Aspects of quantum tunneling with gravity: Towards solution of negative mode problem

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Based on:

M. Koehn, G. L. and J.-L. Lehners, Phys. Rev. D **92** (2015); arXiv:1504.04334 [hep-th]

Work in progress, and earlier published papers.

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Plan of the talk

- Euclidean approach to metastable vacuum decay: QM & QFT
- Vacuum decay: inclusion of gravity
- Negative mode problem in tunnelling transitions with gravity
- Concluding remarks

Bounces and false vacuum decay

Coleman (1977)

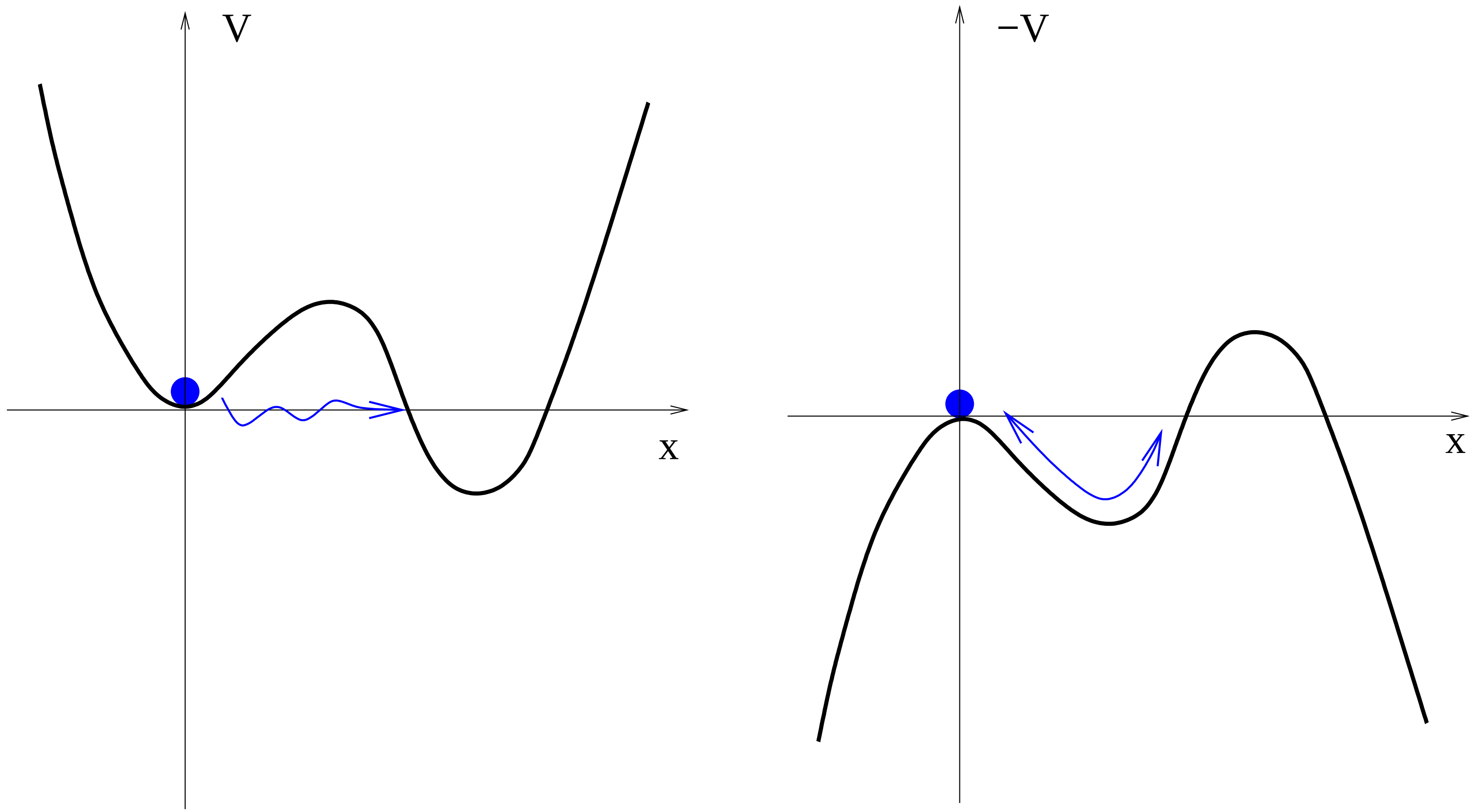


Figure 1: Tunneling in asymmetric double well potential.

Euclidean approach to tunneling

Coleman (1977)

In the semiclassicall approximation, summing multibounce configuration one finds correction to ground state energy $E_0 = \hbar\omega/2$ in the following form

$$E = E_0 - \hbar K \mathrm{e}^{-S/\hbar}[1 + O(\hbar)] \,, \tag{1}$$

where, $S = \int d\eta [\frac{1}{2}(\frac{dx}{d\eta})^2 + V(x)]$ is the Euclidean action on the bounce solution $\bar{x}(\eta)$ and pre-exponential factor K is given by (Gaussian) integration of the exponential of quadratic action of linear perturbations $S^{(2)} = \frac{1}{2}\int d\eta \delta x[-\partial_\eta^2 + V'']\delta x$,

$$K = \frac{1}{2}(\frac{S}{2\pi\hbar})^{1/2}(\frac{det'[-\partial_\eta^2 + V''(\bar{x})]}{det[-\partial_\eta^2 + \omega^2]})^{-1/2} \,. \tag{2}$$

There is exactly one tunnelling negative mode in the spectrum of linear perturbations about the bounce solution, since (translational) zero energy wave function $\psi_0 \sim \frac{d\bar{x}}{d\eta}$ of corresponding Schrödinger equation has a node. i.e. $K = i\Gamma$.

