Evgeny Epelbaum, RUB, Germany

Fifth Autumn School & Workshop of RTN, Tbilisi, Georgia

Nuclear theory
at the precision frontier

The landscape of computational nuclear physics

Ultimate goal: predictive & systematically improvable QCD-based theory for nuclei/ nuclear reactions/nuclear matter with quantified uncertainties

The method: chiral EFT for nuclear forces/currents + ab-initio „few"-body approaches

[Faddeev-Yakubovski, No Core Shell Model, Quantum Monte Carlo, Lorentz Integral Transform, Coupled Cluster, Lattice, self-consistent Gorkov-Green's functions,…]

Open questions:

- quantitative understanding of Nd scattering and light nuclei (3NF problem)
- systematic overbinding of heavier nuclei (A ~ 40): too soft interactions?
- is it possible to describe heavy nuclei without additional fine tuning?
- nuclei on the edge of stability, exotics (e.g. tetra-neutron?)
- interface with lattice QCD

Strategies:

- high orders, no fine tuning in LECs, no tuning to heavy nuclei, error analysis
  EE, Krebs, Meißner; Low Energy Nuclear Physics International Collaboration (LENPIC)
- allow for some fine tuning in LECs and fit to heavy nuclei, error analysis
  The Oak Ridge group: Ekström, Carlsson, Wendt, Papenbrock, Hagen, …
- interactions optimized for specific few-body methods, e.g. local forces & QMC
  Gezerlis, Tews, EE, Gandolfi, Hebeler, Nogga, Schwenk, Piarulli, Girlanda, Schiavilla, Navarro Perez, …

Chiral Effective Field Theory

Chiral Perturbation Theory: expansion of the scattering amplitude in powers of Q,

Weinberg, Gasser, Leutwyler, Meißner, ...

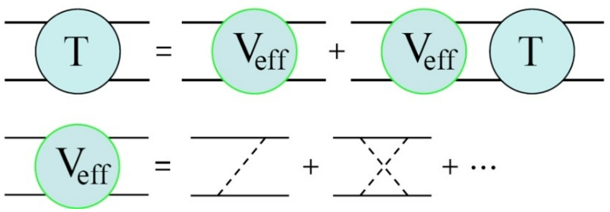
$$Q = \frac{\text{momenta of external particles or } M_\pi \sim 140 \text{ MeV}}{\text{breakdown scale } \Lambda_b}$$

Write down $L_{eff}\,[\pi, N, \ldots]$,
identify relevant diagrams at a given order,
do Feynman calculus,
fit LECs to exp data,
make predictions…

Chiral EFT for nuclear systems: expansion for nuclear forces + resummation (Schrödinger eq.)

Weinberg, van Kolck, Kaiser, EE, Glöckle, Meißner, Entem, Machleidt, Krebs, ...

$$\Big[\Big(\sum_{i=1}^{A}\frac{-\vec{\nabla}_i^2}{2m_N}+\mathcal{O}(m_N^{-3})\Big)+\underbrace{V_{2N}+V_{3N}+V_{4N}+\ldots}_{\text{derived in ChPT}}\Big]|\Psi\rangle = E|\Psi\rangle$$

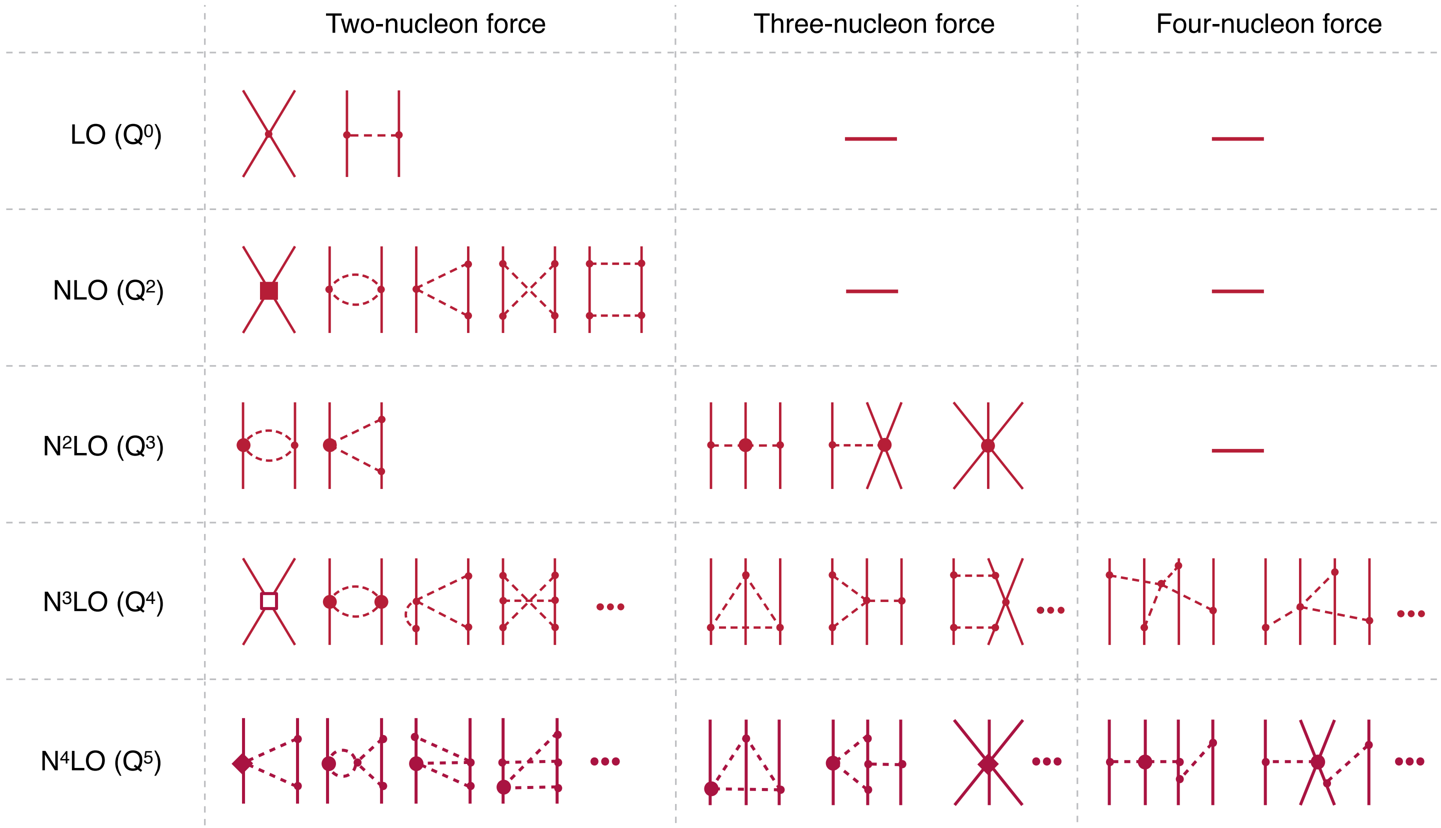


- systematically improvable
- unified approach for $\pi\pi$, πN, NN
- consistent many-body forces and currents
- error estimations

Notice:

- derivation of nuclear forces is not just calculation of Feynman diagrams; have to deal with non-uniqueness and renormalizability… **[more details in the lectures]**
- nonperturbative treatment of chiral nuclear forces in the Schrödinger equation requires the introduction of a finite cutoff **[alternatively, use semi-relativistic approach, EE, Gegelia, et al. '12…'15]**

Chiral expansion of the nuclear forces



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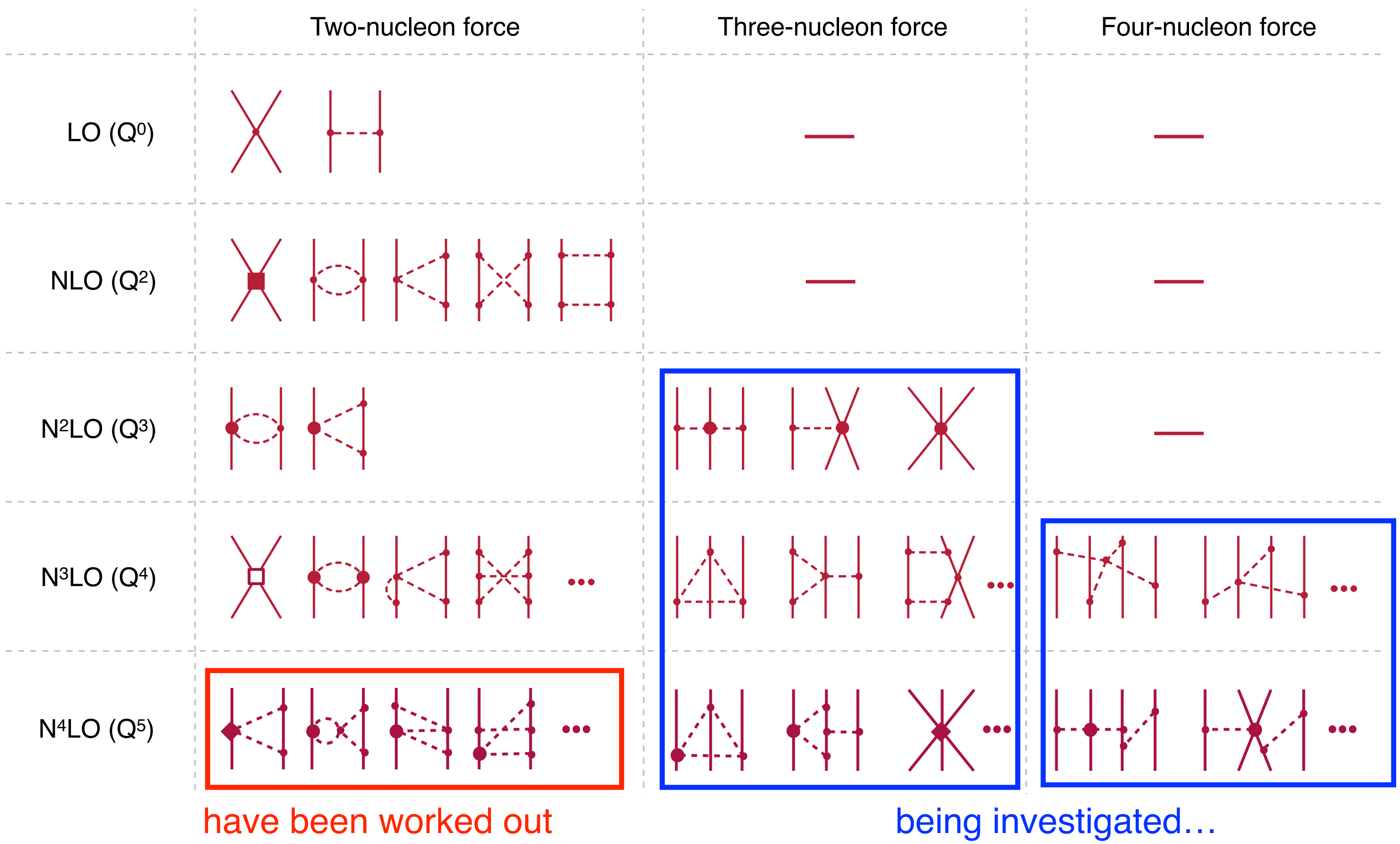

