Strong coupling constant from hadronic τ decays

Badri Magradze[1]

[1] A. Razmadze Mathematical Institute of Iv. Javakhishvili
Tbilisi State University

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The strong coupling constant $\alpha_s = g^2/(4\pi)$ is one of the fundamental parameters of the Standard Model, one of the three gauge couplings, a "fundamental constant" of nature. Like other constants, it must be determined from the experiment. The precise determination of α_s is one of the most important aims of particle physics.

The value of α_s is of fundamental importance for our understanding of QCD and the standard model. It is an important input to precision studied of potential discrepancies between experiment and theory relevant to searches for beyond the Standard Model physics.

It has been determined experimentally in a larger number of independent processes, over a wide range of scales. Its variation over the range $1.78\,GeV < \mu < M_z$ is in excellent agreement with QCD, –a highly non-trivial test of the theory. Predictions for its energy dependence based on the renormalization group equations were confirmed. Critical tests of the standard model and the discovery of physics beyond the Standard Model essentially depend on the precise numerical value of α_s.

The reference value for the coupling constant is commonly given at the scale $M_z = 91.187$ GeV. A world average value of the the $\overline{\text{MS}}$ scheme coupling in 2016

$$\alpha_s(M_z^2) = 0.1181 \pm 0.0011.$$

Bethke 2016

One of the highest-precision low-energy determination comes from finite energy sum rule (FESR) analyses of hadronic τ decay non-strange data. The accuracy of the experimental data, for the invariant mass distributions, available from **ALEPH** and **OPAL** collaborations is very high.

ALEPH 1998,2005,2008, 2013/14

OPAL 1998

On the theoretical side very accurate results are available, the τ-decay rate is calculated up to order α_s^4

P.A. Baikov, K.G. Chetyrkin J.H. Kuhn, (2008)

■ τ is the only lepton heavy enough to decay into hadrons. The characteristic scale of the process is the mass of the τ, $m_\tau = 1.7768\,\text{GeV},$. The perturbative QCD is still applicable, since $\alpha_s(m_\tau) \approx 0.300$.

Main ingredients of the pQCD calculations is the renormalization group improved perturbation theory and the Wilson's Operator Product Expansion (OPE) that provide a systematic approximation scheme for high-energy calculations of physical quantities in the space-like region ($q^2 = -Q^2 < 0$).

In the time-like region perturbation theory cannot be used directly. The calculations is performed owing to the hypothesis of **global quark-hadron duality** suggested by

E. Poggio, H. Quinn, and S. Weinberg 1977,Phys.Rev.D**13** 1958-1968 (1976)

For more recent formulation of the duality see

Shifman, M.A.Quark-hadron duality. Boris Ioffe Festschrift. At the Frontier of Particle Physics,2001

In the pioneering work

E. Braaten, S. Narison, A. Pich, Nucl. Phys. B **373** (1992) 581.

was demonstrated the applicability of pQCD for description the inclusive semi-hadronic decays of the tau.

■ Central quantity to be of interest is the τ decay rate into hadrons, normalized to the lepton decay rate

$$R_\tau(s_0) = \frac{\Gamma(\tau^- \to \nu_\tau + hadrons)}{\Gamma(\tau^- \to \nu_\tau e^- \bar{\nu}_e)} \qquad (1)$$

where $s_0 = m_\tau^2$: $m_\tau = 1.77682\,\mathrm{GeV}$. At the parton level

$$R_\tau = N_c(|V_{ud}|^2 + |V_{us}|^2) \approx 3$$

We shall be concerned only with the decays into non-strange hadrons in the vector channel

$$R_\tau(s_0)|_{non-strange,V} = 6S_{EW}|V_{ud}| \int_0^{s_0} \frac{d\,R_\tau(s)}{d\,s} d\,s \qquad (2)$$

where

$$\frac{d\,R_\tau(s)}{d\,s} = w_\tau(s, s_0) v_1(s) \qquad (3)$$

with

$$w_\tau(s, s_0) = \frac{1}{s_0}(1 - s/s_0)^2(1 + 2s/s_0)$$

and

$$v_1(s) = 2\pi\, \mathrm{Im}\, \Pi^{(1)}(s + \imath\epsilon)$$

The theoretical analysis of $R_{\tau,V}(s_0)$ involves the two-point correlation function for the vector color singlet quark currents, (isovector-vector current)

$$V_\mu(x) = \bar{u}(x)\gamma_\mu d(x)$$

$$\Pi_{\mu\nu}(q^2) = \imath \int d^4x \, \mathrm{exp}(\imath qx)\langle 0|T(V_\mu(x)V_\nu^+(0))|0\rangle \qquad (4)$$

$$\Pi_{\mu\nu}(q^2) = (-g_{\mu\nu}q^2 + q_\mu q_\nu)\Pi^{(1)}(q^2) + q_\mu q_\nu \Pi^{(0)}(q^2),$$

A powerful method to evaluate QCD predictions for the τ decays is the Finite Energy Sum Rules (FESR)

N.V. Krasnikov, A.A. Pivovarov, A.N. Tavkhelidze, Z. Phys. C **19** (1983) 301.