Finally, the decay probability per unit time of the unstable state is given by

$$\begin{aligned} \Gamma &= -2\mathrm{Im}E/\hbar \\ &= (\frac{S}{2\pi\hbar})^{1/2} \left| \frac{det'[-\partial_\lambda^2 + V''(\bar{x})]}{det[-\partial_\lambda^2 + \omega^2]} \right|^{-1/2} \mathrm{e}^{-S/\hbar} \; [1 + O(\hbar)] \; . \end{aligned} \tag{3}$$

In the 1988 NPB article “Quantum Tunneling And Negative Eigenvalues,” Coleman arrives to strong conclusion: “There may exist solutions in other ways like bounces and which have more than one negative eigenvalue, but, even if they do exist, they have nothing to do with tunnelling.”

These quantum mechanical results could be generalized for

- Field theory in flat space-time Frampton (1976); Coleman, Callan and Coleman (1977)
- Field theory with gravity Coleman and De Luccia (1980)

Scalar field in flat space-time

The decay rate per unit volume and unit time is

$$\gamma = \Gamma/VT = \left(\frac{S_{cl}[\varphi]}{2\pi}\right)^2 |\mathcal{D}|^{-1/2} \exp\{-S_{cl}[\varphi] - S_{ct}[\varphi]\} \ , \tag{4}$$

to one-loop accuracy. The coefficient $\mathcal{D}$ here is defined as

$$\mathcal{D}[\varphi] \equiv \frac{\det'(-(\partial/\partial\tau)^2 - \Delta + V''(\varphi))}{\det(-(\partial/\partial\tau)^2 - \Delta + V''(0))} = \frac{\det'(\mathcal{M})}{\det(\mathcal{M}^{(0)})} \ . \tag{5}$$

The prime in the determinant implies omitting of the four translation zero modes. The counterterm action S_{ct} is necessary in order to absorb the divergences of the one-loop effective action

$$S^{eff}_{1-loop}[\varphi] = \frac{1}{2} \ln |\mathcal{D}[\varphi]| \ . \tag{6}$$

For details see e.g.:

G. Isidori, A. Strumia et all, 2001, 2008, 2012;

J. Baacke and G.L., 2004;

G.V. Dunne et all, 2005, 2006, 2008.

Inclusion of Gravity: Scalar field in curved space-time

The Euclidean action of system composed of scalar field minimally coupled to gravity is

$$S = \int d^4x\sqrt{g}\left[-\frac{1}{2\kappa}R + \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + V(\phi)\right] , \tag{7}$$

where $\kappa = 8\pi G$ is the reduced Newton's constant. For the $O(4)-$symmetric metric ansatz we will use

$$ds^2 = d\eta^2 + \rho^2(\eta)d\Omega_3^2 = a^2(\tau)(d\tau^2 + d\Omega_3^2) , \tag{8}$$

where η is (Euclidean) proper time, τ- conformal time, $\rho(\eta)$ is the scale factor and $d\Omega_3^2$ is metric of unit three-sphere:

$$d\Omega_3^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2(\theta)d\varphi^2) . \tag{9}$$

Corresponding field equations in the proper time are

$$\ddot{\phi} + 3\frac{\dot{\rho}}{\rho}\dot{\phi} = \frac{\partial V}{\partial \phi} , \tag{10}$$

$$\ddot{\rho} = -\frac{\kappa\rho}{3}(\dot{\phi}^2 + V(\phi)) , \tag{11}$$

$$\dot{\rho}^2 = 1 + \frac{\kappa\rho^2}{3}(\frac{\dot{\phi}^2}{2} - V) . \tag{12}$$

just Euclidean version of Friedmann equations.

We assume that potential $V(\phi)$ has characteristic asymmetric double-well shape with local minimum at some $\phi = \phi_-$, local maximum ϕ_{top} and global minimum ϕ_+:

