Commutator of higher spin gauge transformations in linear approach

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Introduction

We construct some special local quartic interaction of two scalars and two spin four fields using standard Noether's procedure.

Setup

The starting lagrangian for our task is

$$S^{\Phi\Phi h^{(4)}}(\Phi, h^{(2)}, h^{(4)}) \quad = \quad S_0(\Phi) + S_1(\Phi, h^{(2)}) + S_1(\Phi, h^{(4)}), \tag{1}$$

where

$$S_0(\Phi) \quad = \quad \frac{1}{2}\int d^d x \partial_\mu \Phi \partial^\mu \Phi, \tag{2}$$

$$S_1(\Phi, h^{(2)}) \quad = \quad \frac{1}{2}\int d^d x h^{(2)\mu\nu}\Big[-\partial_\mu \Phi \partial_\nu \Phi + \frac{\eta_{\mu\nu}}{2}\partial_\lambda \Phi \partial^\lambda \Phi \Big]. \tag{3}$$

$$S_1(\Phi, h^{(4)}) = \frac{1}{4}\int d^d x h^{(4)\mu\nu\alpha\beta}[\partial_\mu \partial_\nu \Phi \partial_\alpha \partial_\beta \Phi - \eta_{\mu\nu}\partial_\alpha \partial^\gamma \Phi \partial_\beta \partial_\gamma \Phi]. \tag{4}$$

Gauge transformations

Gauge transformatinos

$$\delta_1 \Phi(x) = \varepsilon^{\mu\nu\lambda}(x)\partial_\mu \partial_\nu \partial_\lambda \Phi(x), \quad (5)$$

$$\delta_0 h^{\mu\nu\lambda\rho} = \partial^{(\mu}\varepsilon^{\nu\lambda\rho)} = \partial^\mu \varepsilon^{\nu\lambda\rho} + \partial^\nu \varepsilon^{\mu\lambda\rho} + \partial^\lambda \varepsilon^{\mu\nu\rho} + \partial^\rho \varepsilon^{\mu\nu\lambda}, \quad (6)$$

$$\delta_0 h^{\mu\nu}_{(2)} = \partial^{(\mu}\epsilon^{\nu)}, \quad (7)$$

$$\epsilon^\nu = \partial_\alpha \partial_\beta \varepsilon^{\nu\alpha\beta}. \quad (8)$$

We use physical traceless and transverse gauge for our external spin four field:

$$\partial_\mu h^{\mu\nu\lambda\rho} = 0, \quad (9)$$

$$h^{\mu\lambda\rho}_\mu = 0, \quad (10)$$

Neother equation

Starting from the cubic term

$$L_1 \sim h^{\mu\nu\lambda\rho}\partial_\mu\partial_\nu\Phi\partial_\lambda\partial_\rho\Phi, \tag{11}$$

using known variation:

$$\delta_1\Phi = \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma\Phi, \tag{12}$$

$$\delta_0 h^{\mu\nu\lambda\rho} = \partial^{(\mu}\varepsilon^{\nu\lambda\rho)}, \tag{13}$$

we try to solve functional equation:

