheta}{2}\right), & u^2v^2 = 1 \end{cases}$$

- Under this symmetry chiral spinors Φ, Ψ and vector X *trialy* "rotate" into each other
- Triality is also connected to the symmetry of D_4 ($SO(4,4)$, $SO(8)$) Dynkin diagram



Previous results: automorphism group G_2^{NC} of $\mathbb{O}'$

G_2^{NC} is interconnected with geometric symmetries

$$s = \lambda_0 + x_n j^n + x_0 I + \lambda_n J^n$$

When extra dimensions are held constant $\lambda_k = \text{const}$ then G_2^{NC} Casimir operator reduces to

$$C_2 = C_L - \lambda^2 C_P$$

where C_L is the Lorentz group Casimir operator

$$C_L = 3\left(t\frac{\partial}{\partial t} + \sum_k x_k \frac{\partial}{\partial x_k}\right) + x^2\frac{\partial^2}{\partial t^2} + \sum_k t^2 \frac{\partial^2}{\partial x_k^2} + 2t\sum_k x_k \frac{\partial^2}{\partial t \partial x_k} + \sum_{i,j,k} |\epsilon_{ijk}| \left(x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} - x_i^2 \frac{\partial^2}{\partial x_j^2}\right)$$

and C_P is a Poincare group Casimir operator

$$C_P = \frac{\partial^2}{\partial t^2} - \sum_k \frac{\partial^2}{\partial x_k^2}$$

Dirac equation: standard formulation

In standard form Dirac equation is

$$i\gamma^\mu D_\mu \Psi = m\Psi$$

where $D_\mu = \partial_\mu - iA_\mu$ and

$$\gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & \sigma^0 \end{pmatrix}, \qquad \gamma^n = \begin{pmatrix} 0 & \sigma^n \\ -\sigma^n & 0 \end{pmatrix}.$$

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We label components as

$$\Psi = \begin{pmatrix} y_0 + i\ell_3 \\ -\ell_2 + i\ell_1 \\ y_3 + i\ell_0 \\ y_1 + iy_2 \end{pmatrix}$$

Dirac equation: inside $\mathbb{O}'$ pseudo-analyticity condition

If we take

$$s = \lambda_0 + x_n j^n + x_0 I + \lambda_n J^n$$
$$\Psi = \ell_0 + y_n j^n + y_0 I + \ell_n J^n$$
$$\frac{\partial}{\partial s} = \frac{\partial}{\partial \lambda^0} + \frac{\partial}{\partial x^n} j^n + \frac{\partial}{\partial x^0} I + \frac{\partial}{\partial \lambda^n} J^n$$
$$\mathcal{A} = A_0 I + A_n j^n$$

then the equation

$$J_3 \left(\frac{\overrightarrow{\partial}}{\partial s} \Psi \right) + I \mathsf{conj}_{I,J} \left(\mathcal{A} \Psi \right) = -m \Psi$$

is the Dirac equation if Ψ is constant in λ. Without the potential $\mathcal{A}$ this is pseudo-analyticity condition

$$\frac{\overrightarrow{\partial}}{\partial s} \Psi = -J_3 m \Psi$$

or just analyticity if the field is massless.

Summary and conclusion

- We claim split octonionic algebra is important in physics and it exhibits multiple properties important in physical systems
- Triality might describe fermion-fermion-boson interaction
- Describing Dirac equation is a step towards constructing triality based supersymmetric theory

References: *THANKS FOR YOUR ATTENTION!*

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