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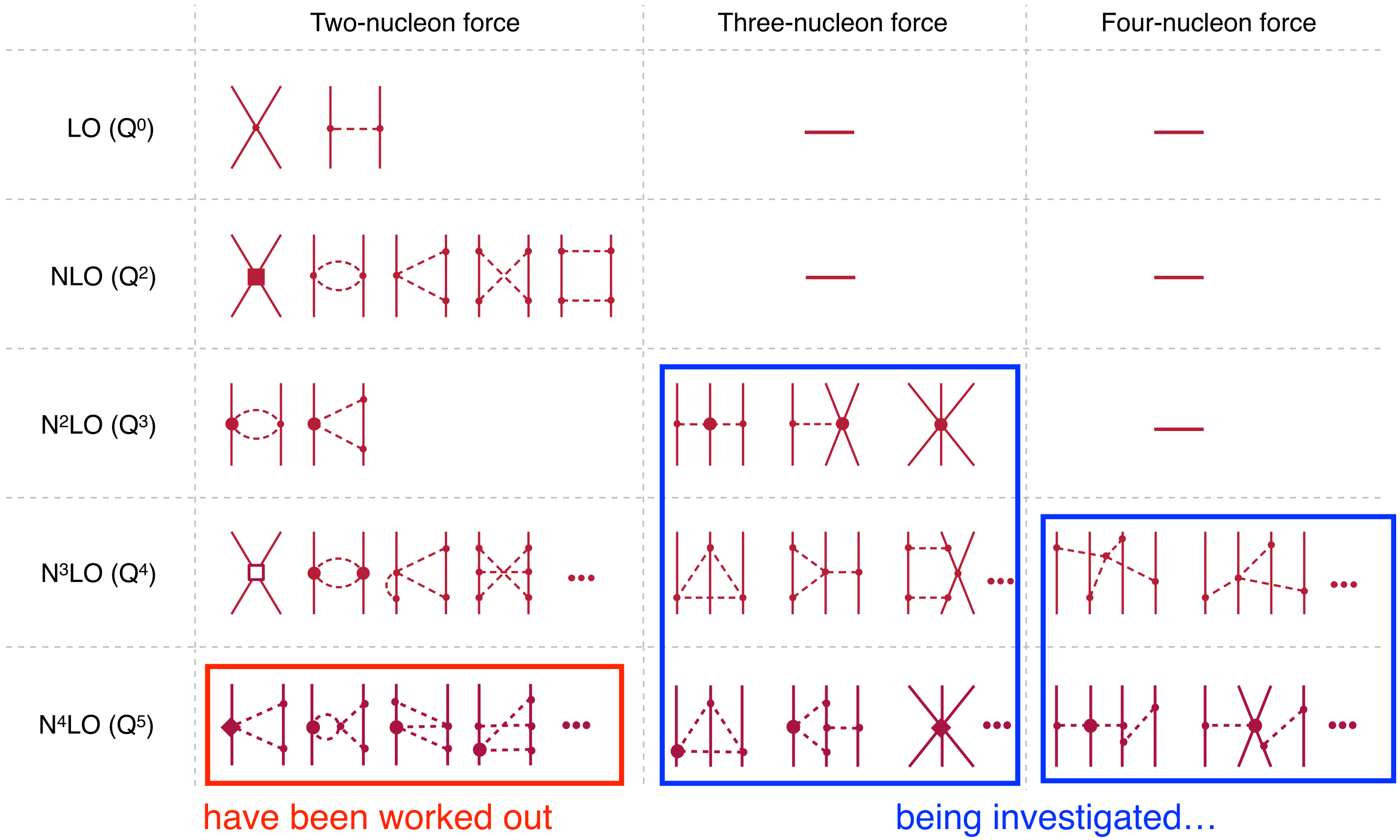


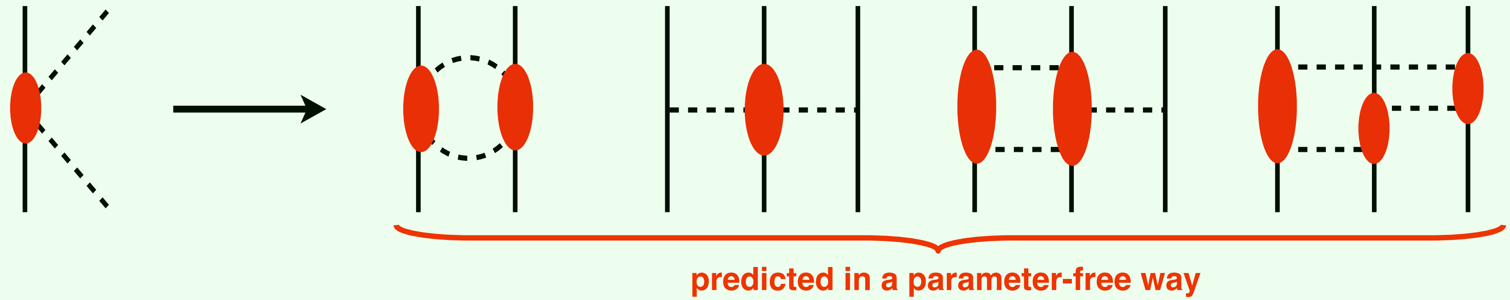
Second-generation chiral NN potentials up to N⁴LO (partially N⁵LO)

— **semilocal** (local r-space regularization of OPEP & TPEP, Gaussian cutoff for contacts)
Epelbaum, Krebs, Meißner, EPJA 51 (2015) 53; PRL 115 (2015) 122301

— **nonlocal** (spectral function regularization for TPEP + nonlocal regulator)
Entem, Machleidt, Nosyk, PRC 96 (2017) 024004

The long-range part of the nuclear force

Long-range nuclear forces are completely determined by the
chiral symmetry of QCD + experimental information on πN scattering



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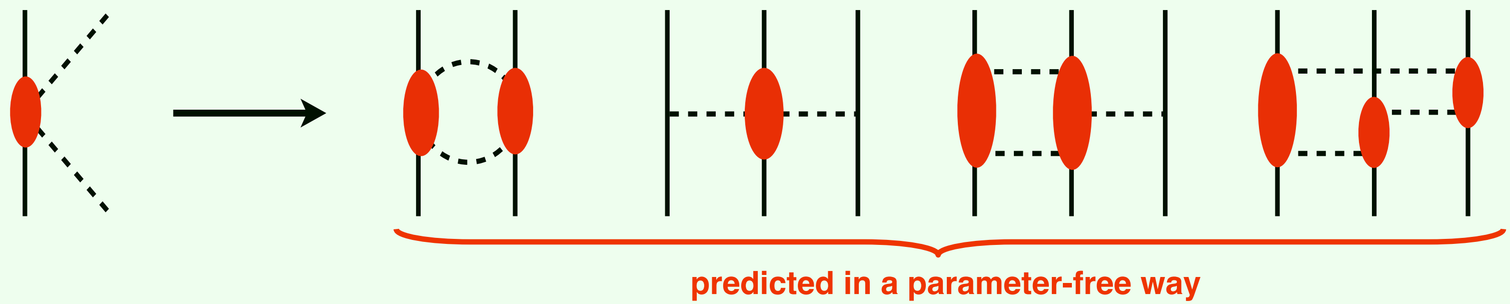
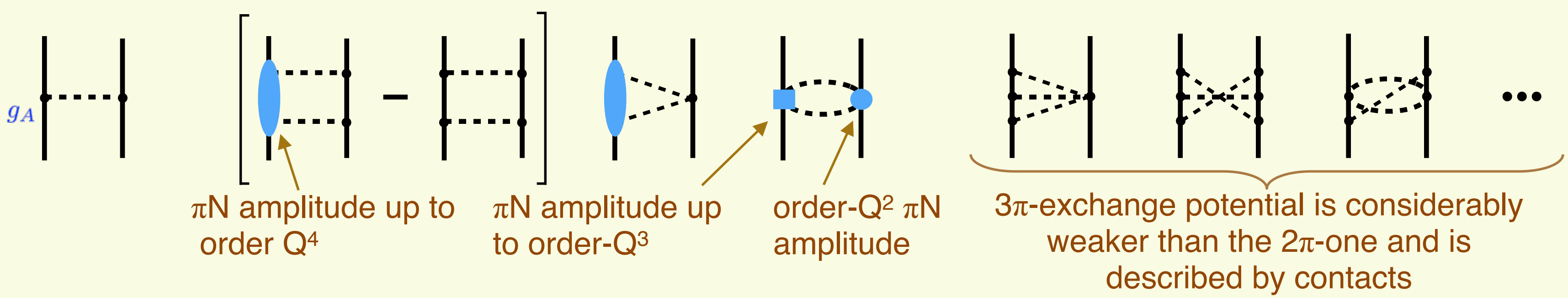


The long-range NN force up to N⁴LO [Q⁵]

Ordonez et al.; Kaiser; EE, Krebs, Meißner; Entem, Machleidt; …



The TPE potential can be derived by taking the phase-space integral of the πN amplitudes computed in ChPT (Lorentz-transformed to the proper kinematics…) Kaiser ’00