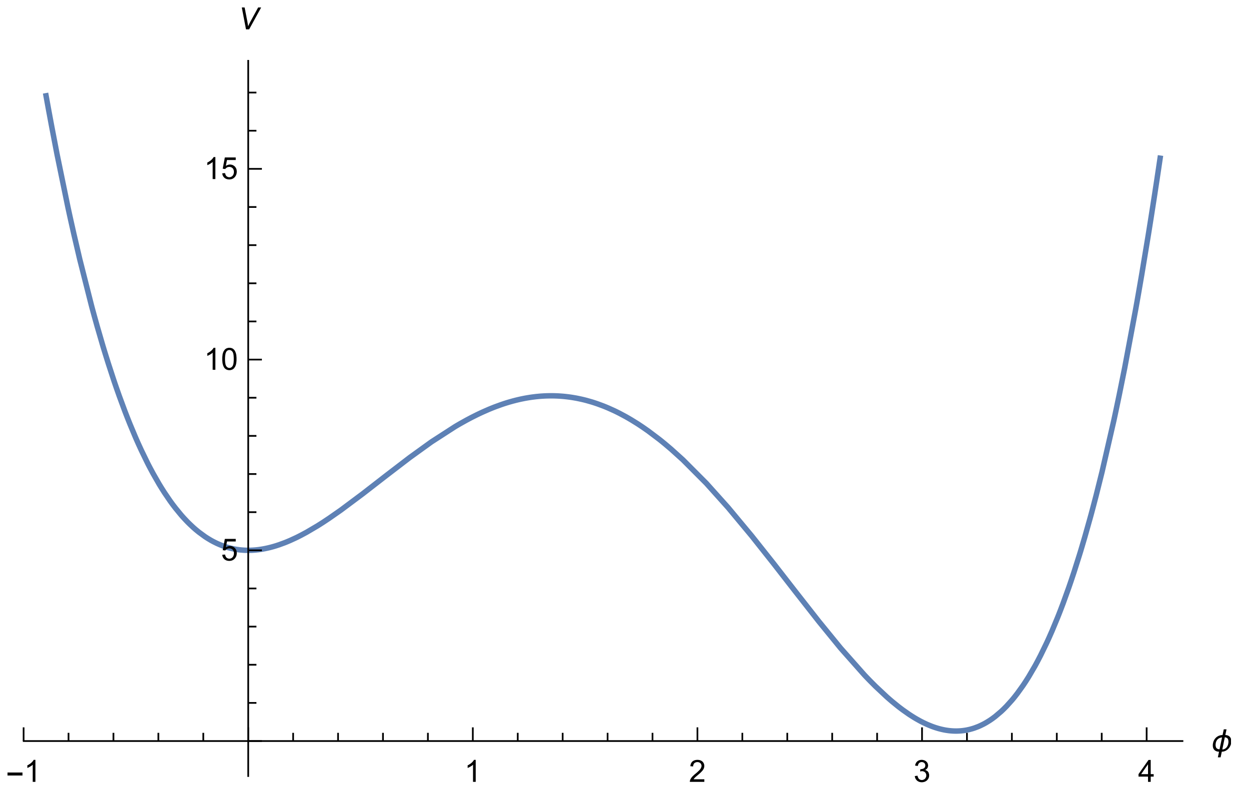


Figure 2: Scalar field potential $V(\phi)$.

Coleman and De Luccia (1980)

The metastable vacuum decay rate with gravity per unit volume and unit time is

$$\gamma = \mathcal{A} e^{-B} \ , \tag{13}$$

where $B = S_E(\phi_{bounce}) - S_E(\phi_{false})$ and $\mathcal{A}$ is pre-exponential factor.

In the thin wall approximation there are two important results

I. dS $\longrightarrow$ flat case, $V(\varphi_-) = \epsilon, \ V(\varphi_+) = 0$, gravity $\Uparrow$ probability

$$B = \frac{B_0}{[1 + (\rho_0/2\Lambda)^2]^2} \tag{14}$$

II. Flat $\longrightarrow$ AdS case, $V(\varphi_-) = 0, V(\varphi_+) = -\epsilon$, gravity $\Downarrow$ probability

$$B = \frac{B_0}{[1 - (\rho_0/2\Lambda)^2]^2} \tag{15}$$

where ρ_0 and B_0 are bubble radius and decay coefficient in the absence of gravity and $\Lambda = (\kappa\epsilon/3)^{-1/2}$.

Special classical solutions in Euclidean quantum gravity

Hawking and Moss (1982)

1. The Hawking-Moss solution is a 4-sphere corresponding to scalar field sitting on the top of the potential barrier

$$\phi(\eta) = \phi_{top}, \quad \rho(\eta) = \mathcal{H}_{top}^{-1}\, sin(\mathcal{H}_{top}\eta)\ , \tag{16}$$

with $\mathcal{H}_{top} = \sqrt{\kappa V(\phi_{top})/3}$. Note that the Euclidean time η varies in finite interval $\eta = (0, \eta_f)$.

Coleman and De Luccia (1980)

2. The Coleman-De Luccia bounce is a deformed 4-sphere. It starts with some $\phi = \phi_0$ at $\eta = 0$ close to ϕ_-, stops at $\eta = \eta_f$ close to ϕ_+ and obeys the regularity conditions

$$\rho(0) = \dot{\phi}(0) = 0, \quad \rho(\eta_f) = \dot{\phi}(\eta_f) = 0\ . \tag{17}$$

Bousso and Linde (1998), Balek and Demetrian (2004)

Hackworth and Weinberg (2005)

B.-H. Lee, C. H. Lee, W. Lee and C. Oh (2010, 2012)