The correlation function $\Pi(q^2)$ is an analytic function in the whole complex $z = q^2$ plane but the cut $z \in (0, \infty)$. Let $w(z)$ be any analytic function in the cut plane, then FESR follows from the Cauchy's theorem for the product $w(z)\Pi(z)$.

$$\int_0^{s_0} w(s) v_1(s)\, d\, s = -\frac{\pi}{\imath} \oint_{|z|=s_0} w(z)\Pi(z)\, dz \tag{5}$$

$$\int_0^{s_0} w(s) v_1(s)\, d\, s = -\frac{1}{4\imath\pi} \oint_{|z|=s_0} \frac{w_1(z)}{z} D(-z)\, dz \tag{6}$$

where $D(Q^2)$ ($Q^2 = -q^2$) is the Adler function

$$D(Q^2) = D(-q^2) = -4\pi^2 q^2 \frac{d}{q^2} \Pi(q^2) \tag{7}$$

and

$$w_1(z) = \int_{s_0}^{z} w(z) d\, z. \tag{8}$$

In QCD $\Pi(q^2)$ can be represented as

$$\Pi(Q^2)_{QCD} = \Pi(Q^2)_{pQCD} + \Delta(q^2)|_{DV} \tag{9}$$

where $\Delta(q^2)|_{DV}$ is non-perturbative duality violating contribution which will be ignored in the sequel and

$$\begin{aligned} \Pi(Q^2)|_{pQCD} &= \Pi(Q^2)|_{PT} + \Pi(Q^2)|_{OPE} & (10) \\ D(Q^2)|pQCD &= D(Q^2)|_{PT} + D(Q^2)|_{OPE} \end{aligned}$$

$$(11)$$

where

$$D(Q^2)|_{PT} = \sum_{n=0} d_n \left(\frac{Q^2}{\mu^2}\right) \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n \tag{12}$$

where α_s is the running coupling parameter of QCD. Using Renormalization Group invariance

$$D(Q^2)|_{PT} = \sum_{n=0} K_n \left(\frac{\alpha_s(Q^2)}{\pi}\right)^n \tag{13}$$

with

$$K_0 = K_1 = 1,\ K_2 = 1.6398,\ K_3 = 6.3710,\ K_4 = 49.0757.$$

Baikov P. A., Chetyrkin K. G., Kuhn J. H.(2008)Phys.Rev.Lett.101: 012002.

the running coupling $\alpha_s(Q^2)$ is solved from the RG equation

$$Q^2 \frac{d}{dQ^2} \alpha_s(Q^2) = \beta(\alpha_s(Q^2)) = \sum_{k=0} \beta_k \alpha_s(Q^2)^{k+2}$$

the last coefficient β_4 was calculated recently

Baikov P. A., Chetyrkin K. G., Kuhn J. H.(2017)Phys.Rev.Lett. 082002.

The RG invariance and the analyticity cannot be combined unambiguously. The most popular methods are

1) **fixed order perturbation theory** (FOPT)
2) **contour improved perturbation theory** (CIPT)
A.A. Pivovarov, Z. Phys. C **53** (1992) 461
F. Le Diberger, A Pich Phys. Lett. B **286** (1992) 147.
3) **The analytic approaches to pQCD**
D.V. Shirkov, I. Solovtsov 1998

Arguments have been given that **CIPT, conceptually, is not well defined**: the predictions obtained within CIPT presumably may be distorted due to the non-physical Landau singularities which present in the running coupling. The Kallen-Lehmann Analyticity of the correlators is a strong consequence of the general principles of the QFT.

■ Conceptual and practical problems in RG improved perturbation theory

From general principles of local field theory (Lorentz-invariance, causality, positivity of energy, unitarity etc) follows the cut-plane analyticity for the physical quantities.