Pion-nucleon scattering

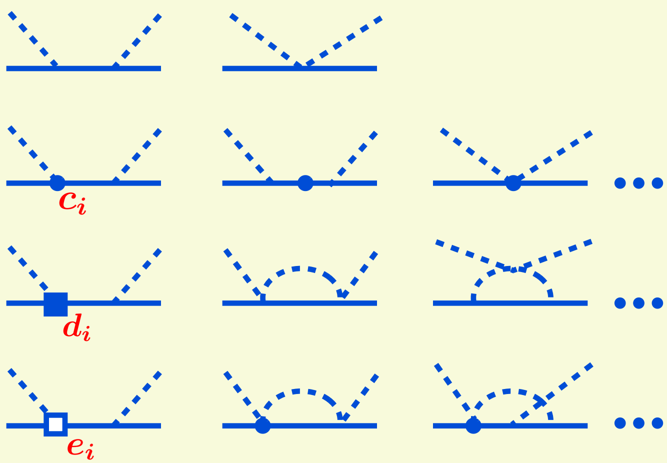
Chiral expansion of the pion-nucleon scattering amplitude up to Q^4

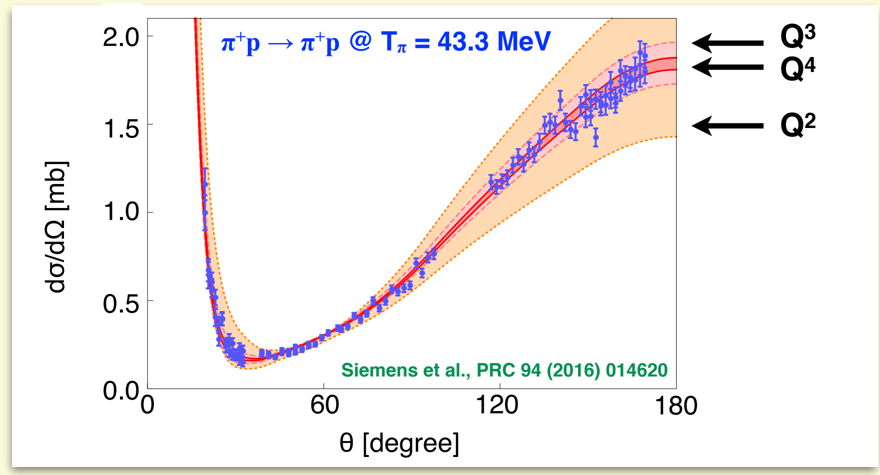
Order Q:

Order Q^2:

Order Q^3:

Order Q^4:





HB ChPT with and without Δ(1232) DOF

Fettes, Meißner ’98, ’00; Krebs, Gasparyan, EE ’12

— comparison with the Karlsruhe-Helsinki (KH) and SAID (GWU) partial-wave analyses…

Covariant baryon ChPT using the IR framework

Becher, Leutwyler ’00; Hoferichter et al. ’10

Covariant baryon ChPT using the EOMS scheme

Alarcon, Camalich, Oller ’13; Chen, Yao, Zheng’13

Covariant baryon ChPT using the EOMS scheme with explicit Δ(1232) DOF

Yao, Siemens, Bernard, EE, Gasparyan, Gegelia, Krebs, Meißner ’16; Siemens, Bernard, EE, Gasparyan, Krebs, Meißner ’16,’17

— also without relying on PWA (i.e. applied to real data) and in combination with the reaction $\pi N \rightarrow \pi\pi N$

Pion-nucleon scattering

Pion-nucleon Roy-Steiner equations

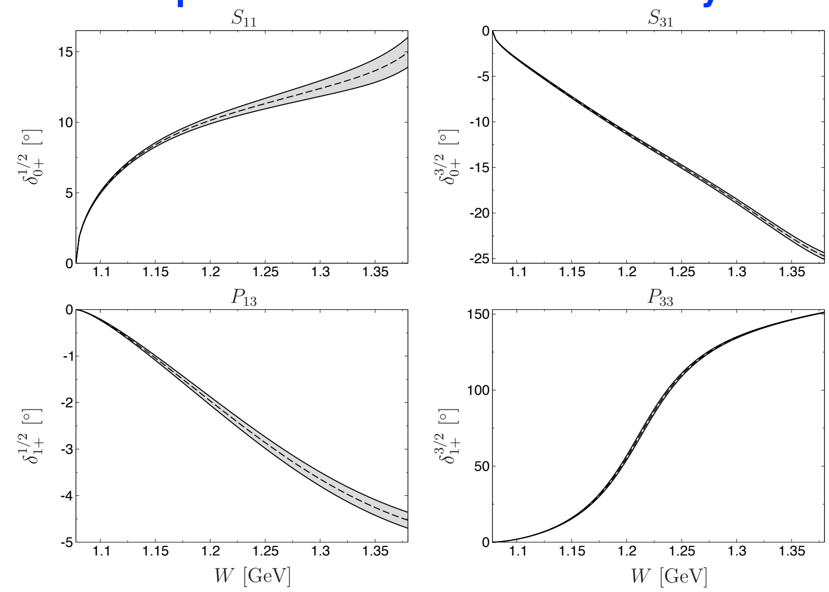
Dietsche et al., JHEP 1206 (12) 043; Hoferichter et al., Phys. Rept. 625 (16) 1

Integral equations in the form of dispersion relations which incorporate constraints from analyticity, unitarity & crossing symmetry

Input: S-,P-waves at high energy, inelasticities, D- & higher waves, scatt. lengths (had. atoms)

Output: reliable results for S-,P-waves with systematic uncertainties; subthreshold coefficients, determination of the σ-term…

πN phase shifts from the RS analysis



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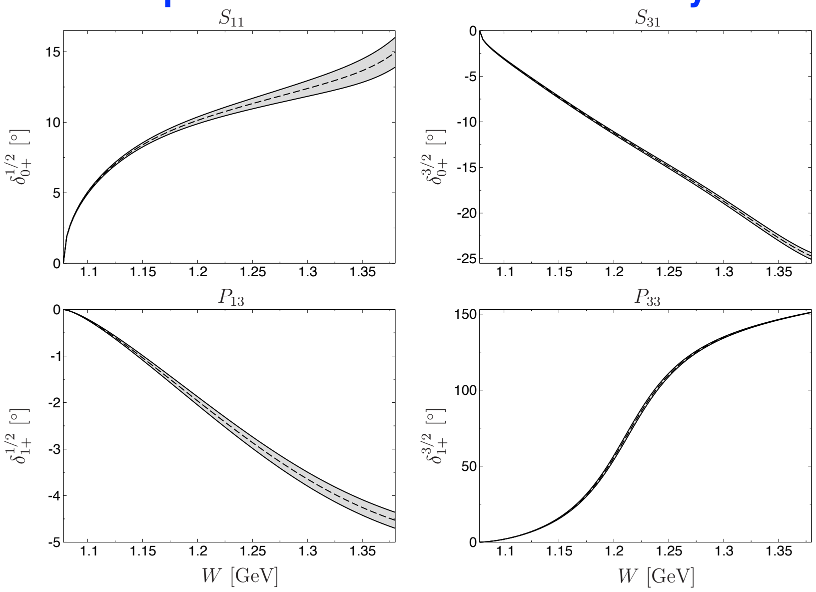

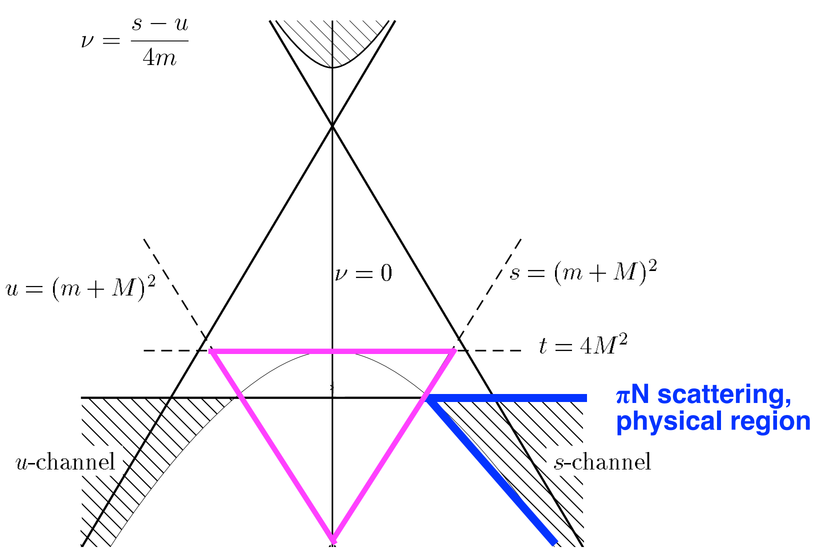

Matching ChPT to πN Roy-Steiner equations

Hoferichter, Ruiz de Elvira, Kubis, Meißner, PRL 115 (2015) 092301

- χ expansion of the πN amplitude expected to converge best within the Mandelstam triangle

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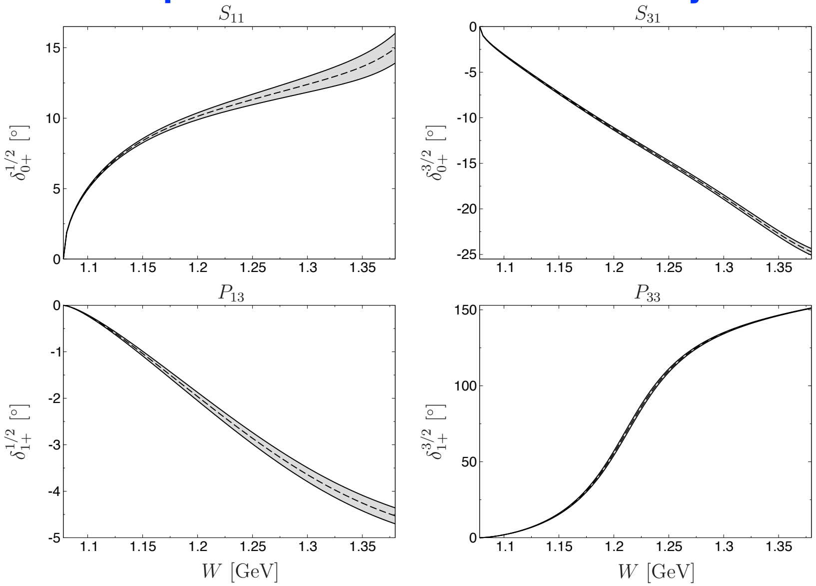