3. Oscillating bounces and instantons, solutions in which the scalar field passes over the barrier more then once.

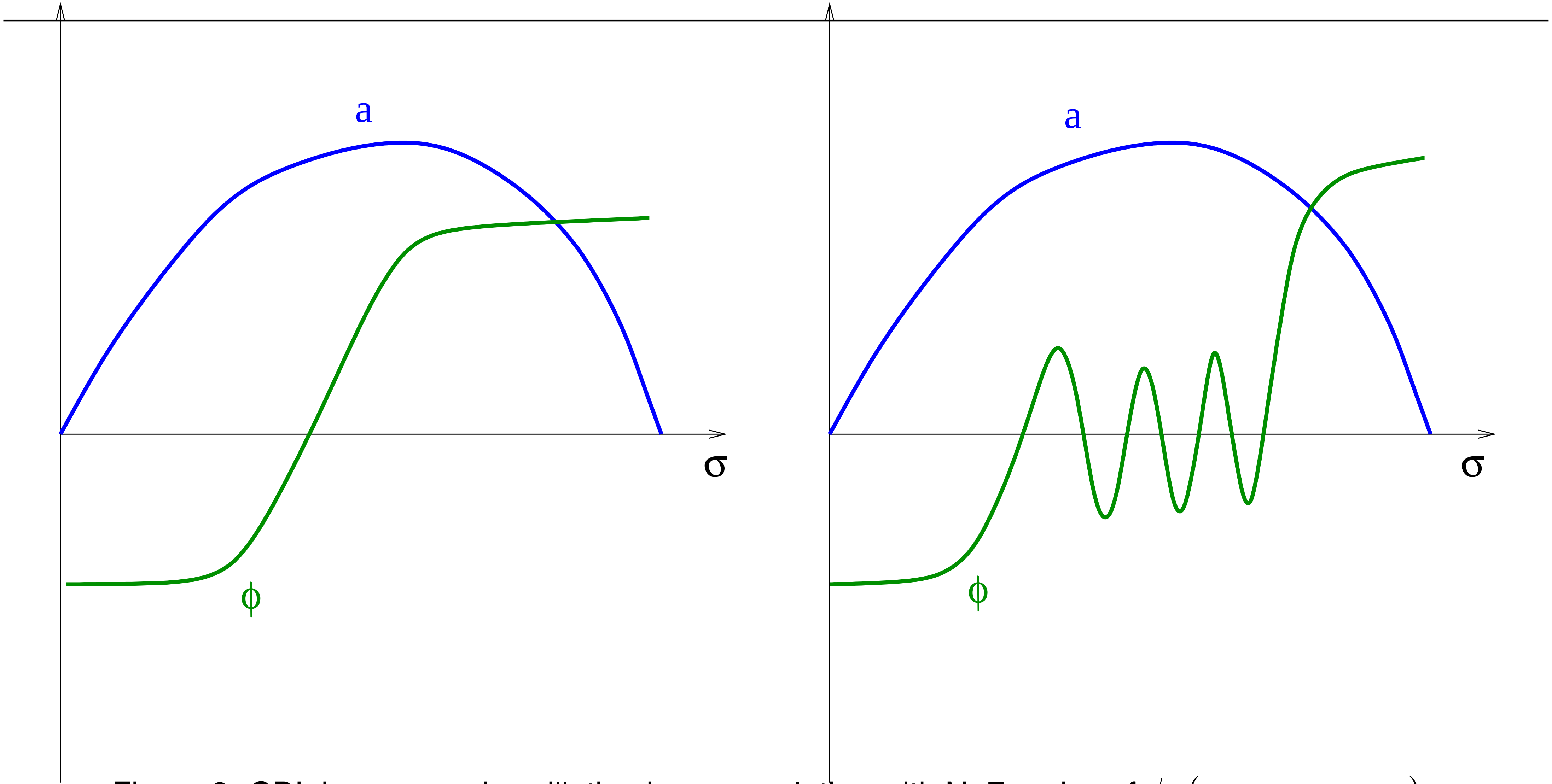


Figure 3: CDL bounce and oscillating bounce solution with N=7 nodes of ϕ, $(\sigma \equiv \eta, a \equiv \rho)$.

Negative mode problem

The Euclidean action of system composed of scalar field minimally coupled to gravity is

$$S=\int d^4x\sqrt{g}\left[-\frac{1}{2\kappa}R+\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi+V(\phi)\right]\ , \tag{18}$$

where $\kappa=8\pi G$ is the reduced Newton's constant. For the $O(4)-$symmetric metric ansatz we will use

$$ds^2=d\eta^2+\rho^2(\eta)d\Omega_3^2=a^2(\tau)(d\tau^2+d\Omega_3^2)\ , \tag{19}$$

where η is (Euclidean) proper time, τ- conformal time, $\rho(\eta)$ is the scale factor and $d\Omega_3^2$ is metric of unit three-sphere.

We expand the metric and the scalar field over a $O(4)-$symmetric background

$$\begin{aligned} ds^2 &= a(\tau)^2\left[(1+2A(\tau))d\tau^2+\gamma_{ij}(1-2\Psi(\tau))dx^idx^j\right], \\ \phi &= \varphi(\tau)+\Phi(\tau), \end{aligned} \tag{20}$$

where τ is the conformal time, a and φ are the background field values and A,Ψ and Φ are small perturbations. $^\cdot\equiv d/d\eta,'\equiv d/d\tau$.

Expanding the total action to second order in perturbations and using the background equations of motion, we find

$$S = S^{(0)}[a,\varphi] + S^{(2)}[A,\Psi,\Phi] \ , \tag{21}$$

where $S^{(0)}$ is the action of the background solution and $S^{(2)}[A,\Psi,\Phi]$ is the quadratic action for scalar $O(4)-$symmetric perturbations given by the Lagrangian:

$$\begin{aligned} {}^{(s)}\mathcal{L} = \frac{1}{2\kappa} a^2 \sqrt{\gamma} \Big[& -6\Psi'^2 + 6\mathcal{K}\Psi^2 + \kappa(\Phi'^2 + a^2 \frac{\delta^2 V}{\delta\varphi\delta\varphi}\Phi^2 + 6\varphi'\Psi'\Phi) \\ & -(2\kappa\varphi'\Phi' - 2\kappa a^2 \frac{\delta V}{\delta\varphi}\Phi + 12\mathcal{H}\Psi' + 12\mathcal{K}\Psi)A - 2(\mathcal{H}' + 2\mathcal{H}^2 + \mathcal{K})A^2 \Big] . \end{aligned}$$

Note that the variation with respect to A gives the first order (constraint) equation

$$2\kappa\varphi'\Phi' - 2\kappa a^2 \frac{\delta V}{\delta\varphi}\Phi + 12\mathcal{H}\Psi' + 12\mathcal{K}\Psi + 4(\mathcal{H}' + 2\mathcal{H}^2 + \mathcal{K})A = 0 \ .$$

To obtain unconstrained (physical) degree of freedom one should impose gauge condition and solve constraints.

Khvedelidze, G.L., Tanaka (2000)

In frame of the Hamiltonian approach, fixing the gauge by $A = 0$, $\Pi_\Psi = 0$ one obtains unconstrained quadratic action for one physical dynamical degree of freedom for $\mathcal{K} = +1$ as

$$S_E^{(2)}[\Phi] = 2\pi^2 \int \rho^3(\eta) d\eta \Big[\frac{1}{2Q(\eta)} \dot{\Phi}^2 + \frac{1}{2} U[\varphi(\eta), \rho(\eta)] \Phi^2 \Big] \, , \tag{22}$$

where the factor Q was given by

$$Q := 1 - \frac{\kappa \rho^2 \dot{\varphi}^2}{6} \, , \tag{23}$$

and the potential U is expressed in terms of the bounce solution as

$$U[\varphi(\eta), \rho(\eta)] \equiv \frac{V''(\varphi)}{Q} + \frac{2\kappa\dot{\varphi}^2}{Q} + \frac{\kappa}{3Q^2} \Big(6\dot{\rho}^2 \dot{\varphi}^2 + \rho^2 V'^2(\varphi) - 5\rho\dot{\rho}\dot{\varphi} V'(\varphi) \Big) \, . \tag{24}$$

The exact form of the fluctuation operator depends on the choice of a weight function, which can be specified by defining a norm.

