A Two-Potential Model For The Pion Vector Form Factor

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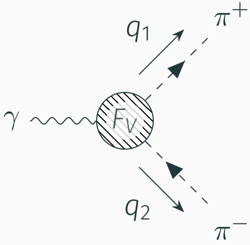
November 11, 2021

Aspects of Symmetry, Tbilisi

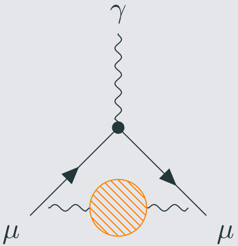
Introduction

The pion vector form factor

Definition



Significance to $g_\mu - 2$

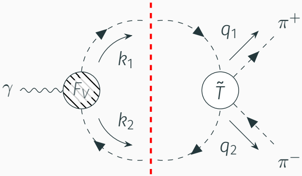


[Aoyama et al., 2020]

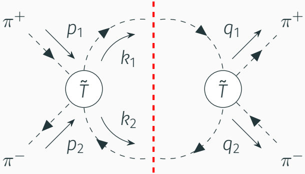
$$\langle \pi^+(q_1)\pi^-(q_2)|J^\mu|0\rangle = e(q_1 - q_2)^\mu F_V(s)$$

Elastic regime: discontinuities due to the $\pi\pi$ intermediate state

Form factor:



Transition matrix:



[Cutkosky, 1960]

$$\text{disc}\,[F_V(s)] = 2i\,\text{Im}\,[F_V(s)] = 2i\sigma(s)\tilde{t}_1^*(s)F_V(s)$$

$$\text{disc}\left[\tilde{t}_1(s)\right] = 2i\sigma(s)\left|\tilde{t}_1^*(s)\right|^2$$

$$\text{where}\quad \sigma(s) = \frac{1}{16\pi}\sqrt{1-\frac{4m_\pi^2}{s}}$$

$$\text{Im}\,[F_V(s)] \in \mathbb{R} \implies \arg\tilde{t}_1(s) = \arg F_V(s)$$ [Watson, 1954]

Elastic regime: the Omnès solution

Transition matrix: phase shift representation

$$\tilde{t}_1(s) = \frac{1}{\sigma(s)} \sin(\tilde{\delta}_1(s)) e^{i\tilde{\delta}_1(s)}$$

$\tilde{\delta}_1$ can be derived from Roy equations [Caprini et al., 2012].

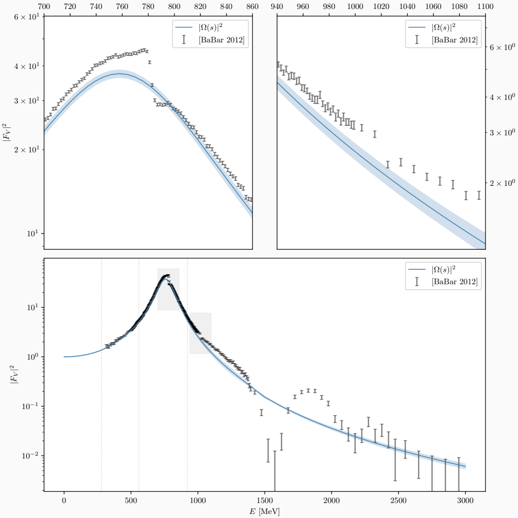
Form factor: Omnès-Muskhelishvili solution [Omnès, 1958]

$$F_V(s) = \Omega[\tilde{\delta}_1](s) P_A(s)$$

$$\text{where } \Omega[\tilde{\delta}_1](s) = \exp\left\{ \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\tilde{\delta}_1(s')}{s' - s - i\epsilon} \right\}$$

and $P_A(s)$ can be approximated as a polynomial at low energies.

Elastic regime: the Omnès solution



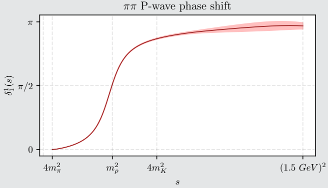
The Omnès solution compared to the experimental data for $F_V(s)$ from [BaBaR, 2012].