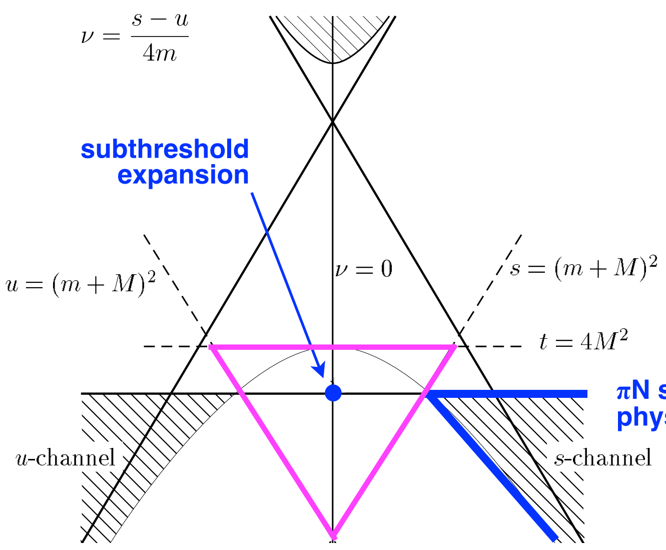

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- Subthreshold coefficients (from RS analysis) provide a natural matching point to ChPT

$$\bar{X} = \sum_{m,n} x_{mn}\, \nu^{2m+k} t^n, \qquad X = \{A^\pm, B^\pm\}$$

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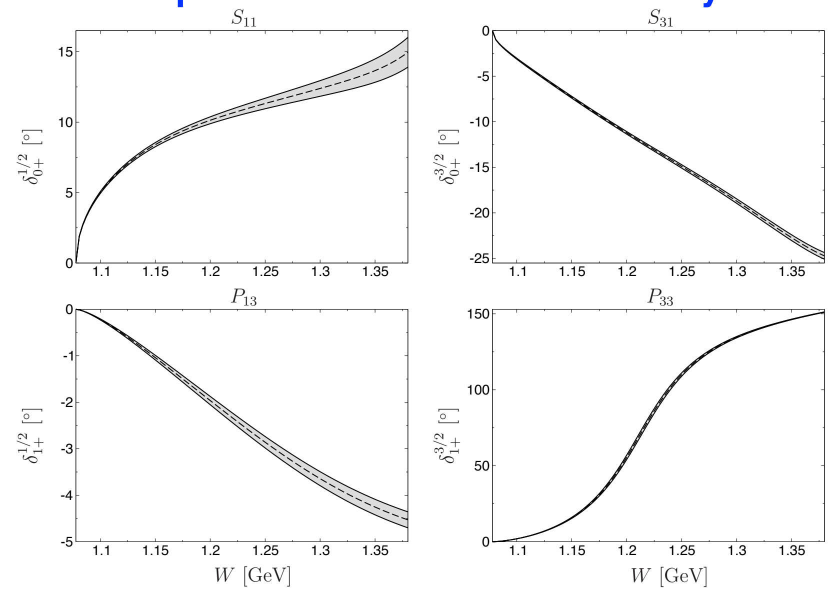

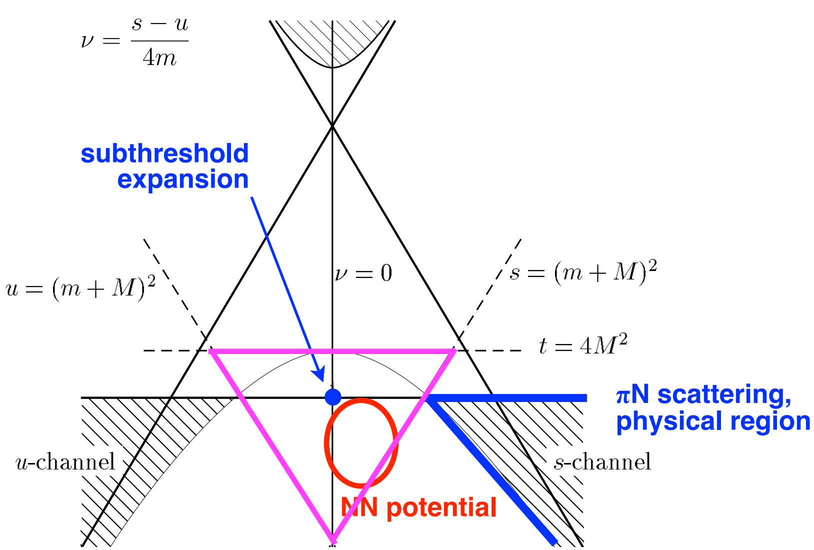

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$$\bar{X} = \sum_{m,n} x_{mn}\, \nu^{2m+k} t^n, \qquad X = \{A^{\pm}, B^{\pm}\}$$

- Closer to the kinematics relevant for nuclear forces…

Determination of the low-energy constants

Relevant LECs (in GeV^{-n}) extracted from πN scattering

	c_1	c_2	c_3	c_4	$\bar{d}_1 + \bar{d}_2$	$\bar{d}_3$	$\bar{d}_5$	$\bar{d}_{14} - \bar{d}_{15}$	$\bar{e}_{14}$	$\bar{e}_{17}$	
$[Q^4]_{\mathbf{HB,\,NN}}$, GW PWA	-1.13	3.69	-5.51	3.71	5.57	-5.35	0.02	-10.26	1.75	-0.58	Krebs, Gasparyan, EE, PRC85 (12) 054006
$[Q^4]_{\mathbf{HB,\,NN}}$, KH PWA	-0.75	3.49	-4.77	3.34	6.21	-6.83	0.78	-12.02	1.52	-0.37	
$[Q^4]_{\mathbf{HB,\,NN}}$, Roy-Steiner	-1.10	3.57	-5.54	4.17	6.18	-8.91	0.86	-12.18	1.18	-0.18	Hoferichter et al., PRL 115 (15) 092301
$[Q^4]_{\mathbf{covariant}}$, data	-0.82	3.56	-4.59	3.44	5.43	-4.58	-0.40	-9.94	-0.63	-0.90	Siemens et al., PRC94 (16) 014620

Notice:

— some LECs show sizable correlations (especially c_1 and c_3)…

— KH PWA and Roy-Steiner LECs lead to comparable results in the NN sector

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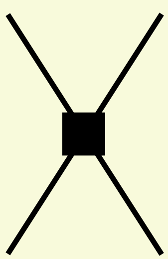
The short-range part of the nuclear force (contact interactions)

Organizational principle for contact terms according to NDA (Weinberg's counting)



LO [Q^0]: 2 operators (S-waves)

NLO [Q^2]: + 7 operators (S-, P-waves and ε_1)

N^{2}LO [Q^3]: no new isospin-conserving operators

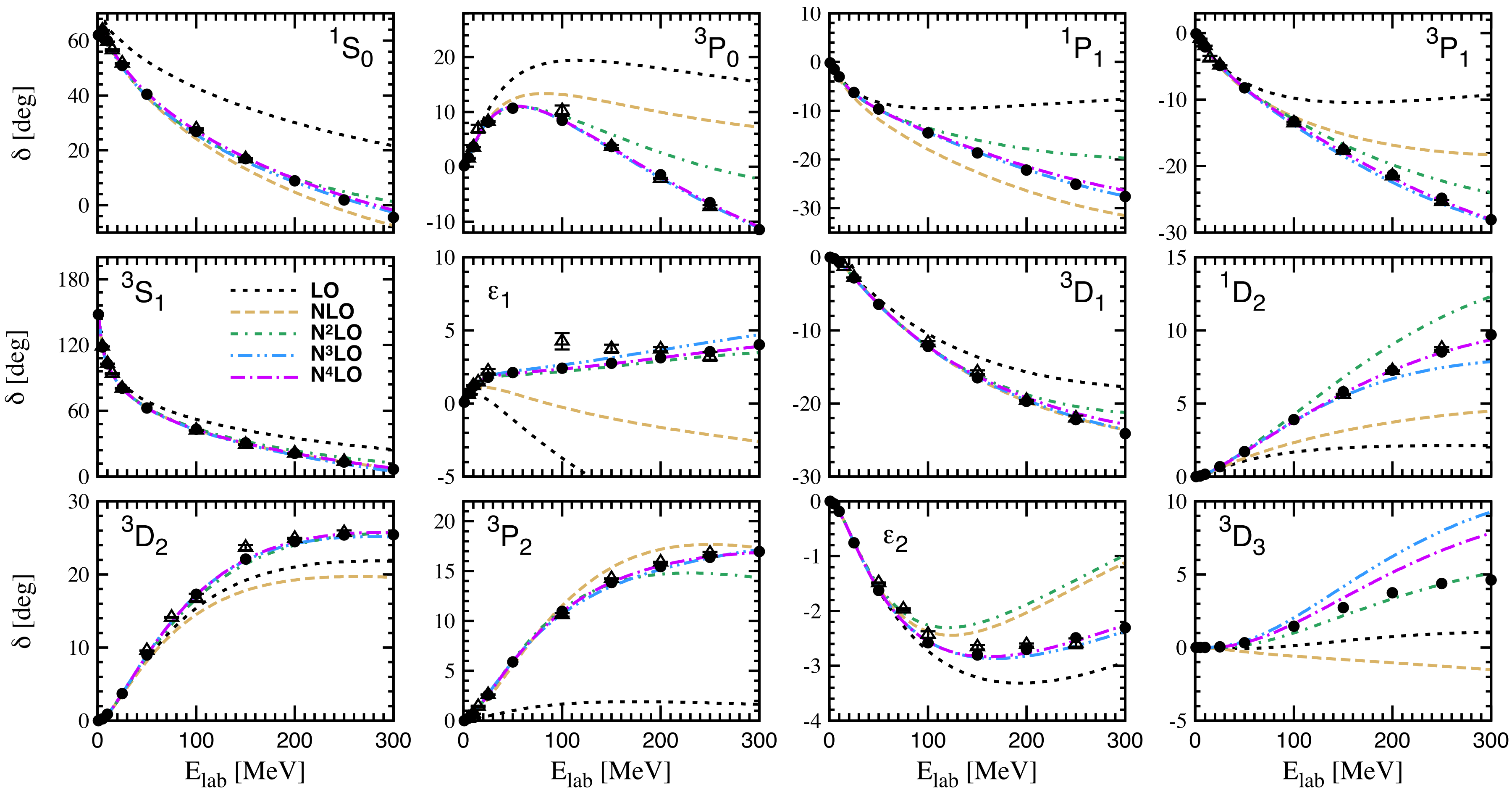
N^{3}LO [Q^4]: + 12 operators (S-, P-, D-waves and ε_1, ε_2)