$$\delta_1 L_1(\Phi, h^{(4)}) + \delta_0 L_2(\Phi, h^{(4)}) = 0, \tag{14}$$

First order variation

$$
\begin{aligned}
\delta_1(h^{\mu\nu\lambda\rho}\partial_\mu\partial_\nu\Phi\partial_\lambda\partial_\rho\Phi) &= \frac{1}{3}\delta_1(h^{\mu\nu\lambda\rho}J^{(4)}_{\mu\nu\lambda\rho}) = \delta_1 h^{\mu\nu\lambda\rho}\partial_\mu\partial_\nu\Phi\partial_\lambda\partial_\rho\Phi \\
&+ \frac{1}{50}\left[\varepsilon^{\mu(\alpha\beta}\partial_\mu h^{\gamma\nu\lambda\rho)} - \partial_\mu\varepsilon^{(\alpha\beta\gamma}h^{\nu\lambda\rho)\mu}\right]J^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} \\
&+ \frac{1}{5}\left[\partial_\alpha\varepsilon^{\mu\nu(\beta}\partial_\mu\partial_\nu h^{\gamma\lambda\rho)\alpha} - \partial_\mu\partial_\nu\varepsilon^{\alpha(\beta\gamma}\partial_\alpha h^{\lambda\rho)\mu\nu}\right]J^{(4)}_{\lambda\rho\beta\gamma} \\
&+ \frac{2}{15}\left[\partial_\alpha\partial_\beta\partial_\gamma\varepsilon^{(\mu\nu\lambda}h^{\rho)\alpha\beta\gamma} - \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho}\right]J^{(4)}_{\mu\nu\lambda\rho} \\
&+ \frac{1}{5}[\partial_\mu\partial_\nu\varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho} - \partial_\mu\partial_\nu\partial_\gamma\varepsilon^{\alpha\beta(\lambda}\partial_\alpha\partial_\beta h^{\rho)\mu\nu\gamma}]J^{(2)}_{\lambda\rho} \\
&+ \frac{1}{5}\partial_\mu\partial_\nu\partial_\lambda\partial_\rho\varepsilon^{\alpha\beta\gamma}\partial_\alpha h^{\mu\nu\lambda\rho}J^{(2)}_{\beta\gamma}. \qquad (15)
\end{aligned}
$$

First order variation

- From second line of (15) follows that *we cannot integrate Noether's equation without introduction of the cubic interaction with a gauge field of spin 6 coupled to the spin 6 current:*

$$h_{(6)}^{\nu\lambda\rho\alpha\beta\gamma} J^{(6)}_{\nu\lambda\rho\alpha\beta\gamma}. \tag{16}$$

- From third and fourth lines we see that $J^{(4)}$ terms arose with different weight $\frac{1}{5}$ and $\frac{2}{15}$. But we will see below that they should come with same weight to complete integration for interaction terms.
- In last two lines we have three unwanted $J^{(2)}$ terms. We should discover way to get rid of them.

Final variation

After long calculations we present the final variation obtained by the tuning procedure.

$$\begin{aligned}
\delta_1(h^{\mu\nu\lambda\rho}\partial_\mu\partial_\nu\Phi\partial_\lambda\partial_\rho\Phi) &= \frac{1}{3}\delta_1(h^{\mu\nu\lambda\rho}J^{(4)}_{\mu\nu\lambda\rho}) = \delta_1 h^{\mu\nu\lambda\rho}\partial_\mu\partial_\nu\Phi\partial_\lambda\partial_\rho\Phi \\
&+ \frac{1}{50}\left[\varepsilon^{\mu(\alpha\beta}\partial_\mu h^{\gamma\nu\lambda\rho)} - \partial_\mu\varepsilon^{(\alpha\beta\gamma}h^{\nu\lambda\rho)\mu}\right]\tilde{J}^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} \\
&+ \frac{1}{6}\left[\partial_\alpha\varepsilon^{\mu\nu(\beta}\partial_\mu\partial_\nu h^{\gamma\lambda\rho)\alpha} - \partial_\mu\partial_\nu\varepsilon^{\alpha(\beta\gamma}\partial_\alpha h^{\lambda\rho)\mu\nu}\right]J^{(4)}_{\lambda\rho\beta\gamma} \\
&+ \frac{1}{6}\left[\partial_\alpha\partial_\beta\partial_\gamma\varepsilon^{(\mu\nu\lambda}h^{\rho)\alpha\beta\gamma} - \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h^{\mu\nu\lambda\rho}\right]J^{(4)}_{\mu\nu\lambda,\rho}
\end{aligned} \tag{17}$$

$$\tilde{J}^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} = J^{(6)}_{\nu\lambda\rho\alpha\beta\gamma} + \frac{1}{9}\partial_{(\alpha}\partial_\beta J^{(4)}_{\gamma\nu\lambda\rho)} + \frac{1}{3}\partial_{(\nu}\partial_\lambda\partial_\rho\partial_\alpha J^{(2)}_{\beta\gamma)}. \tag{18}$$