Khvedelidze, G.L. and Tanaka (2000)

G.L. (2000)

Gratton and Turok (2001)

When $Q > 0$, after redefinition of variables corresponding Schrödinger equation (diagonalizing quadratic action Eq. (22)) reads:

$$-\frac{d^2}{d\eta^2}q + W[\rho(\eta), \varphi(\eta)]q = Eq \ , \tag{25}$$

and it was shown to have one boundstate, i.e corresponding Coleman-De Luccia bounce has exactly one negative mode.

Koehn, G.L. and Lehners (2015)

For general Q, with the natural choice of the norm

$$||\Phi||^2 \equiv \int d^4x \sqrt{g}\, \Phi^2 = 2\pi^2 \int d\eta\, \rho(\eta)^3\, \Phi^2 \ . \qquad (26)$$

The fluctuation equation diagonalizing the quadratic action Eq. (22) then has the form

$$-\frac{1}{Q}\frac{d^2\Phi}{d\eta^2} + \left(\frac{\dot{Q}}{Q^2} - \frac{3\dot{\rho}}{\rho Q}\right)\frac{d\Phi}{d\eta} + U\Phi = \lambda\Phi \ , \qquad (27)$$

with the potential U given in Eq. (24). Note that the function $Q \to 1$ at the ends of the interval $[0, \eta_{max}]$, but for some bounces it can become negative for some interval of η.

Main results:

1. If $Q > 0$, one finds exactly one tunnelling negative mode for all bounces

2. If Q becomes negative in some interval

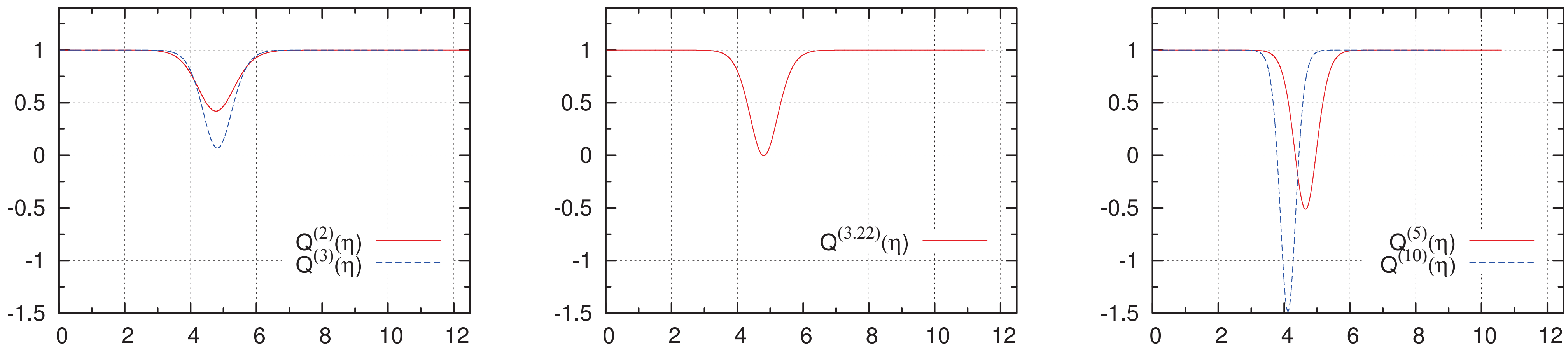
a). Solution of pulsation equation and first derivative is regular across $Q = 0$ points.

b). On top of one "tunneling" negative mode, an infinite tower of additional negative modes arise with support in $Q < 0$ region.

Example potential used by Lee & Weinberg (2014)

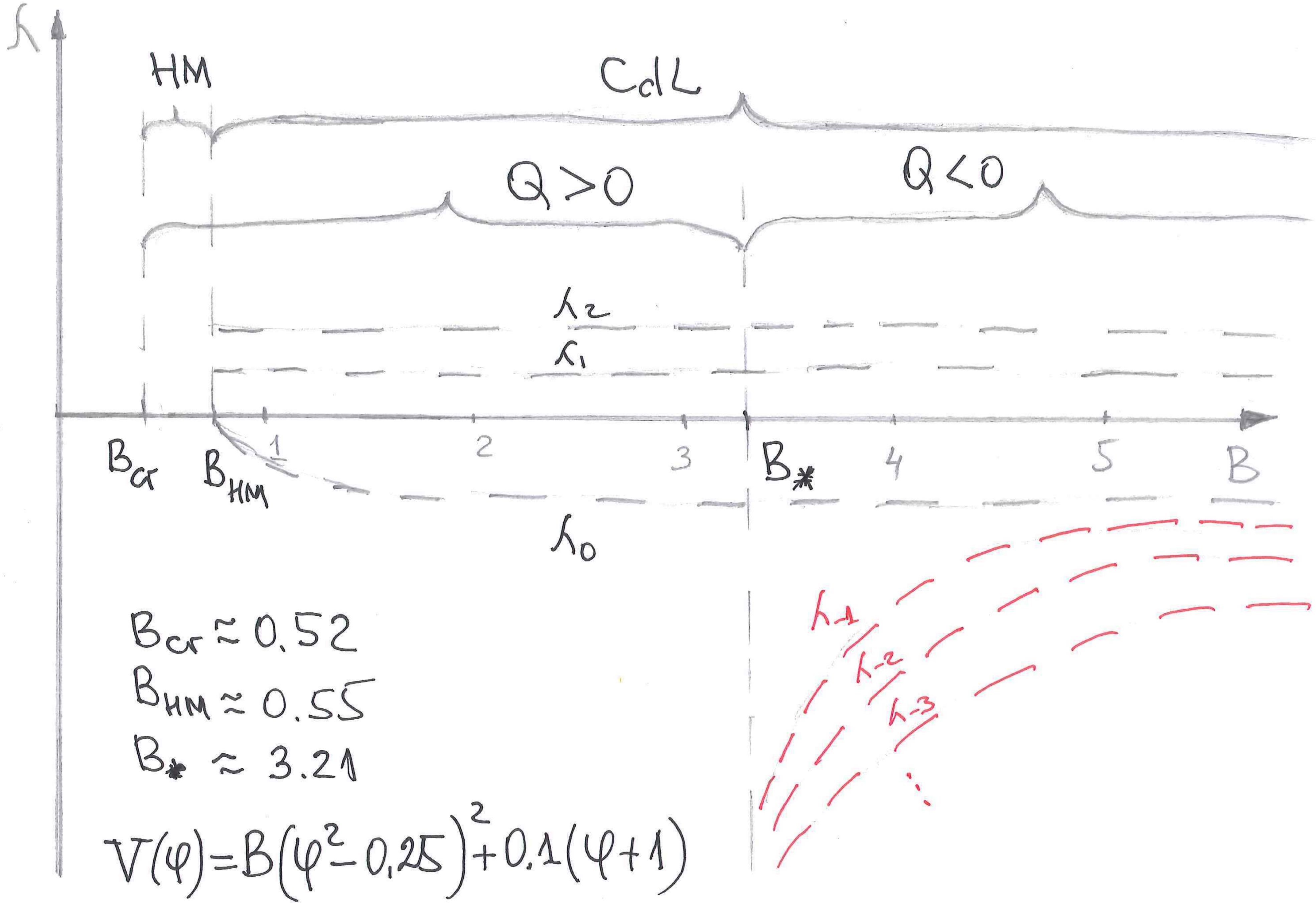
$$V(\varphi) = B(\varphi^2 - 0.25)^2 + 0.1(\varphi + 1) \; , \tag{28}$$