N^{4}LO [Q^5]: no new isospin-conserving operators

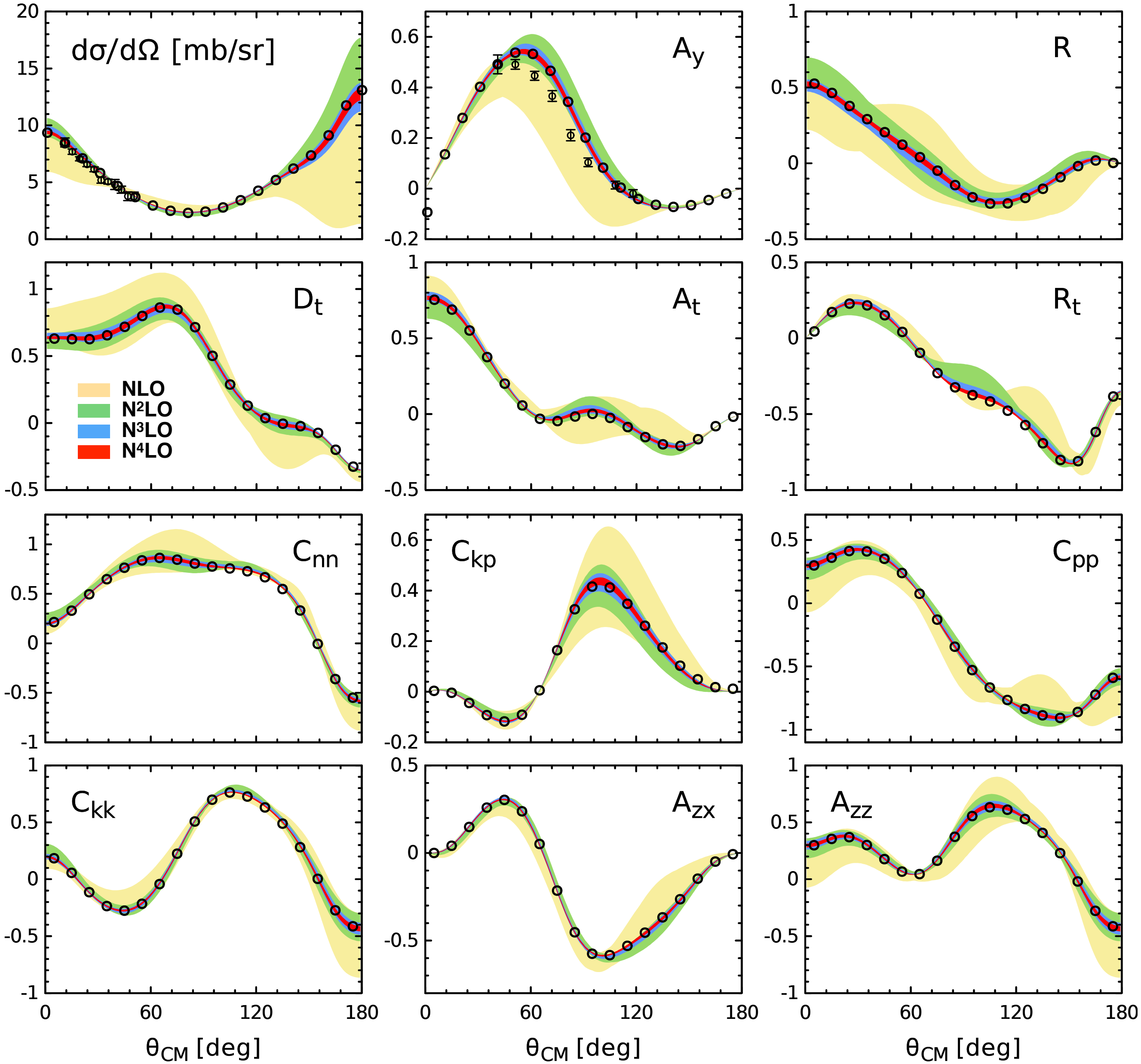
Preliminary results

Patrick Reihert et al., in preparation

Convergence of the chiral expansion for np phase shifts [Λ = 450 MeV]



Proton-neutron scattering observables at E_{lab} = 143 MeV



Description of NN scattering data [Λ = 450 MeV]

E_{lab} bin	LO	NLO	N^2LO	N^3LO	N^4LO	N^4LO^+
neutron-proton scattering data						
0 – 100	73	2.2	1.2	1.08	1.08	1.07
0 – 200	62	5.4	1.8	1.09	1.08	1.06
0 – 300	75	14	4.4	1.99	1.18	1.10
proton-proton scattering data						
0 – 100	2300	10	2.1	0.91	0.88	0.86
0 – 200	1780	91	33	2.00	1.42	0.95
0 – 300	1380	89	38	3.42	1.67	0.99
	2 LECs	+ 7 + 1 IB LECs		+ 12 LECs	+ 1 IB LEC	+ 4 LEC

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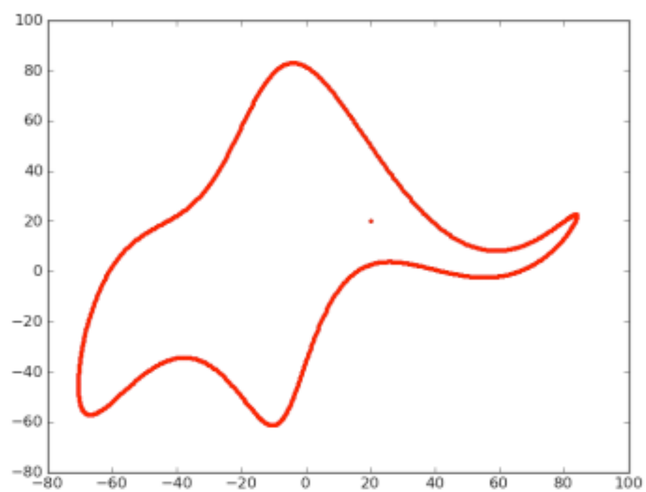
> With four parameters I can fit an elephant, and with five I can make him wiggle his trunk.
>
> John von Neumann



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	2 LECs	+ 7 + 1 IB LECs		+ 12 LECs	+ 1 IB LEC	+ 4 LEC

no new LECs → (NLO → N^2LO)

no new LECs → (N^3LO → N^4LO)

Clear evidence of the (parameter-free) chiral 2π-exchange!

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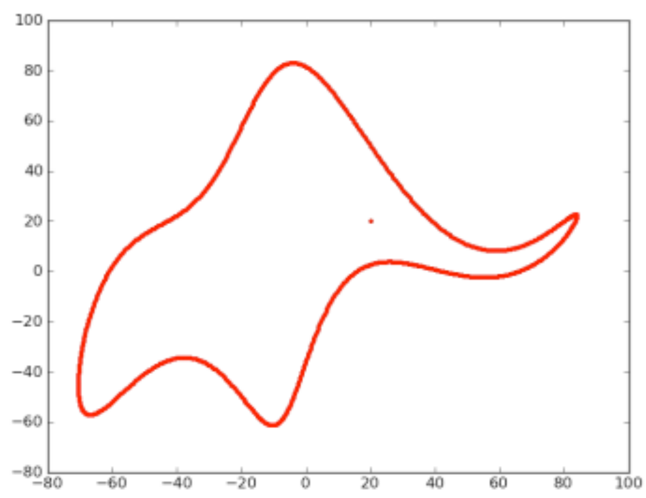


High-precision NN potentials versus chiral N^{4}LO$^+$ [Λ = 450 MeV]

χ^2 per datum for the description of the np and pp scattering data

$E_{\rm lab}$ bin	CD Bonn$_{(43)}$	Nijm I$_{(41)}$	Nijm II$_{(47)}$	Reid93$_{(50)}$	N^{4}LO$^+$$_{(27+1)}$, this work
neutron-proton scattering data					
0 – 100	1.08	1.07	1.08	1.09	1.07
0 – 200	1.08	1.07	1.07	1.09	1.06
0 – 300	1.09	1.09	1.10	1.11	1.10
proton-proton scattering data					
0 – 100	0.88	0.87	0.87	0.85	0.86
0 – 200	0.98	0.99	1.00	0.99	0.95
0 – 300	1.01	1.05	1.06	1.04	0.99

- for the first time, chiral NN potential reaches the precision and even outperforms the most sophisticated phenomenological potentials!
- at the same time, the number of adjustable parameters is reduced by ~ 40%
  $\longrightarrow$ yet another evidence of the importance of the 2π-exchange!
- our results can be regarded as a new PWA and provide quantification of statistical and systematic uncertainties in the extracted phase shifts.

