PAVING A WAY FROM THE REFINED CHERN-SIMONS TO THE TOPOLOGICAL STRINGS

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This presentation is based on our joint work

M. Avetisyan and R. Mkrtchyan,

"On partition functions of refined Chern-Simons theories on S^3,"

Journal of High Energy Physics, vol. 2021, Oct 2021.

arXiv:2107.08679, (2021)

Universal representation of non-refined Chern-Simons on S^3

$$Z_{CS}(\kappa) = Vol(Q^\vee)^{-1}(\delta)^{-\frac{r}{2}}\prod_{\alpha_+} 2\sin\pi\frac{(\alpha,\rho)}{\delta}$$

R.Mkrtchyan, A.Veselov (2012), R.Mkrtchyan (2013)

Algebraic data <-> universal parameters

Universal dimension

Universal quantum dimension

$$Z^U_{CS}(\kappa) = \left(\frac{t}{\delta}\right)^{\frac{dim\mathfrak{g}}{2}} exp\left(-\int_0^\infty \frac{dx}{x(e^x-1)}\left(f\left(\frac{x}{\delta}\right)-f\left(\frac{x}{t}\right)\right)\right)$$

Vogel's universality

Universal formulae

Vogel's projective coordinates/plane

$$(\alpha, \beta, \gamma),$$
$$t = \alpha + \beta + \gamma$$

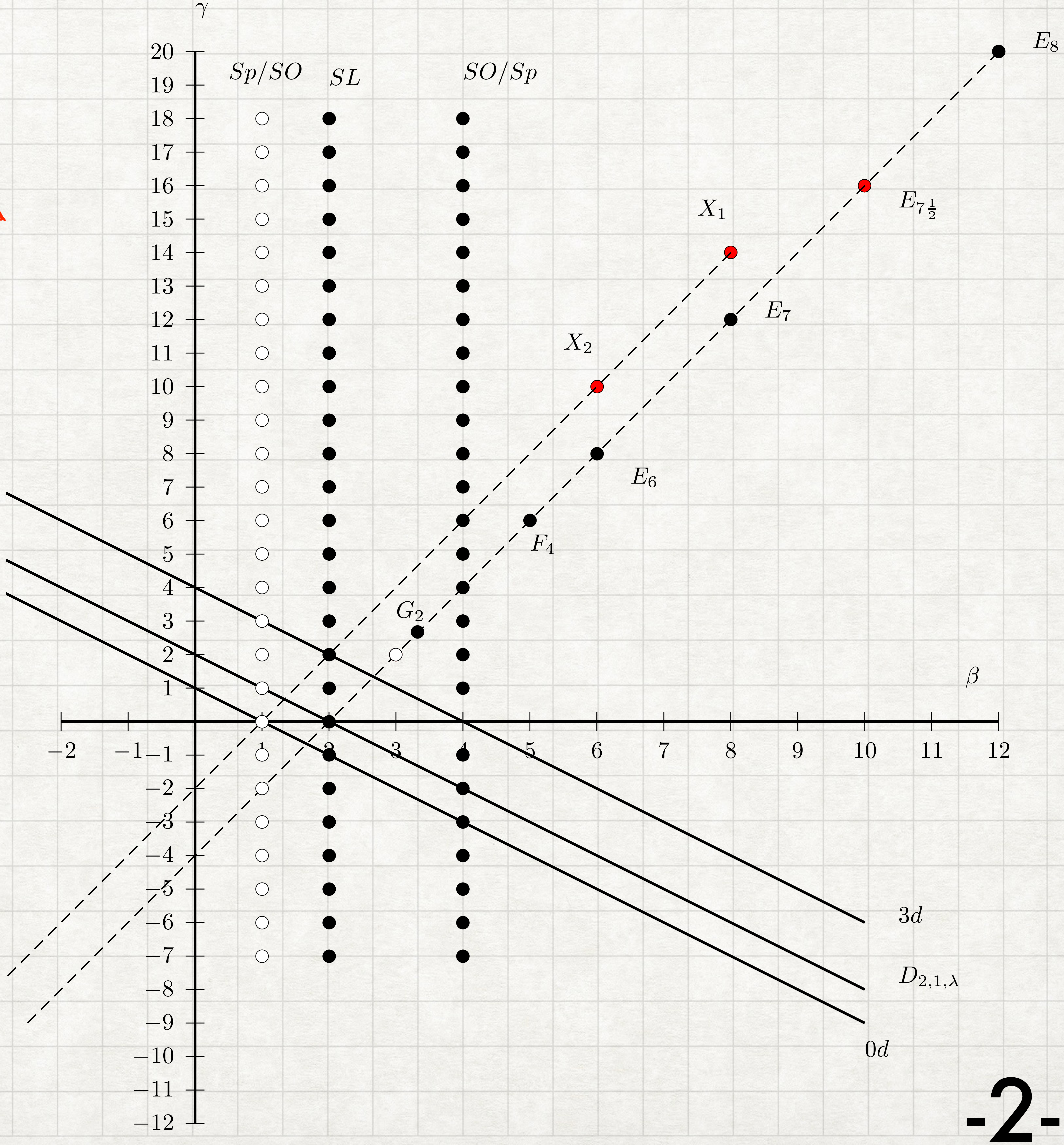
$$f(x, \alpha, \beta, \gamma) = \frac{\sinh(x\frac{\alpha - 2t}{4})}{\sinh(x\frac{\alpha}{4})} \frac{\sinh(x\frac{\beta - 2t}{4})}{\sinh(x\frac{\beta}{4})} \frac{\sinh(x\frac{\gamma - 2t}{4})}{\sinh(x\frac{\gamma}{4})}$$