and corresponding factor Q for different values of parameter B:



(a) $B = 2$ and $B = 3$ (b) $B = 3.22$ ($Q < 0$ briefly) (c) $B = 5$ and $B = 10$

Figure 4: The kinetic pre-factor $Q(\eta)$ for the background potential (28) and different values of B. For $B \approx 3.22$ the factor Q becomes negative along some interval of Euclidean time η.


λ
HM
CdL
Q>0
Q<0
λ_2
λ_1
B_{cr}
B_{HM}
1
2
3
B_*
4
5
B
λ_0
λ_{-1}
λ_{-2}
λ_{-3}
$B_{cr} \simeq 0.52$
$B_{HM} \simeq 0.55$
$B_* \simeq 3.21$
$V(\varphi) = B(\varphi^2 - 0{,}25)^2 + 0{,}1(\varphi + 1)$

Other recent results

Hakjoon Lee and Erick Weinberg (2014)

Working in Lagrangian approach with gauge invariant variable χ,

$$\chi \equiv \dot{\rho}\Phi - \rho\dot{\varphi}\Psi \quad \text{and factor} \quad Q_{LRT} := \dot{\rho}^2 - \frac{\kappa\rho^2\dot{\varphi}^2}{6} = 1 - \frac{\kappa\rho^2}{3}V \ , \tag{29}$$

Lee and Weinberg solved numerically with Mathematica pulsation equation for concrete potentials and arrived to conclusion that type A bounces have tunneling negative mode whereas type B bounces don't!

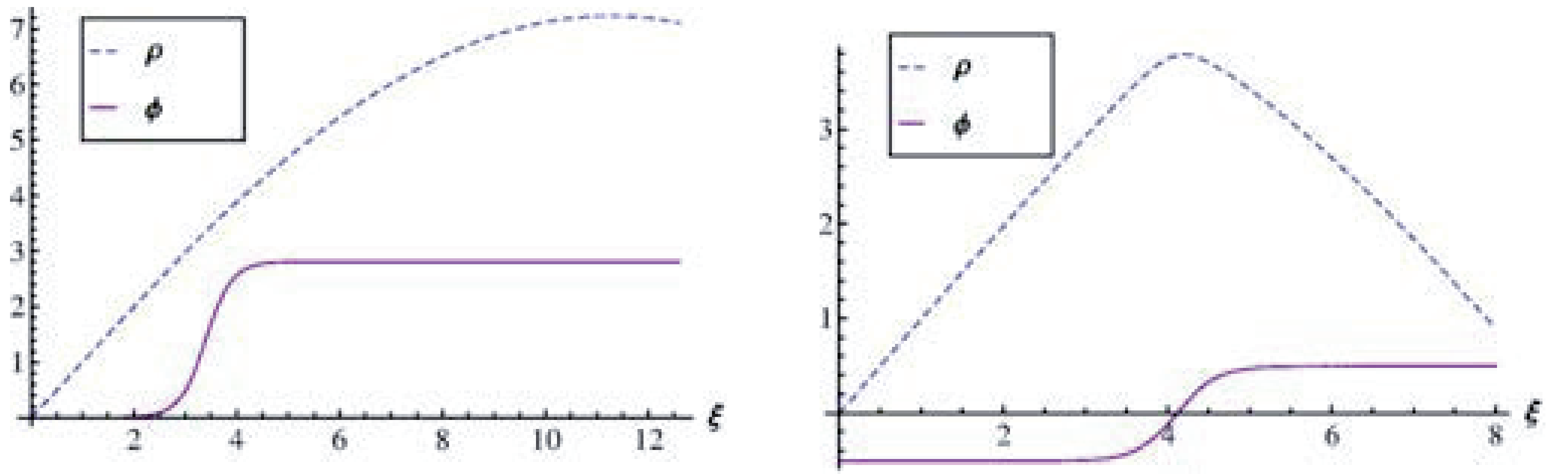


Figure 5: Type A bounces (left panel) and type B (right panel).