Important results

Now we can start to integrate the last three lines of expression (17)

$$S_2(\Phi, h^{(4)}) = \int d^d x \Big\{ \frac{1}{10} h_\mu^{\alpha\beta\gamma} h^{\nu\lambda\rho\mu} \tilde{J}^{(6)}_{\nu\lambda\rho\alpha\beta\gamma}$$
$$-\frac{2}{3} h_\mu^{\alpha\beta\gamma} \partial_\alpha \partial_\beta h^{\mu\nu\lambda\rho} J^{(4)}_{\nu\lambda\rho\gamma} + \frac{1}{2} \partial_\nu h_\mu^{\alpha\beta\gamma} \partial_\alpha h^{\mu\nu\lambda\rho} J^{(4)}_{\lambda\rho\beta\gamma} - \frac{1}{4} \partial^\alpha h^{\beta\gamma}_{\mu\nu} \partial_\alpha h^{\mu\nu\lambda\rho} J^{(4)}_{\lambda\rho\beta\gamma}$$
$$-\partial^\beta h^{\alpha\gamma}_{\mu\nu} \partial_\alpha h^{\mu\nu\lambda\rho} J^{(4)}_{\lambda\rho\beta\gamma} + \frac{1}{3} \partial^\beta h^\gamma_{\mu\nu\lambda} \partial^\alpha h^{\mu\nu\lambda\rho} J^{(4)}_{\rho\alpha\beta\gamma} - \frac{1}{4} h^{\beta\gamma}_{\mu\nu} h^{\lambda\rho\mu\nu} \Box J^{(4)}_{\lambda\rho\beta\gamma} \Big\}, \tag{19}$$

and linear on spin four gauge field transformations fixed by Noether's procedure:

$$\delta_1 h^{\mu\nu\lambda\alpha\beta\gamma}_{(6)} = \varepsilon^{\rho(\alpha\beta} \partial_\rho h^{\gamma\mu\nu\lambda)} + \partial^{(\alpha} \varepsilon^{\beta\gamma}_\rho h^{\mu\nu\lambda)\rho}, \tag{20}$$

$$\delta_1 h^{\mu\nu\lambda\rho} = \varepsilon^{\alpha\beta\gamma} \partial_\alpha \partial_\beta \partial_\gamma h^{\mu\nu\lambda\rho} + \partial^{(\mu} \varepsilon_\gamma^{|\alpha\beta|} \partial_\alpha \partial_\beta h^{\nu\lambda\rho)\gamma} + \partial^{(\mu} \partial^\nu \varepsilon^{|\alpha|}_{\beta\gamma} \partial_\alpha h^{\lambda\rho)\beta\gamma}$$
$$+\partial^{(\mu} \partial^\nu \partial^\lambda \varepsilon_{\alpha\beta\gamma} h^{\rho)\alpha\beta\gamma}, \tag{21}$$

$$\delta_1 h^{\beta\gamma}_{(2)} = \partial_\mu \partial_\nu \partial_\lambda \partial_\rho \varepsilon^{\alpha\beta\gamma} \partial_\alpha h^{\mu\nu\lambda\rho}. \tag{22}$$

Expressing variation with Higher spin Christoffel symbols

$$\delta_1^{(\varepsilon)} h_{\mu\nu\lambda\rho} = \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta\partial_\gamma h_{\mu\nu\lambda\rho} + \partial_{(\mu}\varepsilon^{\alpha\beta\gamma}\partial_{|\alpha}\partial_\beta h_{\gamma|\nu\lambda\rho)} + \partial_{(\mu}\partial_\nu\varepsilon^{\alpha\beta\gamma}\partial_{|\alpha} h_{\beta\gamma|\lambda\rho)} + \partial_{(\mu}\partial_\nu\partial_\lambda\varepsilon^{\alpha\beta\gamma} h_{\rho)\alpha\beta\gamma}. \quad (23)$$