$$dim\mathfrak{g} = lim_{x\to 0} f(x) = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma}$$

P. Vogel (1999),

B.Westbury (2004)

R. Mkrtchyan, A.Veselov (2012)



Outputs of universal representation of $Z_{CS}(\kappa)$

Chern-Simons on S^3

$Z_{CS}(\kappa)$

R.Mkrtchyan, A.Veselov (2012),

R. Mkrtchyan (2013)

Universal

Chern-Simons on S^3

$Z^U_{CS}(\kappa)$

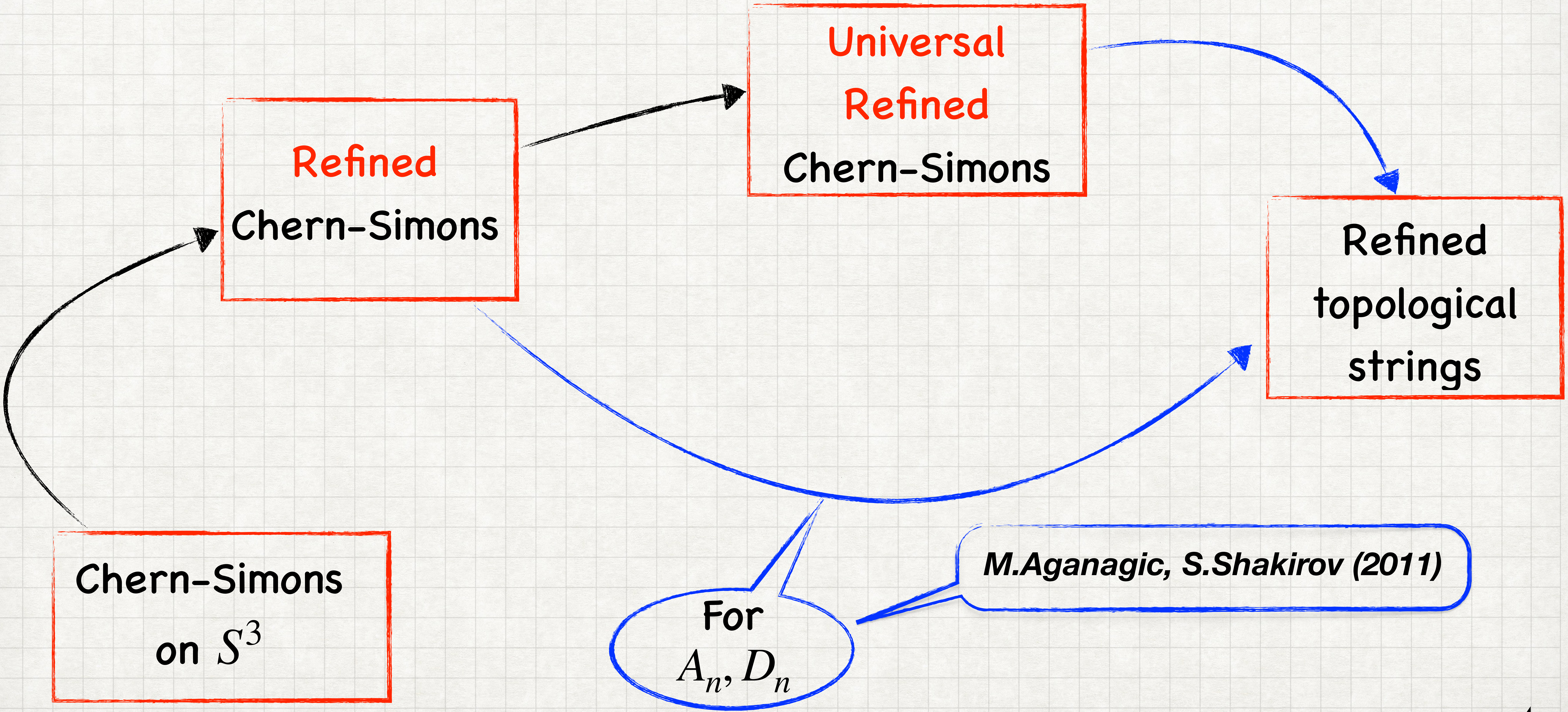
R.Mkrtchyan, (2013, 2014, 2020)