Historical Summary of the negative mode problem

Year / Authors	Tunneling negative mode	Additional negative modes
1985, G.L., Rubakov Tinyakov	Not discussed	Infinitely many
1992, Tanaka, Sasaki	None	None
2000, Khvedelidze, G.L., Tanaka	One	Not discussed (only $Q > 0$ case)
2000, Gratton, Turok	One	Not discussed (only $Q > 0$ case)
2006, G.L.	N for the N-th oscillating bounce	Not discussed (only $Q > 0$ case)
2014, Lee, Weinberg	One for type A / None for type B	Infinitely many
2015, Koehn, G.L., Lehners	One	Infinitely many in $Q < 0$ case

Tab 1. Conclusion about the number of negative modes reached in different investigations.

Concluding remarks

1. We have found that with the proper reduction scheme CdL solution has exactly one negative mode for $Q > 0$ backgrounds, as it should be for proper bounce.

2. The oscillating instantons and bounces with N nodes have exactly N homogeneous negative modes in their spectrum of linear perturbations. Existence of more than one negative modes makes obscure the relation of these oscillating bounce solutions to the false vacuum decay processes.

3. When $Q < 0$ along the bounce:

a). Solution of pulsation equation is smooth across $Q = 0$ points.

b). Single tunneling negative mode continues to exist.

c). Additional, infinite number of negative modes uppears.

Challenge:

- How to interpret an infinite tower of additional negative modes for $Q < 0$ cases:

Their existence and significance remain mysterious even after more than 30 years.

Thank you for attention!

G.L., Rubakov, Tinyakov (1985)

The I approach (Lagrangian):

Fixing the gauge with the condition $\Psi = 0$ and eliminating A with the help of the constraint equation we obtain the unconstrained quadratic action in the form

$$S^{(2)}_{LRT} = \int \frac{a^4\sqrt{\gamma}}{2Q_{LRT}} \left[\frac{\mathcal{H}^2}{a^2}\Phi'^2 - \frac{\kappa\varphi'}{3}\frac{\delta V}{\delta\varphi}\Phi'\Phi \right.$$
$$\left. + \left(\frac{\kappa a^2}{6}(\frac{\delta V}{\delta\varphi})^2 + Q_{LRT}\frac{\delta^2 V}{\delta\varphi\delta\varphi} \right)\Phi^2 \right] d\tau d^3x, \quad (30)$$

with

$$Q_{LRT} := \mathcal{H}^2 - \frac{\kappa{\varphi'}^2}{6} = \mathcal{K} - \frac{\kappa a^2}{3}V \; . \quad (31)$$

Tanaka and Sasaki (1992)

The II approach (Hamiltonian):

Fixing the gauge by $\Phi = 0$ and eliminating Π_Φ (matter degrees of freedom), after some canonical transformation one gets the quadratic part of the Euclidean action ($\mathcal{K}$ is the curvature parameter):

$$S^{(2)} = \frac{(1-4\mathcal{K})}{2} \int \Big[(\frac{dq}{d\tau})^2 + U q^2 \Big] \sqrt{\gamma} d^3x d\tau \ , \tag{32}$$

with a potential U depending on the background fields

$$U = \frac{\kappa}{2} {\varphi'}^2 + \varphi' (\frac{1}{\varphi'})'' + 1 - 4\mathcal{K} \ . \tag{33}$$

We see that quadratic action for the homogeneous harmonic has “wrong” overall sign. To overcome this problem it was suggested that analytic continuation $q \to -iq$ is performed while integrating over this mode.