Beyond the 2N system

LENPIC Collaboration

Goal: precision tests of chiral nuclear forces & currents in light nuclei

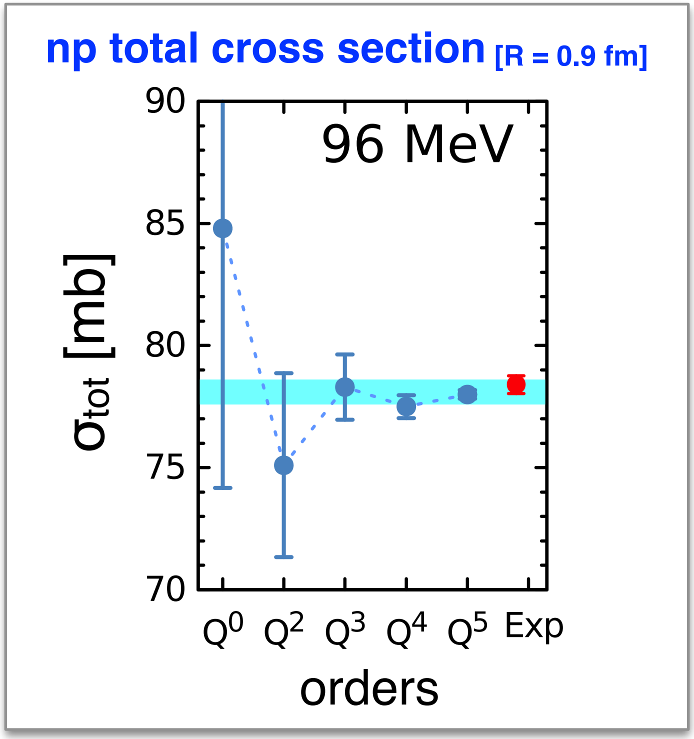
Strategy: go to high orders, do not compromise the πN LECs, no fine tuning to heavy nuclei, careful error analysis

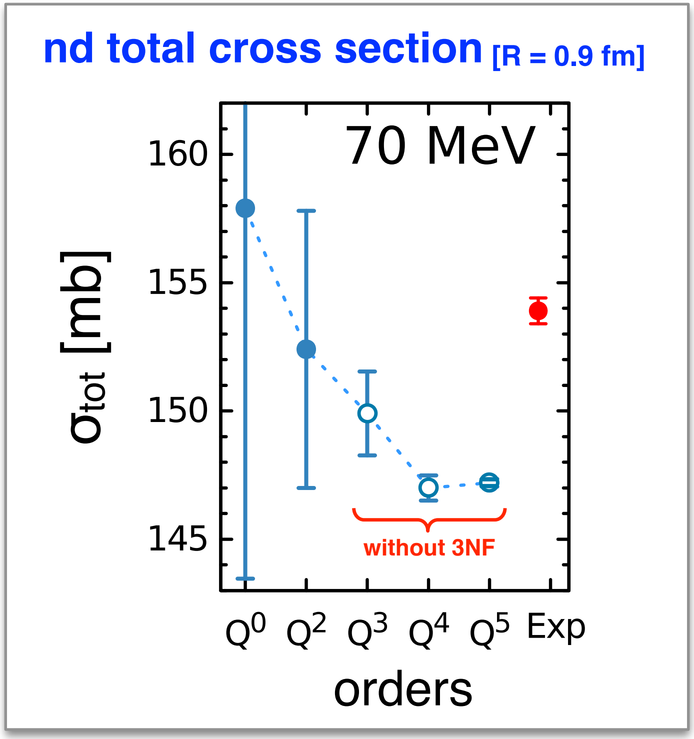
LENPIC: Low **E**nergy **N**uclear **P**hysics **I**nternational **C**ollaboration

Few-N results without 3NF

LENPIC Collaboration (Binder et al.), PRC 93 (2016) 04402

Is there evidence for missing 3N forces effects? Yes!



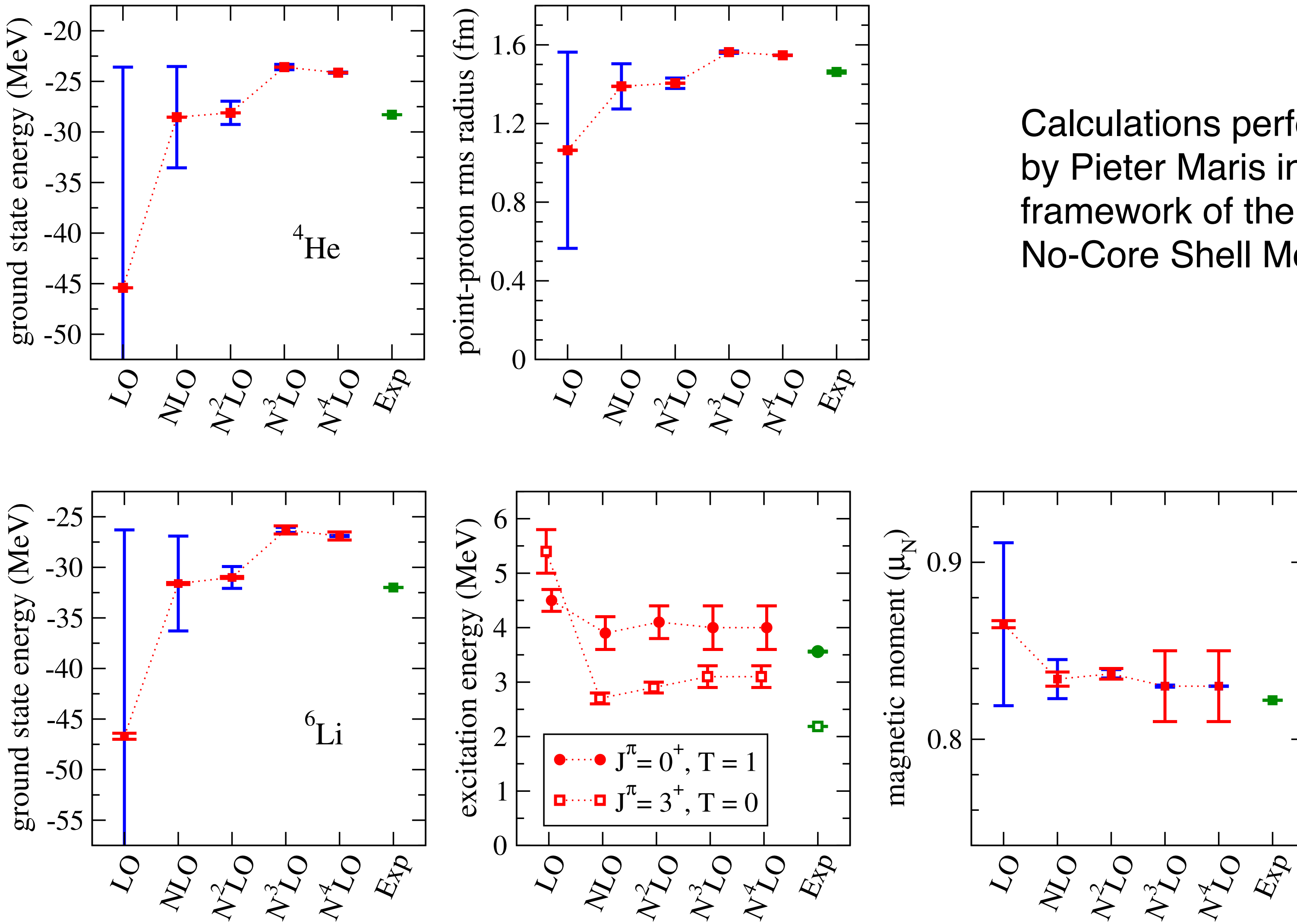


- Discrepancies between theory and data well outside the range of quantified uncertainties
  → **clear evidence for missing 3NF effects**
- Magnitude of the required 3NF contributions matches well the estimated size of N^2LO terms
  → **consistent with the chiral power counting**

LENPIC: Low Energy Nuclear Physics International Collaboration

Few-N results without 3NF

LENPIC Collaboration (Maris et al.), EPJ Web of Conf. 113 (2016) 04015

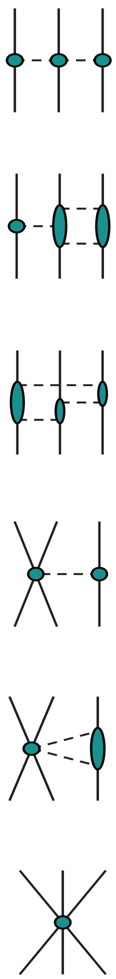


Calculations performed by Pieter Maris in the framework of the No-Core Shell Model

LENPIC: Low Energy Nuclear Physics International Collaboration

LENPIC RUB universität bonn TECHNISCHE UNIVERSITÄT DARMSTADT JAGIELLONIAN UNIVERSITY IN KRAKOW STATE JÜLICH FORSCHUNGSZENTRUM Kyutech OHIO STATE UNIVERSITY IPN ORSAY TRIUMF OAK RIDGE National Laboratory

Chiral expansion of the 3NF



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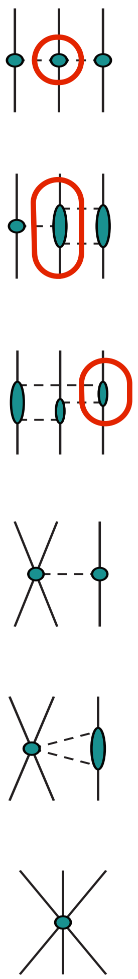
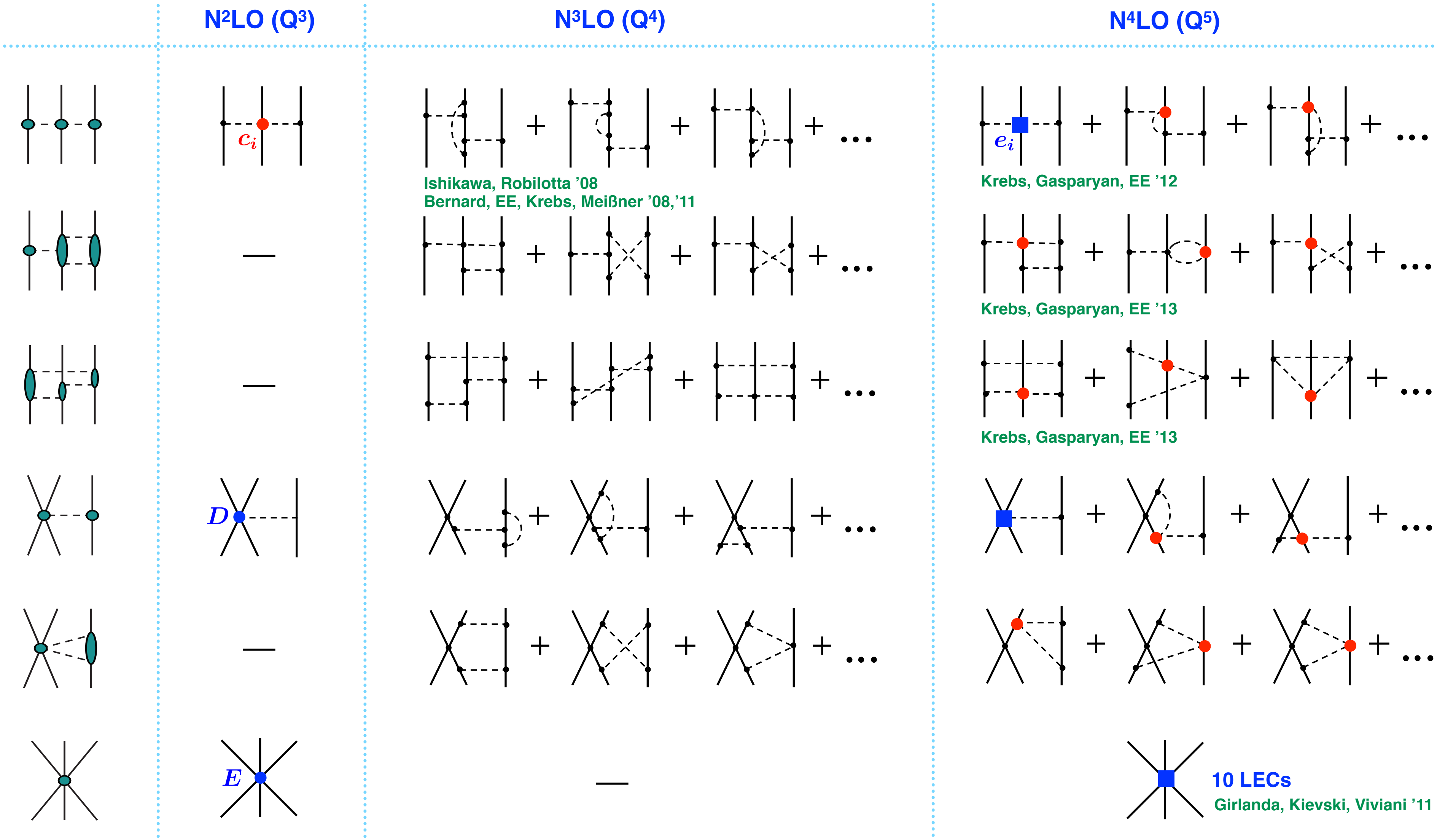