$$\delta_1^{(\varepsilon)} h_{\mu\nu\lambda\rho} = \varepsilon^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) + \partial_{(\mu}\Lambda_{\nu\lambda\rho)}(\varepsilon, h), \quad (24)$$

$$\Lambda_{\nu\lambda\rho}(\varepsilon, h) = \varepsilon^{\alpha\beta\gamma}\partial_\alpha\partial_\beta h_{\gamma\nu\lambda\rho} + \frac{1}{2}\left[\partial_{(\nu}\varepsilon^{\alpha\beta\gamma}\partial_{|\alpha} h_{\beta\gamma|\lambda\rho)} - \varepsilon^{\alpha\beta\gamma}\partial_{(\nu}\partial_{|\alpha} h_{\beta\gamma|\lambda\rho)}\right] + \frac{1}{3}\left[\partial_{(\nu}\partial_\lambda\varepsilon^{\alpha\beta\gamma} h_{\rho)\alpha\beta\gamma} + \varepsilon^{\alpha\beta\gamma}\partial_{(\nu}\partial_\lambda h_{\rho)\alpha\beta\gamma} - \frac{1}{2}\partial_{(\nu}\varepsilon^{\alpha\beta\gamma}\partial_\lambda h_{\rho)\alpha\beta\gamma}\right], \quad (25)$$

Expressing variation with Higher spin Christoffel symbols

$$\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) = \partial_\alpha\partial_\beta\partial_\gamma h_{\mu\nu\lambda\rho} - \frac{1}{3}\partial_{<\alpha}\partial_\beta\partial_{(\mu} h_{\nu\lambda\rho)\gamma>} + \frac{1}{3}\partial_{<\alpha}\partial_{(\mu}\partial_\nu h_{\lambda\rho)\beta\gamma>} - \partial_{(\mu}\partial_\nu\partial_\lambda h_{\rho)\alpha\beta\gamma}, \tag{26}$$

Two type of trivial terms

Initial gauge transformation regulated by Noether's procedure

$$\delta^{(\epsilon)} h_{\mu\nu\lambda\rho} = (\delta_0^{(\epsilon)} + \delta_1^{(\epsilon)} + \delta_2^{(\epsilon)} + \dots) h_{\mu\nu\lambda\rho}, \tag{27}$$

for commutator on the linear level on gauge field we obtain:

$$\left\{ [\delta^{(\omega)}, \delta^{(\epsilon)}] h_{\mu\nu\lambda\rho} \right\}_1 = ([\delta_1^{(\omega)}, \delta_1^{(\epsilon)}] + \delta_0^{(\omega)} \delta_2^{(\varepsilon)} - \delta_0^{(\varepsilon)} \delta_2^{(\omega)}) h_{\mu\nu\lambda\rho}. \tag{28}$$

Two type of trivial terms

- Symmetrized full derivatives from composed gauge parameter linear in gauge fields $\partial_{(\mu}\tilde{\Lambda}_{\nu\lambda\rho)}(\varepsilon, \omega, h$ $(\varepsilon \leftrightarrow \omega)$.
- The terms which can be classified as a second part of r.h.s of (28): $\delta_0^{(\omega)}\delta_2^{(\varepsilon)}h_{\mu\nu\lambda\rho} - (\varepsilon \leftrightarrow \omega)$, and we can throw them out also to understand algebra of two δ_1 transformations.