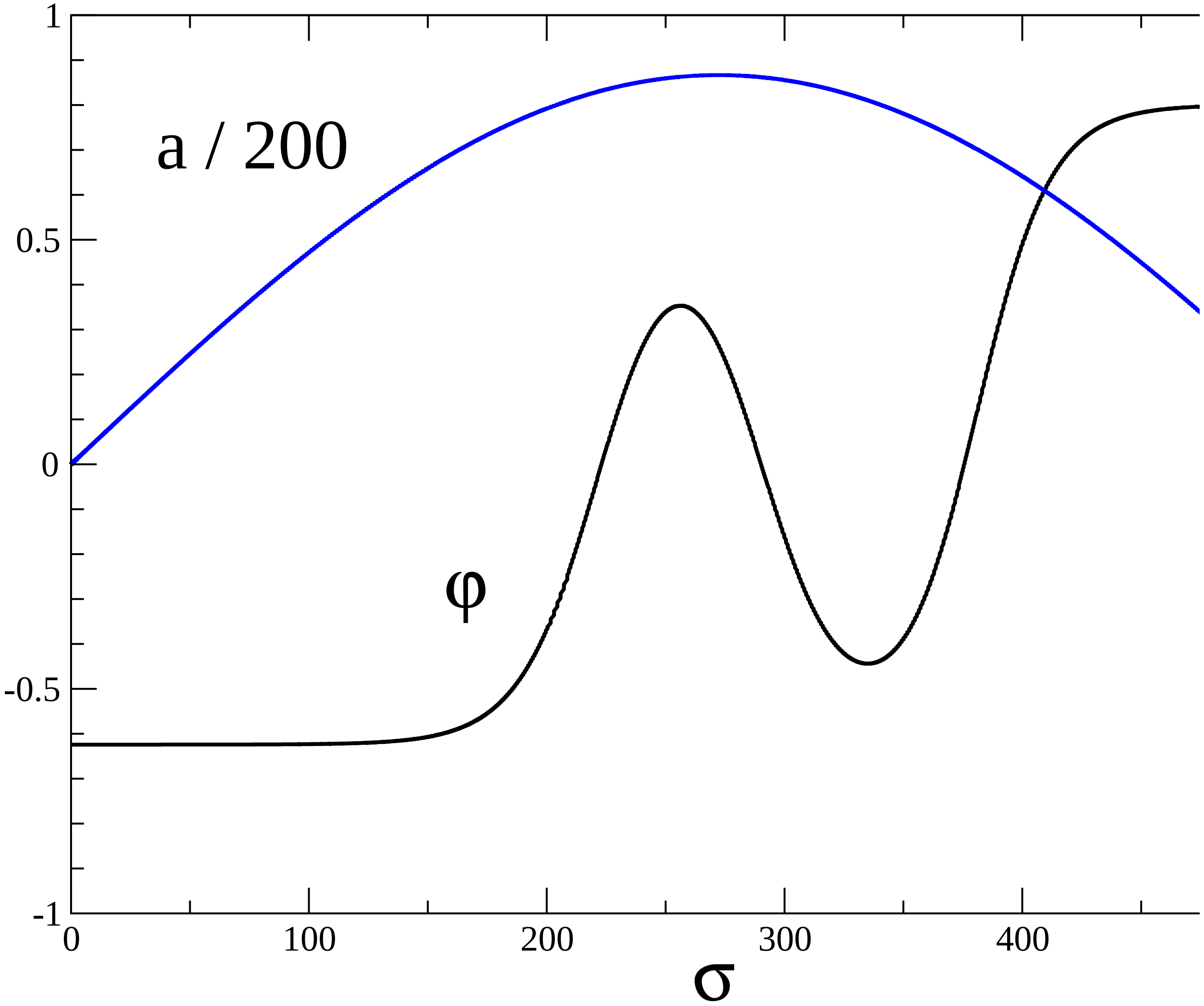


Figure 6: Oscillating bounce solution with three nodes of φ.

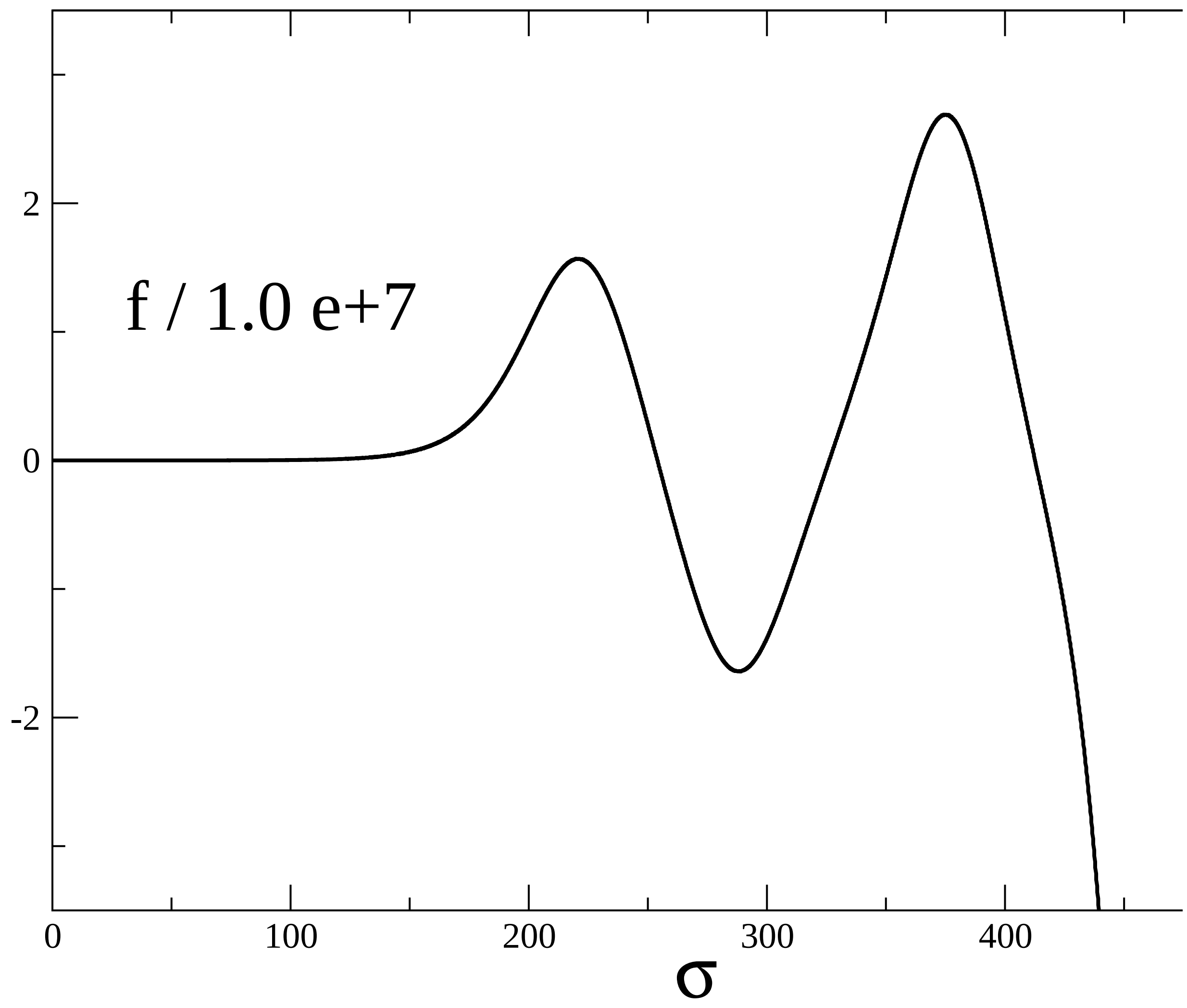


Figure 7: Zero energy wave function f of Schrödinger equation of linear perturbations about oscillating bounce solution with three nodes of φ.

Note that by definition at the local maximum, φ_{top}, the second derivative of potential is negative, $V''(\varphi_{top}) < 0$.

For $-4 < \frac{V''(\varphi_{top})}{\mathcal{H}^2_{top}} < 0$ CDL bounce does not exists and HM has one negative modes.

For $\frac{V''(\varphi_{top})}{\mathcal{H}^2_{top}} < -4$ CDL comes to existence and has one negative mode, whereas HM gets extra negative modes.