However, the running coupling violates the cut-plane analyticity because of the **"Landau ghost pole"** problem:
The one-loop order running coupling

$$\alpha_s(Q^2) = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)} \approx \frac{\Lambda^2}{\beta_0(Q^2 - \Lambda^2))} \tag{14}$$

■Analytic or dispersive approaches to pQCD

To overcome this obstacle analytic or dispersive approaches to perturbative QCD are being developed.

Redmond 1959 Phys.Rev.: dispersive approach to QED

The most prominent analytic approach: the Analytic Perturbation Theory (APT)

D.V. Shirkov, I.L. Solovtsov, Phys. Rev. Lett. **79** (1997) 1209.

K.A. Milton, I.L. Solovtsov, O.P. Solovtsova, Phys. Lett. **B415** (1997) 104.

The Adler function (or correlator) satisfies the dispersion relation

$$D(Q^2) = Q^2 \int_0^\infty \frac{2v_1(s)ds}{(s+Q^2)^2}, \qquad (15)$$

the inversion formula reads

$$v_1(s) = \frac{1}{4\pi\imath} \oint_{-s-\imath\epsilon}^{-s+\imath\epsilon} \frac{D(z)}{z} d\,z, \tag{16}$$

$$D_{pt}(Q^2) = D_0(1 + \sum_{n=1}^{\infty} d_n \alpha_s^n(Q^2)),$$

$$D_{an}(Q^2) = D_0(1 + \sum_{n=1}^{\infty} d_n A_n(Q^2))$$

$$A_n(Q^2) = \frac{1}{\pi} \int_0^{\infty} \frac{\rho_n(\sigma, f)}{\sigma + Q^2} d\sigma$$

where

$$\rho_n(\sigma) = \text{Im}\{\alpha_s(-\sigma - \imath 0)\}^n,$$

in this framework the analytically improved solution to the RGE is introduced, at the one loop order the analytic APT coupling takes the form

$$A_1(Q^2)|_{1-\text{loops}} = 1/\ln(Q^2/\Lambda^2) + \Lambda^2/(\Lambda^2 - Q^2) \qquad (17)$$

However, it was shown in the work

B.V. Geshkenbein, B.L. Ioffe and K.N. Zyablyuk, Phys.Rev. D64: 093009,2001

That **APT with massless quarks is in strong contradiction with experiment**.

More serious difficulty is that APT violates the OPE. It predicts new power suppressed contributions of ultraviolet origin in the correlators, that are not included in OPE. At present the experiment does not confirm the presence of such terms.

Exact explicit solutions to the RG equation and the Lambert-W function

The analyticity structure of the two-loop and higher order running coupling function was studied in

1. B. Magradze: In proceedings of the 10th International Seminar "QUARKS-98" May 1998
2. E. Gardi, G. Grunberg and M. Karliner June 1998

3. B. Magradze: Proc. A. Razmadze Math.Inst. 118,111 (1998)

4. B. Magradze: Int. J. Mod. Phys. A15,2715 (2000)

5. D. Kourashev, B. Magradze, Theor. Math. Phys. 135, 531 (2003)

6. B. Magradze: Few Body Systems V40, 2006, pp. 71-99

In these papers the solutions to the RGE were expressed in terms of the Lambert W function. The Lambert W function is the multivalued solution of

$$W_k(z)\,\exp\{W_k(z)\} = z, \tag{18}$$

the branches of W are denoted $W_k(z)$, $k = 0, \pm 1, \ldots$ The relevant branch of $W(z)$ which determines the coupling depends on the number of light quark flavours n_f. Thus, for $0 \le n_f \le 8$ and a real positive Q^2 the explicit expression for the $\overline{\mathrm{MS}}$ scheme running coupling at the two-loop order reads

B.A. Magradze Proceedings of A. Razmadze Mathematical Institute **118**, 111,(1998)

$$a_s^{(2)}(Q^2) = -\frac{\beta_0}{\beta_1}\frac{1}{1+W_{-1}(\zeta)} : \quad \zeta = -\frac{1}{eb_1}\left(\frac{Q^2}{\Lambda^2}\right)^{-1/b_1}, \qquad (19)$$

where β_0 and β_1 are the first two β-function coefficients

$$\beta_0 = \frac{1}{4}\left(11 - \frac{2}{3}n_f\right), \quad \beta_1 = \frac{1}{16}\left(102 - \frac{38}{3}n_f\right),$$

$b_1 = \beta_1/\beta_0^2$, $\Lambda \equiv \Lambda_{\overline{\mathrm{MS}}}$ and W_{-1} denotes the branch of the Lambert W function.