Input phase shift $\tilde{\delta}_1(s)$ [Caprini et al., 2012].

Extrapolated to π via

$$\tilde{\delta}_1(s > s_c) = \pi + \left(\tilde{\delta}_1(s_c) - \pi\right)\left(\frac{\lambda^2 + s_c}{\lambda^2 + s}\right)$$

$$\lambda = 10\ \text{GeV}, \quad s_c = (1.5\ \text{GeV})^2$$



Our goals

Our goal is to derive a formalism that

- preserves both analyticity and unitarity;
- maps to $\Omega[\tilde{\delta}_1](s)$ at low energies;
- provides an accurate high-energy description;
- includes contributions from coupled channels;
- includes isospin-violating effects (such as ρ mixing with ω and ϕ).

Two-Potential Model

Two-potential model: $\tilde{V}$ and V_R

Let the interaction potential V_{ij} between channels i and j be

$$V_{ij}(s) = \tilde{V}_{ij}(s) + V_{Rij}(s)$$

[Hanhart, 2012] (inspired by [Nakano, 1982]).

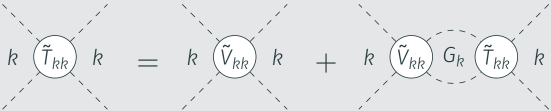
Assumptions:

- $\tilde{V}$ is purely elastic (i.e. only non-zero for $i = j = 1$);
- all long-ranged forces (that induce left-hand cuts) of the first channel are contained in $\tilde{V}$;
- all the deviations from $\tilde{V}$ are assumed to come from s-channel resonances (or possible contact terms);
- no left-hand cuts are allowed in the other channels.

Two-potential model: elastic T-matrix $\tilde{T}$

T-matrix can be split accordingly: $T_{ij}(s) = \tilde{T}_{ij}(s) + T_{Rij}(s)$.

$\tilde{T}$ satisfies the Lippmann-Schwinger equation: $\tilde{T} = \tilde{V} + \tilde{V}G\tilde{T}$.



Here G_k denotes the propagation of channel k.

For instance, in the case of the 2π channel ($k = 1$), we have

$$G_1 = \int \frac{d^4 l}{(2\pi)^4} \left| p_1, p_2 \right\rangle \frac{1}{(p_1^2 - m_\pi^2 + i\epsilon)} \frac{1}{(p_2^2 - m_\pi^2 + i\epsilon)} \left\langle p_1, p_2 \right|$$

where p_1 and p_2 are the momenta of pions with $p_1 - p_2 = l$ and $(p_1 + p_2)^2 = s$.

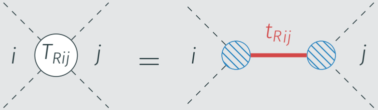
Two-potential model: resonance T-matrix T_R

Vertices Γ_k are defined as

$$\Gamma_{\text{in}}^{\dagger} = 1 + G\tilde{T}, \qquad \Gamma_{\text{out}} = 1 + \tilde{T}G, \qquad \Gamma_1(s) = \Omega[\tilde{\delta}_1](s)$$

$$\text{For other channels } (i > 1): \quad \Gamma_i(s) = \frac{\Lambda^2}{\Lambda^2 + s}$$

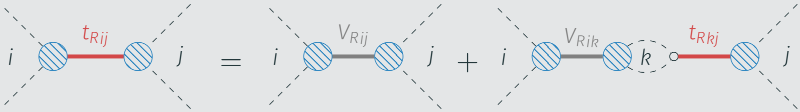
T_R can be written as



$$T_{Rij} = \xi_i \Gamma_{\text{out},i} t_{Rij} \Gamma_{\text{in},j}^{\dagger} \xi_j$$

Two-potential model: resonance t-matrix t_R

T_R can be connected to V_R as



$$t_{Rij} = V_{Rij} + V_{Rik} \underbrace{\xi_k G_k \Gamma_k \xi_k}_{\Sigma_k} t_{Rkj}$$

$$\Longrightarrow \quad t_R = \left[\mathbb{I}_C - V_R \Sigma\right]^{-1} V_R$$

Σ provides propagation of pions in the presence of elastic interaction.