D.Krefl, R.Mkrtchyan (2015)

Gopakumar-Vafa form of partition function of topological strings

$Z_{top.strings}$

Formulation of the problem



Refinement of Chern-Simons on S^3

$$Z_{CS}(\kappa) = Vol(Q^\vee)^{-1}(\delta)^{-\frac{r}{2}}\prod_{\alpha_+} 2\sin\pi\frac{(\alpha,\rho)}{\delta}$$

$$Z^{ref}_{CS}(\kappa,1) = Z_{CS}(\kappa);$$

$$Z^{ref}_{CS}(0,y) = Z_{CS}(0) = 1.$$

$$Z^{ref}_{CS}(\kappa,y) = Vol(Q^\vee)^{-1}\delta^{-\frac{r}{2}}\prod_{m=0}^{y-1}\prod_{\alpha_+} 2\sin\pi\frac{y(\alpha,\rho)-m(\alpha,\alpha)/2}{\delta}$$

It is based on the refined formula of the volume of the fundamental domain of the coroot lattice $Vol(Q^\vee)$

$$Vol(Q^\vee) = (\det(\alpha_i^\vee, \alpha_j^\vee))^{1/2} = t^{-\frac{r}{2}} \prod_{\alpha_+} 2 \sin \pi \frac{(\alpha, \rho)}{t}$$

$$\alpha_i^\vee = \alpha_i \frac{2}{(\alpha_i, \alpha_i)}, \ i = 1,\ldots,r$$

V.Kac, D.Peterson (1984)

$$Vol(Q^\vee) = (ty)^{-\frac{r}{2}} \prod_{m=0}^{y-1} \prod_{\alpha_+} 2 \sin \pi \frac{y(\alpha, \rho) - m(\alpha, \alpha)/2}{ty}$$

$$N = \prod_{k=1}^{N-1} 2 \sin \pi \frac{k}{N}$$

Refined CS partition function rewrites as follows:

$$Z_{CS}^{ref}(\kappa, y) = \left(\frac{ty}{\delta}\right)^{\frac{r}{2}} \prod_{m=0}^{y-1} \prod_{\alpha_+} \frac{\sin \pi \frac{y(\alpha,\rho) - m(\alpha,\alpha)/2}{\delta}}{\sin \pi \frac{y(\alpha,\rho) - m(\alpha,\alpha)/2}{ty}}$$

Refinement parameter (y)

$\alpha + \beta + \gamma$ (t)

Rank of an algebra (r)

Weyl vector (ρ)

Coupling constant (κ)

$\kappa + ty$ (δ)

Positive roots (α_+)

Integral representation of refined Chern-Simons

$$Z_{CS}^{ref}(\kappa, y) = exp\left(-\frac{1}{4}\int_{R_+} \frac{dx}{x} \frac{\sinh\left(x(ty-\delta)\right)}{\sinh\left(xty\right)\sinh\left(x\delta\right)} F_X(2x, y) \right)$$

$$r + \sum_{m=0}^{y-1} \sum_{\alpha_+} \left(e^{2x(y(\alpha,\rho)-m(\alpha,\alpha)/2)} + e^{-2x(y(\alpha,\rho)-m(\alpha,\alpha)/2)} \right)$$

Refined quantum dimension of the adjoint

$$f(x, \alpha, \beta, \gamma, t)$$

Is $F_X(x,y)$ universal?