On the other hand, a running coupling at higher orders may be expanded in powers of the exact (explicitly solved) two-loop order coupling

Kourashev, D.S., Magradze, B.A.: Theor. Math. Phys. **135**, 531 (2003)

$$\alpha_s^{(\mathrm{k-loops})}(Q^2) = \sum_{n=1}^{\infty} \mathcal{C}_n^{(k)} \alpha_s^{(\mathrm{two-loops})n}(Q^2)|_{\mathrm{exact}}, \qquad (20)$$

where the numerical coefficients $\mathcal{C}_n^{(k)}$ are determined in terms of the β-function coefficients (see Appendix A). It has been shown in

Magradze, B.A.: Few-Body Systems **40**,71-99 (2006)

that this series has a sufficiently large radius of convergence in the space of the coupling constants.

The non-perturbative part is the asymptotic series

$$\Pi(s)|_{OPE} \approx \sum_{\mathcal{D}=2,4..} \frac{C_{\mathcal{D}}(-s)\langle \mathcal{O}_{\mathcal{D}}(\mu)\rangle|_{vac}}{(-s)^{\mathcal{D}/2}} \tag{21}$$

$s = q^2$, $C_{\mathcal{D}}(-s)$ is the Wilson coefficient and $< \mathcal{O}_{\mathcal{D}} > |_{vac}$ is the QCD condensate of dimension $\mathcal{D} = 2k$. It can be represented also as

$$\Pi(s)|_{OPE} = \sum_{k=1,2..} \frac{C_{2k}(s)}{(-s)^k} \tag{22}$$

where $C_{2k}(s)$ is the QCD condensate combination of dimension $\mathcal{D} = 2k$. Usually in the Wilson coefficients the weak dependence on s is ignored i.e. $C_{2k}(s) = constant$ They depend on s via $\alpha_s(s)$. Up to logarithmic corrections proportional to α_s^2

$$C_{2k}(s) \approx C_{2k}^{(0)} + C_{2k}^{(1)}\alpha_s(m_\tau^2)$$

Choosing special weights $w(s) = s^n$ $n = 0, 1 \ldots$, in the sum rule we may determine the condensate combinations by perturbation theory and the data.

The approach presented in this talk is motivated by the *ansatz* frequently used in the ITEF (SVZ) QCD sum rule framework. The quark-hadron duality can be implemented via the following *ansatz* for the hadronic spectral function

$$v_1(s) \simeq v_1(s)|_{\text{semi.exp}} = \theta(s_c - s)v_1(s)|_{\text{exp}} + \theta(s - s_c)v_1(s)|_{\text{pQCD}}, \tag{23}$$

R.A. Bertlmann. G. Launer and de Rafael 1985

Peris, S., Perrottet, M., de Rafael, E. 1998

s_c denotes the continuum threshold, the energy squared above which we trust perturbative QCD, $v_1(s)|_{\text{exp}}$ is the spectral function measured on the experiment and $v_1(s)|_{\text{pQCD}}$ is determined by the theoretical model, i.e. QCD.

Let us choose in the FESR the "spectral weights" associated with the spectral moments of the hadronic invariant mass distributions

$$w_{kl}(s,s_0) = \frac{1}{s_0}\left(1-\frac{s}{s_0}\right)^{k+2}\left(\frac{s}{s_0}\right)^{l}\left(1+2\frac{s}{s_0}\right) \tag{24}$$

where $k,l = 0,1\ldots$ we choose $s_0 = m_\tau^2$, and the moments are given by

$$R^{kl}(s_0) = \int_{s_{\rm thr}}^{s_0} w_{kl}(s,s_0) v_1(s) d\,s,$$

taking into account the *ansatz* (20) we write the FESR as

$$\int_{s_c}^{s_0} w_{kl}(s) v_1(s)|_{\rm ex} d\,s = -\frac{1}{4\pi\imath}\left(\oint_{|z|=s_0} - \oint_{|z|=s_c}\right)\frac{w_{1,kl}(-z)}{z} D(z)|_{\rm PT} d\,z \tag{25}$$

where $s_0 = m_\tau^2 > s_c$. In the assumed approximation the integrated OPE contributions cancel on the RHS of the FESR.

Let us assume that at $s = s_c$ the integrated DVs are also small and can be ignored. So that, the RHS of the FESR should be calculable in pure PT.