Two-potential model: self-energy Σ_k

Self-energy for channel k is derived from the discontinuity equation:

$$\text{disc}\,[\Sigma_k] = \text{disc}\,[\xi_k G_k \Gamma_k \xi_k] = 2i\sigma_k \xi_k^2 |\Gamma_k|^2$$

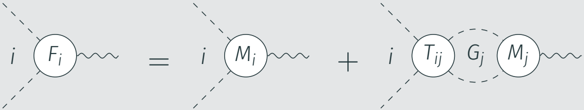
This allows for the integral solution for Σ_k:

$$\Sigma_k(s) = \frac{s}{\pi} \int_{s_{\text{thr},k}}^{\infty} \frac{ds'}{s'} \frac{\sigma_k(s')\xi_k^2(s')|\Gamma_k(s')|^2}{s' - s - i\epsilon}$$

Subtraction constant for Σ_k is absorbed in other model parameters.

Two-potential model: form factor F_i

Form factor F_i can be defined as



$$\xi_i F_i = \xi_i M_i + T_{ij} G_j \xi_j M_j$$

where M_k denote point-like source terms for channel k.

$$F_i = \Gamma_{\text{out},i} [\mathbb{I}_C - V_R \Sigma]^{-1}_{ij} M_j$$

$$\text{disc}\,[F_i] = 2i \sum_{k=1}^{n_C} T^*_{ik} \sigma_k (\xi_k / \xi_i) F_k$$

where n_C is the number of open channels.

Note: Watson's theorem does not hold above inelastic threshold.

Modeling the resonance potential and the source

The resonance potential

$$\bar{V}_{Rij}(s) = -\sum_{l,l'}^{n_R} g_i^{(l)} G^{(l,l')}(s) g_j^{(l')}, \quad V_{Rij}(s) = \bar{V}_{Rij}(s) - \bar{V}_{Rij}(0)$$

where n_R is the number of resonances and

$$G^{(l,l')}(s) = \frac{\delta_{l,l'}}{s - m_l^2}$$

The point-like source term

$$M_k(s) = c_k - \sum_{l,l'}^{n_R} g_k^{(l)} G^{(l,l')} \alpha^{(l')} s$$

$g_k^{(l)}, m_l, c_k, \alpha^{(l)}$ are *real* parameters of the fit.

Isospin-violating effects

ρ mixing with ω and ϕ

$$c_1 \to c_1 \left(1 + \kappa_\omega \frac{s}{s - m_\omega^2 + i m_\omega \Gamma_\omega} + \kappa_\phi \frac{s}{s - m_\phi^2 + i m_\phi \Gamma_\phi} \right)$$

- well-established effect;
- new fitting parameters: κ_ω and κ_ϕ;
- values for $m_{\omega/\phi}, \Gamma_{\omega/\phi}$ are taken from [PDG, 2020];
- M_1 acquires imaginary part $\implies$ breaks unitarity.

Two-potential model: parameters

Fitting parameters:

$$m_l - \ n_R \text{ resonance masses;}$$

$$g_i^{(l)} - \ n_C \times n_R \text{ channel-resonance couplings;}$$

$$\alpha^{(l)} - \ n_R \text{ resonance-photon couplings;}$$

$$c_k - \ n_C - 1 \text{ channel-photon couplings;}$$

$$\kappa_{\omega/\phi} - \ 2 \text{ strength parameters for } \omega/\phi \text{ mixing;}$$

Note: $c_1 = 1$ is fixed by charge conservation, so only $c_2, \ldots, c_{n_C}$ can be adjusted by the fit.