Final Result

The final result for commutator hiding long and tedious calculation:

$$
\begin{aligned}
&[\delta_1^{(\omega)}, \delta_1^{(\varepsilon)}]h_{\mu\nu\lambda\rho} \sim \left[\varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\partial_\eta\omega^{\alpha\beta\gamma} + T^{\alpha\beta\gamma}(\partial,\varepsilon,\omega)\right]\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h) \\
&+3\varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\omega^{\alpha\beta\gamma}R^{(4)}_{\eta\alpha\beta\gamma;\mu\nu\lambda\rho}(h) + \frac{9}{20}\varepsilon_\delta^{\sigma\eta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{(\sigma}R^{(4)}_{\eta\alpha\beta\gamma);\mu\nu\lambda\rho}(h) \\
&+[Rem]_{\mu\nu\lambda\rho}(\varepsilon,\omega,h) - (\varepsilon \leftrightarrow \omega)\,, \qquad (29)
\end{aligned}
$$

where:

$$
\begin{aligned}
T^{\alpha\beta\gamma}(\partial,\varepsilon,\omega) \;=\;& \frac{1}{4}\partial^{(\alpha}\partial^\beta\varepsilon^{\delta\sigma\eta}\delta_0^{(\omega)}h^{\gamma)}_{\delta\sigma\eta} - \frac{5}{48}\partial^{(\alpha}\varepsilon^{\delta\sigma\eta}\partial^\beta\delta_0^{(\omega)}h^{\gamma)}_{\delta\sigma\eta} + \frac{7}{16}\partial^{(\alpha}\varepsilon^{\delta\sigma\eta}\partial_\delta\delta_0^{(\omega)}h_{\sigma\eta}^{\;\;\;\beta\gamma)} \\
&- \frac{1}{16}\partial^\delta\varepsilon^{\sigma\eta(\alpha}\partial^\beta\delta_0^{(\omega)}h^{\gamma)}_{\delta\sigma\eta} + \frac{1}{16}\partial^\delta\varepsilon^{\sigma\eta(\alpha}\partial_\delta\delta_0^{(\omega)}h^{\beta\gamma)}_{\sigma\eta}\,, \qquad (30)
\end{aligned}
$$

Important remarks

1. The first line describes spin four gauge transformation with composite *symmetric rank 3 tensor parameter* in the form

$$[\omega, \varepsilon]^{\alpha\beta\gamma}\Gamma^{(3)}_{\alpha\beta\gamma;\mu\nu\lambda\rho}(h), \tag{31}$$

where

$$[\omega, \varepsilon]^{\alpha\beta\gamma} = \varepsilon^{\delta\sigma\eta}\partial_\delta\partial_\sigma\partial_\eta\omega^{\alpha\beta\gamma} + T^{\alpha\beta\gamma}(\partial, \varepsilon, \omega) - (\varepsilon \leftrightarrow \omega). \tag{32}$$

Important remarks

1. The second line also corresponds to the transformation of the spin four gauge field in respect to gauge transformation with symmetric tensor parameter. But in this case we have *symmetric tensor parameters of rank 4 and 5*, which means that it is transformation coming from gauge field with spin 5 and 6.

$$\Omega_{(4)}^{\eta\alpha\beta\gamma\delta}(\varepsilon,\omega)R^{(4)}_{\eta\alpha\beta\gamma;\mu\nu\lambda\rho}(h), \tag{33}$$

$$\Omega_{(5)}^{\sigma\eta\alpha\beta\gamma\delta}(\varepsilon,\omega)\partial_{(\sigma}R^{(4)}_{\eta\alpha\beta\gamma);\mu\nu\lambda\rho}(h), \tag{34}$$

where

$$\Omega_{(4)}^{\eta\alpha\beta\gamma\delta}(\varepsilon,\omega) = \frac{3}{4}\varepsilon^{\delta\sigma(\eta}\partial_\delta\partial_\sigma\omega^{\alpha\beta\gamma)} - (\varepsilon\leftrightarrow\omega), \tag{35}$$

$$\Omega_{(5)}^{\sigma\eta\alpha\beta\gamma\delta}(\varepsilon,\omega) = \frac{9}{200}\varepsilon^{\delta(\sigma\eta}\partial_\delta\omega^{\alpha\beta\gamma)} - \frac{3}{200}\varepsilon_\delta^{(\sigma\eta}\partial^\alpha\omega^{\beta\gamma)\delta} - (\varepsilon\leftrightarrow\omega). \tag{36}$$