$$F_X(x,y) = r + \sum_{m=0}^{y-1} \sum_{\alpha_+} \left(e^{x(y(\alpha,\rho) - m(\alpha,\alpha)/2)} + e^{-x(y(\alpha,\rho) - m(\alpha,\alpha)/2)} \right)$$

$$F_X(x,1) = f(x,\alpha,\beta,\gamma,t) = \frac{\sinh(x\frac{\alpha-2t}{4})}{\sinh(x\frac{\alpha}{4})} \frac{\sinh(x\frac{\beta-2t}{4})}{\sinh(x\frac{\beta}{4})} \frac{\sinh(x\frac{\gamma-2t}{4})}{\sinh(x\frac{\gamma}{4})}$$

Stands for an algebra

$$F_X(x,y) \stackrel{?}{=} \tilde{F}(x,y,\alpha,\beta,\gamma)$$

Previous knowledge on this question

$$f(x, \alpha, \beta, \gamma, t)$$

$(\alpha, \beta, \gamma, t) \to (\alpha, y\beta, y\gamma, yt)$

D.Krefl, A.Shwartz (2015)

For $X = A_n$ and D_n algebras

$$f(x, \alpha, y\beta, y\gamma, yt) = F_X(x, y)$$

$$F_{A_n,D_n}(x,y) = \frac{\sinh(x\frac{\alpha - 2ty}{4})}{\sinh(x\frac{\alpha}{4})}\frac{\sinh(xy\frac{\beta - 2t}{4})}{\sinh(xy\frac{\beta}{4})}\frac{\sinh(xy\frac{\gamma - 2t}{4})}{\sinh(xy\frac{\gamma}{4})}$$

We extend this result to all simply-laced algebras

$$F_X(x,y) = f(x,\alpha,y\beta,y\gamma,yt) = \frac{\sinh(x\frac{\alpha-2ty}{4})}{\sinh(x\frac{\alpha}{4})}\frac{\sinh(xy\frac{\beta-2t}{4})}{\sinh(xy\frac{\beta}{4})}\frac{\sinh(xy\frac{\gamma-2t}{4})}{\sinh(xy\frac{\gamma}{4})}$$

For $X = A_n$, D_n and $E_n, n = 6,7,8$ algebras

$Z_{CS}^{ref}(\kappa, y)$ is universal for all simply-laced (ADE) algebras

What about non-simply-laced algebras?

For $X = B_n$, C_n ,
and G_2, F_4 algebras

$$F_X(x, y) \neq \text{“Universal-form” function}$$

Ratio of products of sinhs

However, it rewrites as

$$\frac{A_X(x, y)}{B_X(x, y)}$$

Polynomial in e^x:
$$\sum_l e^{xl}$$

Product of the form
$$\prod_a (e^{xa} - 1)$$

An example: for C_n series

$$(e^x+1)e^{xy}\left(e^{2xny}-1\right)\left(e^{2xny+1}-1\right)+$$

$$+\left(e^{2xy}-1\right)\left(e^{xny}-1\right)\left(e^{xny+1}-1\right)\left(e^{2xny+1}-1\right)$$

$$F_{C_n}(x,y)=\frac{A_{C_n}(x,y)}{B_{C_n}(x,y)}$$

$$(e^{2x}-1)\left(e^{2xy}-1\right)$$

What we have overall?

$$Z^{ref}_{CS}(\kappa, y) = exp\left(-\frac{1}{4}\int_{R_+} \frac{dx}{x}\frac{\sinh\left(x(ty-\delta)\right)}{\sinh\left(xty\right)\sinh\left(x\delta\right)} F_X(2x,y)\right)$$

For simply-laced (ADE) algebras

$$f(2x, \alpha, y\beta, y\gamma, yt)$$

For non-simply-laced algebras

$$\frac{\sum_l e^{xl}}{\prod_a (e^{xa}-1)}$$

Future plans

$$Z_{CS}^{ref} = exp\left(-\frac{1}{4}\int_{R_+} \frac{dx}{x}\frac{\sinh\left(x(ty-\delta)\right)\sum_l e^{xl}}{\sinh\left(xty\right)\sinh\left(x\delta\right)\prod_a (e^{xa}-1)}\right)$$

Product of multiple sines: $S_r(z\,|\,\underline{\omega})$ functions

$$\exp\left((-1)^r\frac{\pi i}{r!}B_{r,r}(z\,|\,\underline{\omega}) + (-1)^r\int_{R_+}\frac{dx}{x}\frac{e^{zx}}{\prod_{k=1}^r (e^{\omega_i x}-1)}\right)$$

Refined topological strings ?

THANKS!