Relating perturbative and non-perturbative sectors in Quantum Mechanics

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Outline

- A crash course on QM, imaginary time and Path integrals
- Divergent series and Borel summation
- Trans-series and Dunne-Ünsal formula

Instantons in QM

Consider a spinless particle in a potential $V(x)$

$$L = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - V(x)$$

Feynman integral: The transition amplitude is equal to the sum over all paths joining the world points $(-\frac{t_0}{2}, x_i)$ and $(\frac{t_0}{2}, x_f)$ taken with the weight

$$\exp\left[i \int_{-\frac{t_0}{2}}^{\frac{t_0}{2}} dt\, L \right]$$

$$\langle x_f | \underbrace{e^{-i\overbrace{H}^{\text{Hamiltonian}} t_0}}_{\text{evolution operator}} | x_i \rangle = \underbrace{N}_{\text{normalization factor}} \int [Dx]\, e^{iS[x(t)]}$$

integration over all functions
x(t) with boundary cond.
$x(-t_0/2) = x_i \quad x(t_0/2) = x_f$

Wick rotation :

$$t \to -it$$

$$e^{-iHt} \longrightarrow e^{-Ht}$$

$$S \longrightarrow -\int dt \left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x) \right]$$

$$\downarrow$$

This is equivalent working with an inverted potential.

Instantons are classical solutions to the Wick rotated equations of motion

$$\frac{d^2x}{dt^2} - V'(x) = 0 \quad \leftarrow \text{Euler-Lagrange equation}$$

which have non-trivial topology.

Anharmonic oscillator

Consider one-dimensional anharmonic oscillator

$$H = \underbrace{\frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2}_{\text{exact result}} + \underbrace{g^2 x^4}_{\text{perturbation}}$$

One can construct the ground state energy E_0 order by order in perturbation theory

$$E_0(g) = \lim_{\beta \to +\infty} -\frac{1}{\beta} \ln \operatorname{tr} e^{-\beta H}$$

expand in g before taking the large β limit

Prof. Postdoc Student

$$E_0 = \frac{\omega}{2}\left(1 + c_1 g^2 + c_2 g^4 + \dots\right)$$

Does not define the ground state energy!

Coefficients c_k are factorially divergent at large k

$$c_k \sim \underbrace{(-1)^k}_{\text{alternating}} \left(\frac{1}{3}\omega^3\right)^{-k} k! \qquad k \gg 1$$

[Bender, Wu 1969]

The ground state needs a regularization.

Borel summation

$$\sum_{k=0}^{\infty} c_k g^k \longrightarrow \sum c_k g^k \frac{k!}{k!}$$

$$\longrightarrow \sum c_k g^k \frac{1}{k!} \int_0^{\infty} e^{-s} s^k \, ds$$

$$\longrightarrow \int_{s=0}^{\infty} \sum c_k \frac{1}{k!} (gs)^k e^{-s} \, ds$$

crucial point

Definition :

If $f(x)$ has a divergent perturbative series whose coefficients grow as $a_k \sim k!$

Then one defines the Borel transforms

$$B(f) = \sum \frac{a_n}{n!} s^n$$

Laplace-type transformation gives the function which has the same asymptotic expansion as the original func.

$$S_0 f(x) = \int_0^{\infty} B(f)(sx)\, e^{-s}\, ds$$

Quartic potential :

$$BE_0 = \frac{\omega}{2}\left(1 + \sum_{k=1}^{\infty} \frac{1}{k!} c_k \, g^{2k}\right) = \frac{\omega}{2} f(g^2)$$

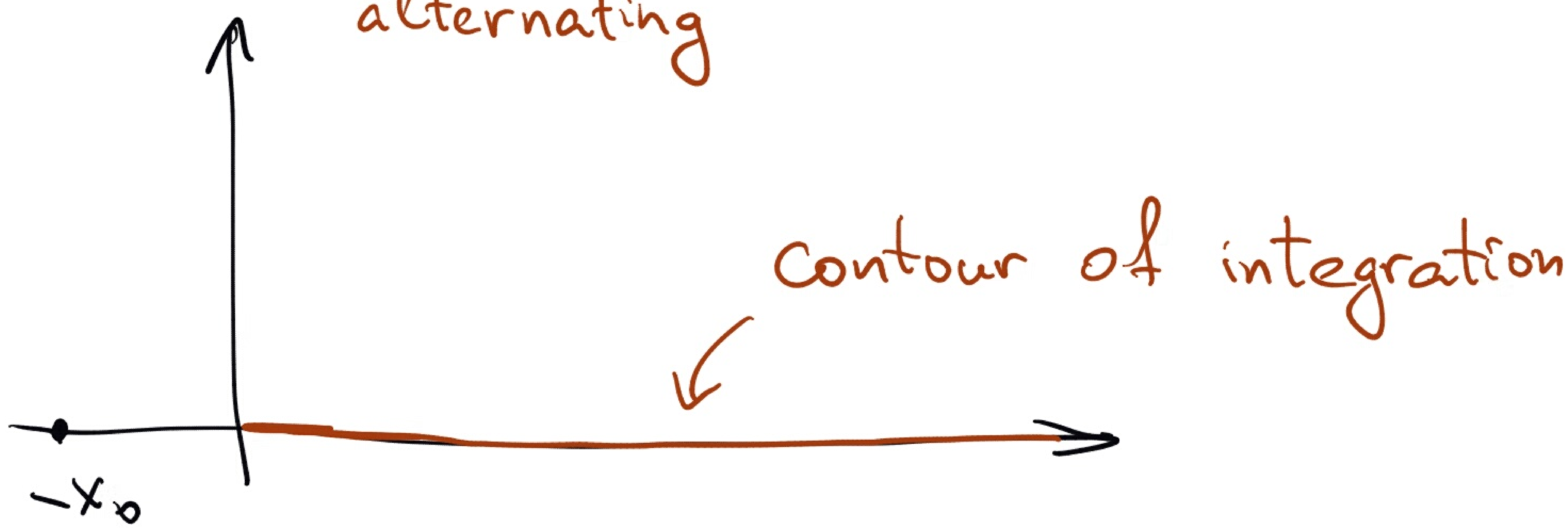
If $f(g^2)$ has no singularities on the real positive semi-axis $g^2 \geq 0$ then one can use Borel summation