Mixed term

$$[Rem]_{\mu\nu\lambda\rho}(\varepsilon,\omega,h) =$$

$$\frac{9}{20}\varepsilon_{\delta}^{\eta\sigma}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{(\mu}R^{(3)}_{\nu\lambda\rho);\eta\beta\gamma}(H^{(3)}_{[\alpha\sigma]}) + \frac{3}{2}\partial_{(\mu}\varepsilon_{\delta}^{\eta\sigma}\partial^{[\delta}\omega^{\alpha]\beta\gamma}R^{(3)}_{\nu\lambda\rho);\eta\beta\gamma}(H^{(3)}_{[\alpha\sigma]}) - \frac{9}{40}\varepsilon_{\delta}^{\eta\sigma}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{(\mu}R^{(3)}_{\nu\lambda\rho);}$$

$$+\frac{3}{8}\varepsilon^{\eta}_{\sigma\delta}\partial^{[\sigma}\partial^{\left[\delta\right.}\omega^{\left.\alpha\right]\beta]\gamma}\partial_{(\mu}\Gamma^{(2)}_{\beta\gamma;\nu\lambda\rho)}(H^{(3)}_{[\eta\alpha]}) + \frac{1}{2}\partial_{(\mu}\varepsilon^{\eta}_{\delta\sigma}\partial^{[\sigma}\partial^{\left[\delta\right.}\omega^{\left.\alpha\right]\beta]\gamma}\Gamma^{(2)}_{\beta\gamma;\nu\lambda\rho)}(H^{(3)}_{[\eta\alpha]})$$

$$+\frac{3}{8}\varepsilon_{\delta}^{\sigma\eta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{(\mu}\partial_{\nu}\Gamma^{(1)}_{\gamma;\lambda\rho)}(H^{(2)}_{[\eta\alpha][\sigma\beta]}) + \frac{1}{2}\partial_{(\mu}\varepsilon_{\delta}^{\sigma\eta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\partial_{\nu}\Gamma^{(1)}_{\gamma;\lambda\rho)}(H^{(2)}_{[\eta\alpha][\sigma\beta]})$$

$$+\frac{3}{4}\partial_{(\mu}\partial_{\nu}\varepsilon_{\delta}^{\sigma\eta}\partial^{[\delta}\omega^{\alpha]\beta\gamma}\Gamma^{(1)}_{\gamma;\lambda\rho)}(H^{(2)}_{[\eta\alpha][\sigma\beta]}) \qquad (37)$$

Mixed term

Now we analyze the third line of (29) or eight terms in expression (37). First of all we see that in this remaining part of commutator our spin four field expressed through the reduced curvatures and Christoffel symbols.
All such a objects possess one (first two lines of (37)) ore two (remaining two lines of (37)) pair of antisymmetrized indices contracted with composed gauge parameter. Therefore the could describe some mixed symmetry field gauge transformation acting on spin four symmetric gauge field. For example first term in (37) we can rewrite in the form:

$$\Omega^{[\alpha\sigma],\eta\beta\gamma}_{[2],(3)}\partial_{(\mu}R^{(3)}_{\nu\lambda\rho);\eta\beta\gamma}(H^{(3)}_{[\alpha\sigma]}) \tag{38}$$

where

$$\Omega^{[\alpha\sigma],\eta\beta\gamma}_{[2],(3)} = \frac{3}{40}(\varepsilon^{\delta(\eta[\sigma}\partial_\delta\omega^{\alpha]\beta\gamma)} - \varepsilon^{(\eta[\sigma}_{\delta}\partial^{\alpha]}\omega^{\beta\gamma)\delta}) \tag{39}$$

Summary

We see that our *commutator of spin four linear on gauge field transformations produce regular terms coming from gauge transformation of symmetric tensors with spin* $s < 6$ and remaining irregular transformation with mixed symmetry gauge field parameters.

Thank you