The dimension two FESR

An independent sum rule follows, in the chiral limit, from the absence of the operator of dimension $d = 2$ in the OPE of the correlator

$$\int_{s_{th}}^{s_c} v_1(s)|_{\text{ex}} d\, s = \frac{1}{4\imath\pi} \oint_{|z|=s_c} \frac{z + s_c}{z} D(z)|_{PT} d\, z \qquad (26)$$

We will assume that the parameter s_c is the same in FESRs (22) and (23).

We will use the updated and corrected data (in 2013/2014) from the **ALEPH** collaboration

M. Davier et al. Eur.Phys.J. C44:2803(2014)

The input values

$$\begin{aligned} m_\tau &= 1.77682 \pm 0.00016\,\text{GeV} \\ B_e &= 0.17818 \pm 0.00032, \\ S_{EW} &= 1.0198 \pm 0.0006 \\ |V_{ud}| &= 0.97418 \pm 0.00019 \end{aligned}$$

Data for the invariant mass distribution $sfm2(s)$ is organized in bins with variable width, the bin number k is centered at sbin(k) and has width dsbin(k). $1 \leq k \leq 80$.

$$v_1(sbin(k)) = \frac{m_\tau^2 sfm2(k)}{6|V_{ud}|S_{EW}100B_e w_T(sbin(k))dsbin(k)} \tag{27}$$

Note that a w_{kl} FESR (23) should be compatible with the dimension-two FESR (24). Each of these FESRs relates the parameters s_c and QCD scale parameter in the $\overline{\mathrm{MS}}$ scheme $\Lambda \equiv \Lambda_{\overline{\mathrm{MS}}}$. The compatibility condition of the two FESRs leads to the system of equations for the parameters

$$F_1^{kl}(s_c, \Lambda) = I_1^{kl}(s_c)|_{\mathrm{ex}} \tag{28}$$

$$F_2(s_c, \Lambda) = I_2(s_c)|_{\mathrm{ex}} \tag{29}$$

where $F_1^{kl}(s_c, \Lambda)$ and $F_2(s_c, \Lambda)$ denote the QCD parts of the w_{kl} and dimension-two FESRs respectively. We seek admissable solutions in the domain

$$1\,\mathrm{GeV}^2 \leq s_c < m_\tau^2, \tag{30}$$

$$0.280\,\mathrm{GeV} \leq \Lambda \leq 0.420\,\mathrm{GeV} \tag{31}$$

With these restrictions with the ALEPH data the system admits a unique solution. In the numerical calculations we have used the

RG equation at four loop order. We employed the very accurate analytic approximation to the four-loop order running coupling determined in terms of the Lambert-W function. We employ N^3LO approximation to the Adler function.

The error analysis

The errors were determined using the system equations (26)-(27) with the covariance matrices provided by **ALEPH**.

Results

We have solved numerically the system (26)-(27)for several w_{kl} weights. We give results obtained using modified FOPT ($FOPT^+$) and CIPT ($CIPT^+$) approaches (based on the w_{kl} FESRs (23)) separately. Results are given in Tables 1 and 2

Table 1. The CIPT^{+} results obtained from the ALEPH τ decay data in the $\overline{\text{MS}}$ scheme, and using $w_{k,l}$ FESRs. The errors are given from the experimental uncertainties only.

(K,L)	$\Lambda_{n_f=3}$ GeV	$\alpha_s(m_\tau)$	s_c GeV^2
(0,0)	$0.349 \pm 0.021_{\text{exp}}$	$0.322 \pm 0.011_{\text{exp}}$	$1.70 \pm 0.03_{\text{exp}}$
(1,0)	$0.339 \pm 0.019_{\text{exp}}$	$0.316 \pm 0.010_{\text{exp}}$	$1.73 \pm 0.03_{\text{exp}}$
(1,1)	$0.344 \pm 0.020_{\text{exp}}$	$0.319 \pm 0.011_{\text{exp}}$	$1.72 \pm 0.03_{\text{exp}}$
(1,2)	$0.348 \pm 0.022_{\text{exp}}$	$0.321 \pm 0.011_{\text{exp}}$	$1.70 \pm 0.03_{\text{exp}}$
(1,3)	$0.358 \pm 0.025_{\text{exp}}$	$0.327 \pm 0.013_{\text{exp}}$	$1.68 \pm 0.04_{\text{exp}}$

Table 2. The same quantities as in Table 1 but obtained using FOPT$^+$.