Assume that $f(x)$ has a pole at $x = -x_0$

$$f(x) = \frac{x_0}{x+x_0} \qquad x_0 = \frac{\omega^3}{3}$$

Then the integral $S(B(f))$ is well-defined

$$\sum \underbrace{(-1)^n}_{\text{alternating}} n!\, g^{2n} = \int_0^\infty \frac{ds\, e^{-s}}{1+g^2 s}$$



non-alternating

$$\sum (+1)^n\, n!\, g^{2n} = \int_0^\infty \frac{ds\, e^{-s}}{1-g^2 s}$$

pole on the contour

$\rightarrow$ Residue calculation

$$\pm i\pi\, e^{-\frac{1}{g^2}}$$

Every coefficient is real but we end up with imaginary

The correct expansion: Trans-series

$$E(g^2) = E_{per}(g^2) + \sum_{k=1}^{\infty} \sum_{\ell} \sum_{p=0}^{\infty} \underbrace{\left(\frac{1}{g^{2N+1}} e^{-\frac{c}{g^2}} \right)^k}_{k\text{-instanton}} \overbrace{\left(\log \frac{c}{g^2} \right)^{\ell}}^{\text{interaction}} \underbrace{c_{k,\ell,p}\, g^{2p}}_{\text{pertur.}}$$

↑ can be obtained from the generalized quantization conditions by taking small g expansion

[Zinn-Justin et al.]

? May we obtain the non-perturbative information from perturbative sector ?

In simple cases the generalized quantization conditions depend on two functions

$B(E,g)$ and $A(E,g)$

perturbative function

non-perturbative instanton function

when $B = N + \frac{1}{2}$

one obtains the usual

Rayleigh - Schrödinger perturbative expansion

includes the instanton action and the instanton fluctuation terms

Quartic potential : Harmonic $+ \overset{\text{negative}}{g} q^4$

$$\frac{1}{\Gamma\left(\frac{1}{2} - B(E,g)\right)} \sim \left(\frac{1}{g}\right)^{B(E,g)} e^{-A(E,g)}$$

[Zinn-Justin et al]

Perturbative and non-perturbative relation

$$\frac{\partial E(B,g)}{\partial B} = -\frac{g}{S}\left[B + g\,\frac{\partial A(E,g)}{\partial g}\right]$$

[Dunne, Ünsal 1306.4405, 1401.5202]

Cubic, quartic potential: [G., Tezgin 1512.08466]

The instanton fluctuation factor

$$H = \exp\left[S \int \frac{dg}{g^2} \left(\frac{\partial E}{\partial B} + \frac{Bg}{S} - 1 \right) \right]$$

Dunne-Ünsal formula

$$\operatorname{Im} E_N \sim H \frac{\partial E}{\partial B}$$

Quartic potential

$$\operatorname{Im} E_0 = -\sqrt{-\frac{2}{\pi g}}\, e^{\frac{1}{3g}} \left[1 + \frac{95}{24} g - \frac{13259}{1152} g^2 + \dots \right]$$

$$\operatorname{Im} E_1 = -\sqrt{\frac{2^5}{\pi (-g)^3}}\, e^{\frac{1}{3g}} \left[1 + \frac{371}{24} g - \frac{3371}{1152} g^2 + \dots \right]$$

[G., Tezgin 1608.08119]

Other potentials:

- Cubic
- Quartic

sine-Gordon, double-well

Dunne-Ünsal formula

- Quintic
- Sextic
- Septic

⋮

Needs to be generalized

Hyperelliptic curve: $y^2 = P_{2N}(x)$

Polynomial of degree N

$$2g = 2N - 2$$

PNP relation works for genus-1 potentials.

For higher genus: Relation to Whitham equations

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