3NF structure functions at large distance are
model-independent and parameter-free predictions
based on χ symmetry of QCD + exp. information on πN system

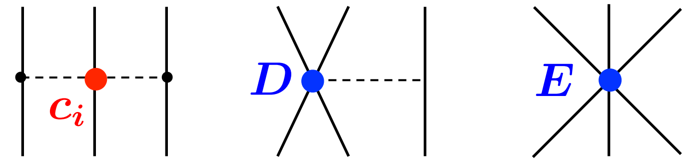
Chiral expansion of the 3NF



Some PRELIMINARY results with 3NF

LENPIC, in progress

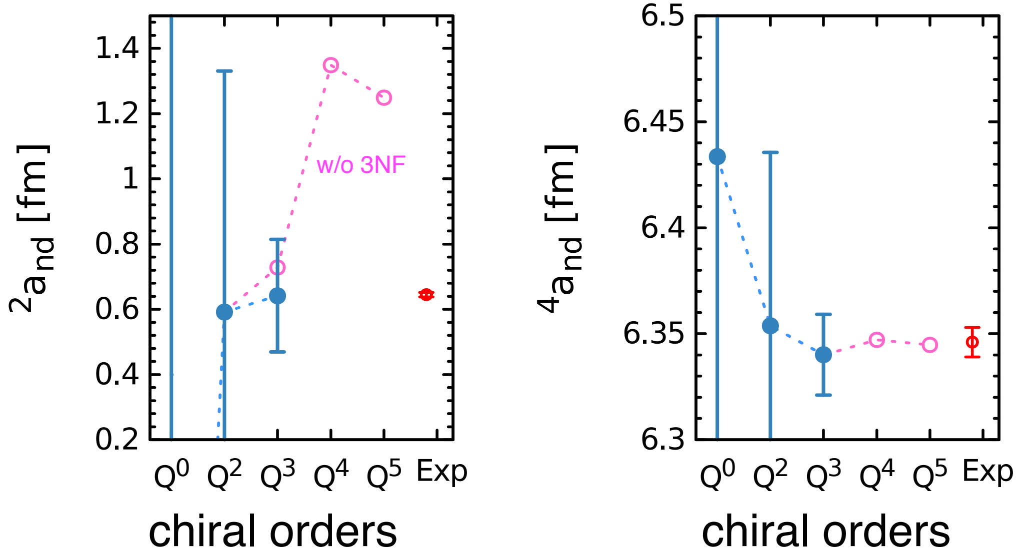
The LECs D, E are determined from the ^{3}H and the Nd cross section minimum @70 MeV (RIKEN data)



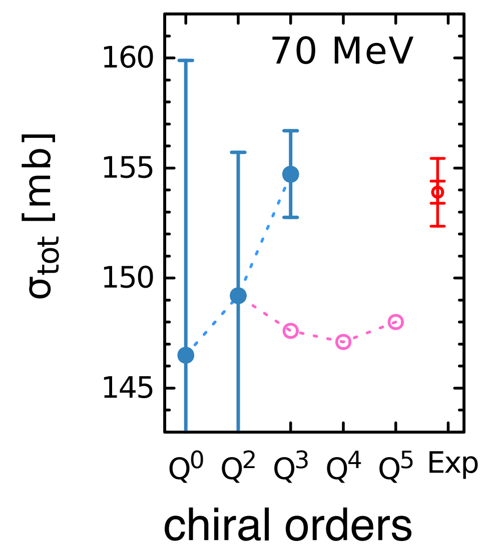
The results are **preliminary:**

— still have to analyze different ways to determine D and E, check other sources of uncertainties, …

nd scattering lengths [R = 1.0 fm]



nd σ_{tot} at 70 MeV [R = 1.0 fm]



LENPIC: Low **E**nergy **N**uclear **P**hysics **I**nternational **C**ollaboration

Nuclear lattice simulations:

A novel ab initio approach to nuclei and nuclear reactions

EE, H. Krebs, T. Lähde, D. Lee, T. Luu, U.-G. Meißner, G. Rupak + post-docs + students

Some recent highlights:

Ab initio calculation of the Hoyle state

EE, H. Krebs, D. Lee, U.-G. Meißner, PRL 106 (11) 192501;
EE, H. Krebs, T.A.Lähde, D. Lee, U.-G. Meißner, PRL 109 (12) 252501

Ab initio calculation of the spectrum and structure of ^{16}O

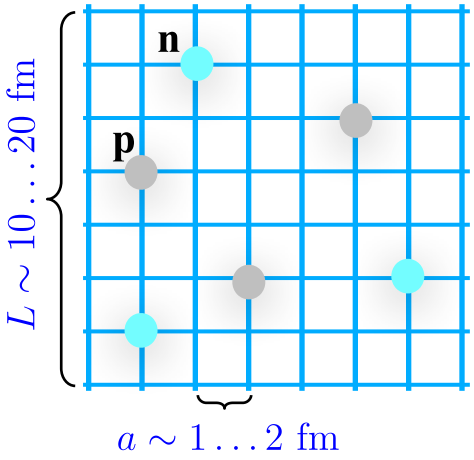
EE, H. Krebs, T. A. Lähde, D. Lee, U.-G. Meißner, G. Rupak, PRL 112 (14) 102501

Lattice EFT for medium-mass nuclei

T. A. Lähde, EE, H. Krebs, D. Lee, U.-G. Meißner, G. Rupak, PLB 732 (14) 110

Symmetry-sign extrapolations

T.A. Lähde, T. Luu, D. Lee, U.-G. Meißner, EE, H. Krebs, G. Rupak, EPJ A51 (15) 92



Ab initio alpha-alpha scattering

Elhatisari, Lee, Rupak, EE, Krebs, Lähde, Luu, Meißner, Nature 528 (2015) 111

nature International weekly journal of science

Ab initio alpha–alpha scattering

Serdar Elhatisari[1], Dean Lee[2], Gautam Rupak[3], Evgeny Epelbaum[4], Hermann Krebs[4], Timo A. Lähde[5], Thomas Luu[1,5] & Ulf-G. Meißner[1,5,6]

Nature **528**, 111–114 (03 December 2015) | doi:10.1038/nature16067

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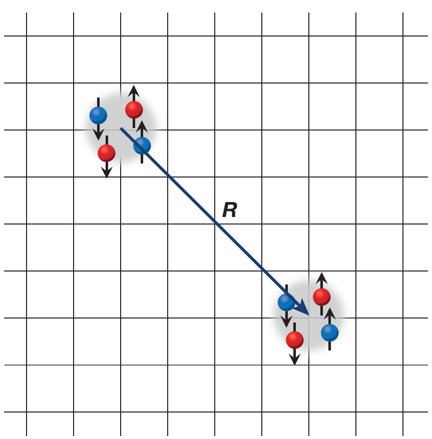
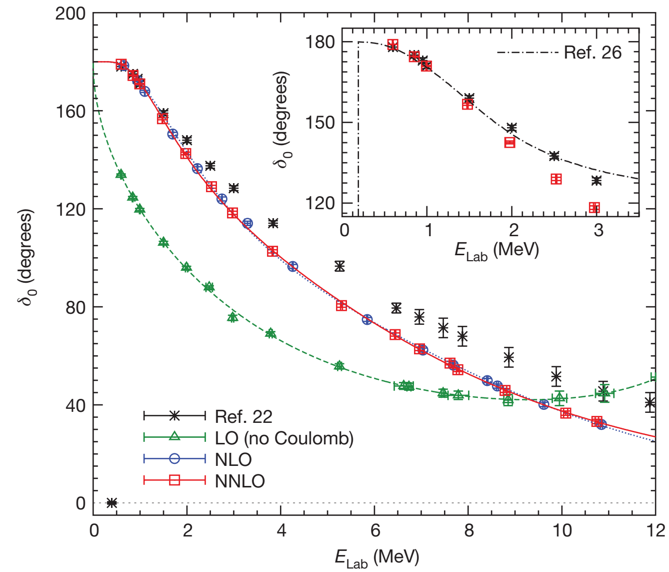
First ab initio calculation of alpha-alpha scattering!