(K,L)	$\Lambda_{n_f=3}$ GeV	$\alpha_s(m_\tau)$	s_c GeV^2
(0,0),	$0.303 \pm 0.024_{\text{exp}}$	$0.298 \pm 0.012_{\text{exp}}$	$1.69 \pm 0.03_{\text{exp}}$
(1,0),	$0.299 \pm 0.022_{\text{exp}}$	$0.296 \pm 0.011_{\text{exp}}$	$1.72 \pm 0.02_{\text{exp}}$
(1,1),	$0.299 \pm 0.024_{\text{exp}}$	$0.296 \pm 0.012_{\text{exp}}$	$1.72 \pm 0.03_{\text{exp}}$
(1,2),	$0.303 \pm 0.024_{\text{exp}}$	$0.298 \pm 0.012_{\text{exp}}$	$1.69 \pm 0.03_{\text{exp}}$
(1,3),	$0.306 \pm 0.028_{\text{exp}}$	$0.299 \pm 0.014_{\text{exp}}$	$1.69 \pm 0.03_{\text{exp}}$

As our best values for $\alpha_s(m_\tau^2)$, we take the values from the $w_{0,0}$ FESR

$$\alpha_s(m_\tau^2)|^{n_f=3}_{\mathrm{FOPT}^+} = 0.298 \pm 0.012|_{\mathrm{ex}}, \quad (32)$$

$$\alpha_s(m_\tau^2)^{n_f=3}_{\mathrm{CIPT}^+} = 0.322 \pm 0.011|_{\mathrm{ex}} \quad (33)$$

Performing evaluation of the α_s values to the Z^0-mass scale

$$\alpha_s(M_z^2)|^{n_f=5}_{\mathrm{FOPT}^+} = 0.1158 \pm 0.0016|_{\mathrm{ex}} + 0.0005|_{ev} \quad (34)$$

$$\alpha_s(M_z^2)|^{n_f=5}_{\mathrm{CIPT}^+} = 0.1189 \pm 0.0013|_{ex} + 0.0005|_{ev} \quad (35)$$

The CIPT$^+$ value here is in good agreement with the recent world summary of the determinations of the strong coupling constant.

Conclusions

▲ We have determined the numerical value for the strong coupling constant $\alpha_s(m_\tau^2)$ from the inclusive semi-hadronic decays of the τ lepton in the vector channel. We analyze the corrected ALEPH data using five FESRs based on the spectral weights $w_{kl}(s)$. in combination with the dimension two FESR. In the w_{kl} FESRS we have chosen the duality region specifically: the integration interval in the energy squared range have been limited from below $0 < s_c < s < m_\tau^2$. This enable us to eliminate the non-perturbative OPE contributions from the FESRs. In standard frameworks the presence of these terms makes the analyzes very difficult. Another advantage is that the non-physical contributions comming from the Landau singularities of the running are also eliminated. Assuming that the duality radius of the dimension two FESR is equal to the continuum threshold s_c used in the w_{kl} FESRs we determine the parameters s_c and α_s numerically.

▲ We presented five result for the parameters $\alpha_s(m_\tau^2)$ and s_c obtained from different w_{kl} FESRs. All these results agree with one another within the errors quoted. We have given for the strong coupling constant values obtained with the FOPT$^+$ and CIPT$^+$ resummation schemes separately. The differences between FOPT and CIPT values are found to be larger than these in other approaches. The CIPT and FOPT values for the parameter s_c obtained within the same w_{kl} FESR are very close, while their values from different FESRs are consistent within errors.

▲ Taking the average of "FOPT" and "CIPT" values for the coupling (corresponding to the w_{00} FESR) we find

$$\alpha_s(m_\tau^2) = 0.310 \pm 0.012_{ex} \pm 0.012_{th} = 0.310 \pm 0.017 \qquad (36)$$

This should be compared with the most recent results from other groups

$$\begin{aligned}
\alpha_s(m_\tau^2) &= 0.332(12)\ \text{M. Davier et.al, Eur.Phys.J.(2014)} \\
&= 0.328(12)\ \text{A. Pich, A.Rodriguez, P.R.D94(2016)} \\
&= 0.301(10)\ \text{D. Boito etal. P.R.D91(2015)} \\
&= 0.309(9)\ \text{including OPAL data D. Boito}
\end{aligned}$$

the agreement is reasonable.

▲Thus we conclude that owing to the optimal choice of the duality radius and weight functions in the FESR, at the duality point $s = s_c$, the duality violating contributions to the w_{kl} FESRs become small and they can be safely ignored. This can achieved only for special class of weights w_{kl}. Contamination of the extracted coupling values from the condensates is also minimized due to restricting the duality region $0 < s_c < s < m_\tau^2$.