Application

Application: channels and factors

Three channels with $I = 1, L = 1$

1. $\pi^+\pi^-$ with $E_{\text{thr},1} = 2m_\pi \approx 279$ MeV;
2. 4π with $E_{\text{thr},2} = 4m_\pi \approx 558$ MeV;
3. $\pi^0\omega$ with $E_{\text{thr},3} = m_\pi + m_\omega \approx 922$ MeV;

Phase space, centrifugal barrier, cross section

For a 2-body channel:

$$\sigma_{ab}(s) = \frac{\sqrt{\lambda(s, m_a^2, m_b^2)}}{16\pi\ s}, \quad \xi_{ab}(s) = \sqrt{\frac{\lambda(s, m_a^2, m_b^2)}{3s}}$$

where $\lambda\left(s, m_a^2, m_b^2\right) = \left(s - (m_a + m_b)^2\right)\left(s - (m_a - m_b)^2\right)$ and $|\vec{p}_{\text{CMS}}| = \sqrt{\lambda/4s}$.

$$\sigma_{e^+e^-\to i} = \frac{e^4}{s^2}\sigma_i(s)\,[\xi_i(s)]^2\,|F_i(s)|^2$$

Application: phase space, centrifugal barrier, vertex

4π channel:

$$\sigma_2(s) = \frac{1}{16\pi}\sqrt{1 - \frac{16m_\pi^2}{s}}^{\,7}$$

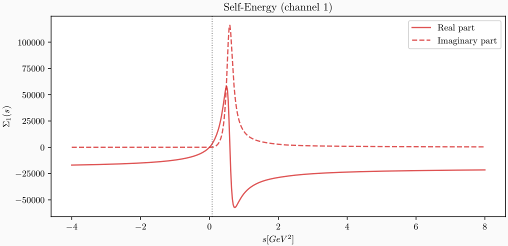
$$\xi_2(s) = \sqrt{\frac{s - 16m_\pi^2}{3}}$$

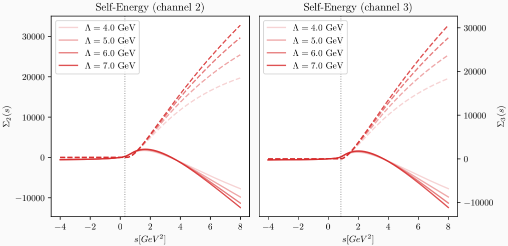
Vertex factor for 4π and $\pi^0\omega$ channels:

$$\Gamma_2(s) = \Gamma_3(s) = \frac{\Lambda^2}{\Lambda^2 + s}$$

$\Lambda \implies$ systematic uncertainties.

Application: self-energy



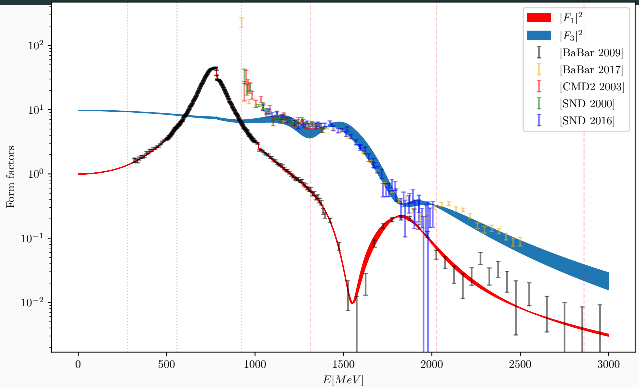


Application: fitting data

1. The $\pi\pi$ P-wave scattering phase: $\tilde{\delta}_1$
   [Caprini et al., 2012];
2. The vector pion form factor: F_V
   [BaBaR, 2012];
3. The $e^+e^- \to \pi^0\omega$ cross section: $\sigma_{e^+e^-\to\pi^0\omega}$
   [BaBaR, 2017, CMD-2, 2003, SND, 2000, SND, 2016];
4. The $\pi\pi$ elasticity parameter: η_1
   [García-Martín et al., 2011];
5. The non-2π over 2π cross section ratio: r
   [Eidelman and Łukaszuk, 2004].