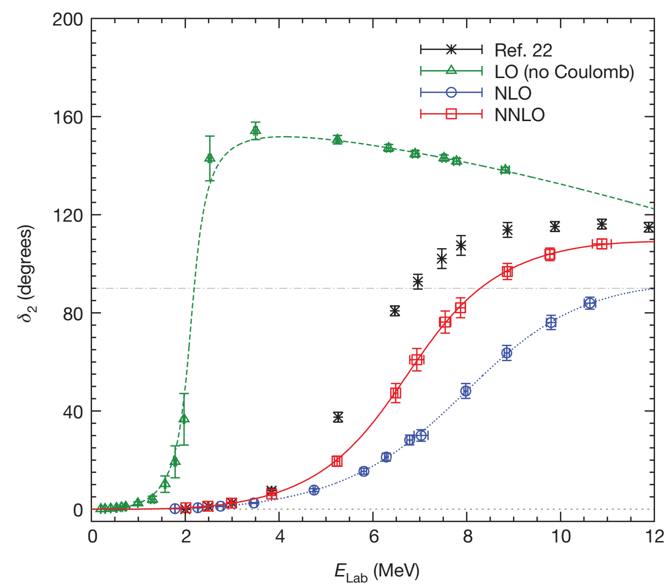
Used lattice EFT to extract the effective Hamiltonian for two interacting α-clusters (adiabatic projection method [A. Rokash et al., PRC 92 (15) 054612])



Phase shifts obtained employing a hard spherical wall boundary at asymptotically large distances

Promising scaling with respect to the number of particles as $\sim (A_1 + A_2)^2$





Summary and outlook

25 years after Weinberg's proposal, the most precise nuclear forces finally come from chiral EFT!

Frontiers & challenges for the near future:

Precision physics beyond the 2N system: challenge the theory

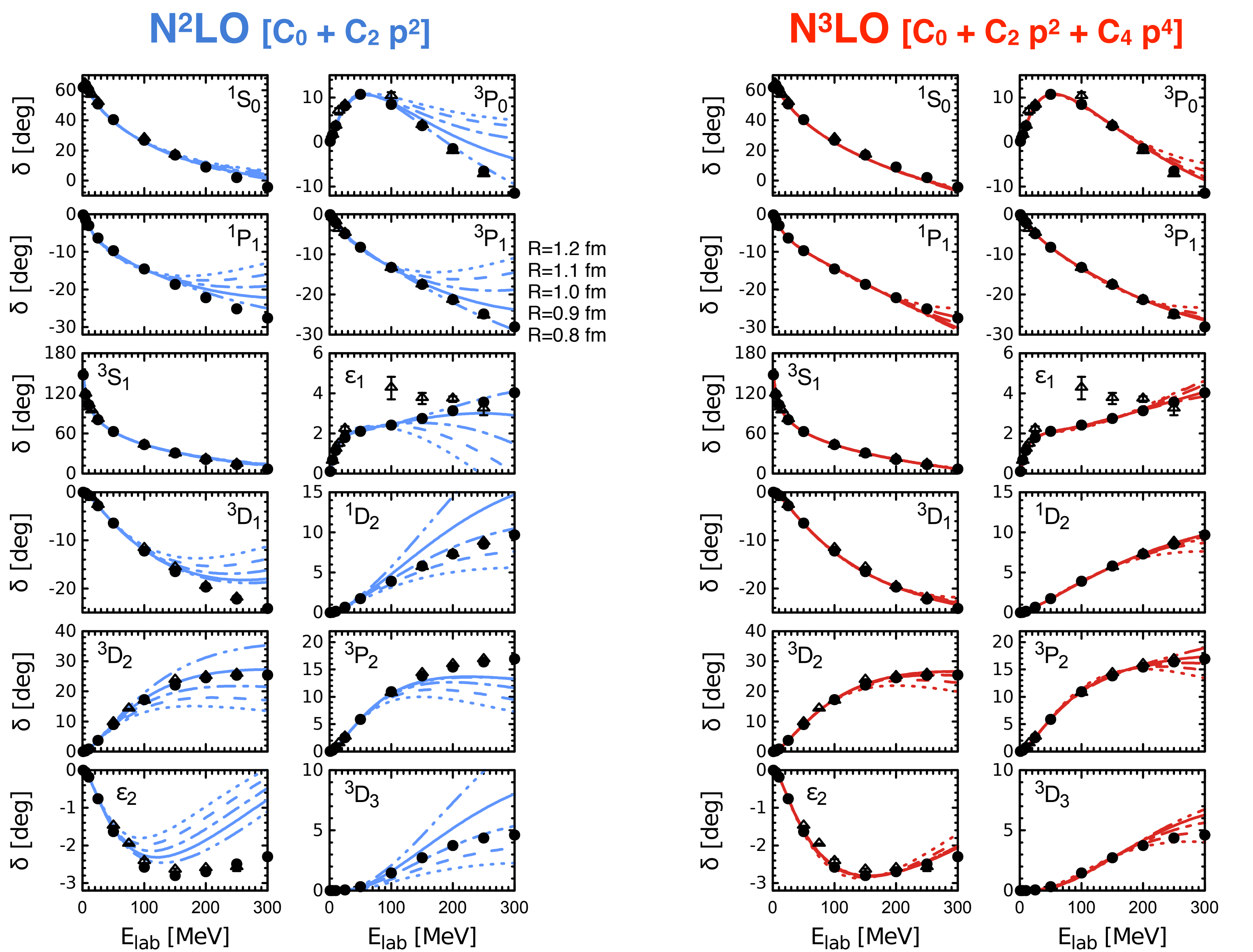
- Test predictive power (N^3LO contributions to 3NF & 4NF are parameter-free, ^3H β-decay is parameter-free up to N^3LO after fixing 3NF@N^2LO, …)
- 3NF & long-standing puzzles in 3N continuum
- Push theory to heavier nuclei (underbinding? radii?)
- More reliable error analysis
- Test different power counting schemes

Chiral EFT as a tool to deal with nuclear effects when looking at physics of/beyond the SM (parity violation, EDM, 0νββ, proton charge radius,…)

EFT for lattice QCD (extrapolations), lattice QCD for EFT (quark mass dependence, „data", …)

spares…

Residual cutoff dependence



Regulator (in)dependence

How do our results depend on the specific form of the regulator $f\left(\frac{r}{R}\right) = \left[1 - \exp\left(-\frac{r^2}{R^2}\right)\right]^n$

and/or additional spectral function regularization $V_C(q) = \frac{2}{\pi} \int_{2M_\pi}^{\Lambda_{SFR}} d\mu \, \mu \, \frac{\rho_C(\mu)}{\mu^2 + q^2}$

Selected phase shifts (in deg.) for different values of Λ_{SFR} and n at $N^3LO_{[R = 0.9\ fm]}$

Lab. energy	NPWA	our result	DR, $n=5$	DR, $n=7$	SFR, 1.0 GeV	SFR, 1.5 GeV	SFR, 2.0 GeV
proton-proton $^1S_0$ phase shift							
10 MeV	55.23	55.22 ± 0.08	55.22	55.22	55.22	55.22	55.22
100 MeV	24.99	24.98 ± 0.60	24.98	24.98	24.98	24.98	24.98
200 MeV	6.55	6.56 ± 2.2	6.55	6.56	6.56	6.56	6.57
neutron-proton $^3S_1$ phase shift							
10 MeV	102.61	102.61 ± 0.07	102.61	102.61	102.61	102.61	102.61
100 MeV	43.23	43.22 ± 0.30	43.28	43.20	43.17	43.21	43.22
200 MeV	21.22	21.2 ± 1.4	21.2	21.2	21.2	21.2	21.2
proton-proton $^3P_0$ phase shift							
10 MeV	3.73	3.75 ± 0.04	3.75	3.75	3.75	3.75	3.75
100 MeV	9.45	9.17 ± 0.30	9.15	9.18	9.18	9.17	9.17
200 MeV	−0.37	-0.1 ± 2.3	−0.1	−0.1	−0.1	−0.1	−0.1
proton-proton $^3P_1$ phase shift							
10 MeV	−2.06	-2.04 ± 0.01	−2.04	−2.04	−2.04	−2.04	−2.04
100 MeV	−13.26	-13.42 ± 0.17	−13.43	−13.41	−13.41	−13.42	−13.42
200 MeV	−21.25	-21.2 ± 1.6	−21.2	−21.2	−21.2	−21.2	−21.2
proton-proton $^3P_2$ phase shift							
10 MeV	0.65	0.65 ± 0.01	0.66	0.65	0.65	0.65	0.65
100 MeV	11.01	11.03 ± 0.50	10.97	11.06	11.07	11.05	11.04
200 MeV	15.63	15.6 ± 1.9	15.6	15.5	15.5	15.5	15.6

→ negligible regulator dependence (compared to the estimated theor. accuracy)

Deuteron properties R=0.9 fm

EE, Krebs, Meißner, arXiv:1412.0142 [nucl-th], arXiv:1412.4623 [nucl-th]

	LO	NLO	N	N	N	empirical
B	2.0235	2.1987	2.2311	2.2246*	2.2246*	2.224575(9)
A	0.8333	0.8772	0.8865	0.8845	0.8844	0.8846(9)
η	0.0212	0.0256	0.0256	0.0255	0.0255	0.0256(4)
r_d	1.990	1.968	1.966	1.972	1.972	1.97535(85)
Q [fm	0.230	0.273	0.270	0.271	0.271	0.2859(3)
P_D	2.54	4.73	4.50	4.19	4.29	

— fast convergence of the chiral expansion (P_D is not observable)

— error estimation (assuming $Q=M_\pi/\Lambda_b$)

A_S: LO: 0.83(5) → NLO: 0.878(13) → N²LO: 0.887(3) → N³LO: 0.8845(8) → N⁴LO: 0.8844(2)

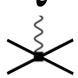
η: LO: 0.021(5) → NLO: 0.026(1) → N²LO: 0.0256(3) → N³LO: 0.0255(1) → N⁴LO: 0.0255

→ theoretical results for A_S,η at N⁴LO are more accurate than empirical numbers

— results for r_d and Q do not take into account MECs and relativistic corrections:

r_d: $|\Delta r_d| \simeq 0.004$ fm [Kohno '83] → predictions in agreement with the data

Q: rel. corrections + 1π-exchange MEC: $\Delta Q \simeq +0.008\ \text{fm}^2$ [Phillips '07] → $Q \simeq 0.279\ \text{fm}^2$

the remaining deviation of 0.007 fm² agrees with the expected size of  [Phillips '07]