Results

Results: issue with the ρ peak



Fitting parameters for 3 channels $[\pi\pi, 4\pi, \pi^0\omega]$ and 3 resonances:

$m_1 = 1313 \pm 13$ MeV	$g_{11} = -0.07 \pm 0.03$	$g_{21} = 0.12 \pm 0.48$	$g_{31} = -4.85 \pm 0.63$
$m_2 = 2027 \pm 27$ MeV	$g_{12} = -5.33 \pm 0.19$	$g_{22} = 2.89 \pm 0.16$	$g_{32} = -24.0 \pm 0.07$
$m_3 = 2860 \pm 51$ MeV	$g_{13} = 1.67 \pm 0.13$	$g_{23} = 25.0 \pm 1.5$	$g_{33} = -8.75 \pm 1.2$
$\alpha_1 = -0.56 \pm 0.01$	$\alpha_2 = -0.002 \pm 0.007$	$\alpha_3 = -0.14 \pm 0.03$	
$c_2 = 12.9 \pm 1.0$	$c_3 = 3.13 \pm 0.25$		
$\kappa_\omega = -0.002 \pm 0.0008$	$\kappa_\phi = 0.0005 \pm 0.0024$	χ^2/d.o.f. $= 3.13$ (*excluding r data set*).	

Removing input $\Omega[\tilde{\delta}_1](s)$

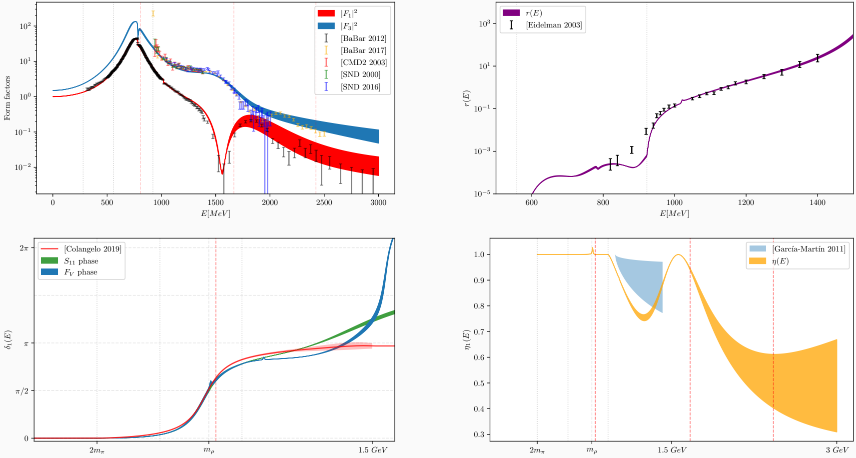
Two-potential model for pion vector form factor [Hanhart, 2012]

- $\Gamma_1 = \Omega[\tilde{\delta}_1](s)$;
- $\tilde{\delta}_1(s)$ is used as input;
- $\pi^0\omega$ couples with ρ via $\pi\pi$;
- contribution of ρ is included in the input phase;

Generalized Gounaris-Sakurai model [Gounaris and Sakurai, 1968]

- $\Gamma_1 = \Lambda^2/(\Lambda^2 + s)$
- $\tilde{\delta}_1(s)$ is part of the χ^2;
- $\pi^0\omega$ couples with ρ directly;
- contribution of ρ must be reconstructed within the model parameters;

Results for Gounaris-Sakurai model



Results for Gounaris-Sakurai model

Fitting parameters (3 channels, 3 resonances):

$m_1 = 806 \pm 2$ MeV	$m_2 = 1667 \pm 23$ MeV	$m_3 = 2423 \pm 36$ MeV
$g_{11} = -6.0 \pm 0.3$	$g_{12} = -0.53 \pm 0.17$	$g_{13} = 1.59 \pm 0.22$
$g_{21} = -0.50 \pm 0.07$	$g_{22} = -4.11 \pm 0.07$	$g_{23} = -7.73 \pm 0.27$
$g_{31} = 4.29 \pm 0.32$	$g_{32} = -24.5 \pm 1.1$	$g_{33} = -24.8 \pm 1.4$
$\alpha_1 = -0.47 \pm 0.02$	$\alpha_2 = -0.343 \pm 0.004$	$\alpha_3 = 0.13 \pm 0.01$
$c_2 = -1.2 \pm 2$	$c_3 = 1.22 \pm 0.16$	
$\kappa_\omega = 0.020 \pm 0.009$	$\kappa_\phi = 0.001 \pm 0.004$	

$\chi^2/\text{d.o.f.} = 5.05$ (*excluding* $\tilde{\delta}$ data set).

$\chi^2/\text{d.o.f.} = 88.32$ (*including* $\tilde{\delta}$ data set).

Conclusions

Two different dispersion-theoretical approaches were applied to $\pi\pi$ scattering in P-wave:

Omnès solution + resonances

- Perfect matching at low energies, scattering phase and elasticity;
- unable to reconstruct ρ peak in the $\pi^0\omega$ cross section;
- could be improved by introducing contact terms in the potential;

Resonance model

- The pion vector form factor as well as the $\pi^0\omega$ cross section data are described simultaneously;
- too simple to accurately reconstruct the scattering phase.

Thank you

Thank you for your attention!

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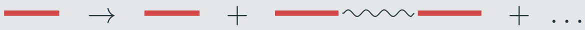
Backup slides

Isospin-violating effects

Resonances mixing with the photon



$$G \to \hat{G} \equiv \left[\mathbb{I}_R - Gs\alpha\alpha^T\right]^{-1} G$$

$$g_i^{(l)} \to \hat{g}_i^{(l)} \equiv g_i^{(l)} - e^2\alpha^{(l)}c_i$$

Isospin-violating effects

Resonances mixing with the photon

$$\bar{V}_{Rij}(s) = -\sum_{l,l'}^{n_R} \hat{g}_i^{(l)} \hat{G}^{(l,l')} \hat{g}_j^{(l')}, \quad V_{Rij}(s) = \bar{V}_{Rij}(s) - \bar{V}_{Rij}(0) - e^2 \frac{c_i c_j}{s}$$

$$M_k(s) = c_k - \sum_{l,l'}^{n_R} \hat{g}_k^{(l)} \hat{G}^{(l,l')} \alpha^{(l')} s$$

- No new parameters;
- To retain normalization at $s = 0$, we must transform c_i as

$$c_i \to c_i \left(1 + \frac{e^2}{\pi} \sum_k c_k^2 \int_{s_{\mathrm{thr},i}}^{\infty} \frac{ds' \sigma_k(s') \xi_k^2(s') |\Gamma_k(s')|^2}{(s')^2}\right)$$

Phase space for a 4-body state

n-particle phase space [PDG, 2020]

$$d\Phi_n = \delta^{(4)}\left(\sum_i p_i - \sum_j q_j\right) \prod_{j=1}^{n} \frac{d^3 q_j}{(2\pi)^3 2E_{\vec{q}_j}}$$

where $E_{\vec{q}_j} = \sqrt{m_j^2 + \vec{q}_j^2}$. p_i are the initial momenta, while q_j are the final ones.

Near-threshold scaling

$$d\Phi_n \sim |\vec{q}_M|^{3n-5}$$

where $|\vec{q}_M|$ is the maximum momentum observed in the final state at given energy $\sqrt{s}$.

For a final state of n identical particles with masses m,

$$|\vec{q}_M| \sim \sqrt{1 - \frac{(nm)^2}{s}}$$