# *NHEG Orbits:*

*Near Horizon Extrmal Geometries*,

*Their Symplectic Symmetry Algebras and Coadjoint Orbits*

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Based on: My Upcoming Work

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# Introduction and Motivation

- Infinite dimensional Lie algebras appear in many places in physics, generically related with diffeomorphisms.

- Diffeo's are typically relevant to theories of gravity.

- A well-known example is the Witt/Virasoro algebra related to $diff(S^1)$.

- There are Lie groups associated with these algebras.

- Given a group/algebra one should ask about its representations and its actions (adjoint or coadjoint) and modules.

- Consider diffeo's of a $d$-dimensional space.
  - Q: its subalgebras/groups?
  - A: diffeo's on any $p$-dimensional ($p < d$) hypersurfaces in it; diffeo's with a prescribed properties or b'dry/falloff,e.g. $\text{BMS}_3$, $\text{BMS}_4$ or Brown-Henneaux.

- Relevance to physics:
  - membrane or $D_p$-brane dynamics
  - A new statement of the Equivalence Principle if one can associate conserved charges to these diffeo's.
  - May be relevant to black hole physics and open up a window to quantum gravity.

- In this talk I focus on a specific subsector of $diff(T^d)$.

- This algebra may be viewed as a higher dimensional extension of the Virasoro algebra/group.

- The algebra we discuss here is called NHEG algebra, as it is related to Near Horizon limit of Extremal black hole Geometry.

- The NHEG Group will be denoted by $\widehat{\text{Diff}_{\vec{k}}(T^d)}$, it comes with a $d$-dimensional vector $\vec{k}$ and a central extension $c$.

- In this work we classify $\widehat{\text{Diff}_{\vec{k}}(T^d)}$ coadjoint orbits and modules.

# Outline

- A quick introduction to Near Horizon Extremal Geometry, NHEG.
- Introduction to NHEG algebra $\widehat{\mathcal{V}_{\vec{k},c}}$ and NHEG group $\widehat{\text{Diff}_{\vec{k}}(T^d)}$.
- Coadjoint orbits, a review on general notion for Virasoro group.
- Classification of $\widehat{\text{Diff}_{\vec{k}}(T^d)}$ coadjoint orbits.
- $\widehat{\mathcal{V}_{\vec{k},c}}$ from $U(1)^d$ current algebra, NHEG Kac-Moody algebra.
- NHEG modules.
- Summary and outlook.

## The NHEG

- Near Horizon limit of Extremal black holes leads to Near-Horizon Extremal Geometry (NHEG).

- The NHEG form a class of solutions to Einstein GR, generically with an $SL(2,\mathbb{R}) \times U(1)^n$ isomtery.

- For simplicity, we focus on $d$ dimensional Einstein vacuum solutions with $SL(2,\mathbb{R}) \times U(1)^{d-3}$ isomtery, with the metric

$$ds^2 = \Gamma(\theta)\left[-r^2 dt^2 + \frac{dr^2}{r^2} + d\theta^2 + \gamma_{ij}(\theta)(d\varphi^i + k^i r dt)(d\varphi^j + k^j r dt)\right]$$

  where $i, j = 1, 2, \cdots, d-3$.

## NHEG Killing vectors

$$\begin{aligned}
\boldsymbol{\xi}_1 &= \partial_t\,,\\
\boldsymbol{\xi}_2 &= t\partial_t - r\partial_r\,,\\
\boldsymbol{\xi}_3 &= \frac{1}{2}(t^2+\frac{1}{r^2})\partial_t - tr\partial_r - \sum_{i=1}^{n}\frac{k^i}{r}\partial_{\varphi^i}\,,\\
\boldsymbol{m}_i &= \partial_{\varphi^i}\,,
\end{aligned}$$

Their algebra

$$[\boldsymbol{\xi}_1,\boldsymbol{\xi}_2]=\boldsymbol{\xi}_1\,,\qquad [\boldsymbol{\xi}_2,\boldsymbol{\xi}_3]=\boldsymbol{\xi}_3\,,\qquad [\boldsymbol{\xi}_1,\boldsymbol{\xi}_3]=\boldsymbol{\xi}_2\,,$$
$$[\boldsymbol{\xi}_a,\boldsymbol{m}_i]=0\,,\quad a\in\{1,2,3\}\ \text{ and, }\ i\in\{1,\dots,d-3\}\,.$$

- The NHEG we consider here are uniquely specified by $(d{-}3)(d{-}2)/2$ parameters [S. Hollands, A. Ishibashi, 2010].

- Among these parameters are $d-3$ angular momenta $J_i$.

- The NHEG has a “bifurcation horizon surface” which are codimension two constant $r, t$ surfaces, with metric

$$ds_H^2 = \Gamma(\theta)\left[d\theta^2 + \gamma_{ij}(\theta)d\phi^i d\phi^j\right].$$

- Constant $\theta$ surfaces on the above is a generic $T^d$.

- $T^d$ part of the geometry involves a constant $d$ dimensional vector $\vec{k}$ with components $k^i$.

- Note: $\vec{k}$ does not show up in $ds_H^2$.

- All interesting physical information about the NHEGs is in the $\vec{k}$.

## Introduction to NHEG algebra $\widehat{\mathcal{V}_{\vec{k},c}}$

- NHEG algebra is a centrally extended subalgebra of smooth diffeomorphisms on a $d$ dimensional torus $T^d$.

- Consdier the vector fields on $T^d$:

$$X = f^i \partial_{\phi^i}$$

$f^i = f^i(\phi^1, \cdots, \phi^d)$ are periodic functions on the torus:

$$f^i(\phi^1, \cdots, \phi^i + 2\pi, \phi^d) = f^i(\phi^1, \cdots, \phi^i, \cdots, \phi^d), \quad i = 1, 2, \cdots, d$$

- Any vector field $X_{\vec{f}}$ is labelled by $n$ functions $\vec{f} = (f_1, \cdots, f_n)$.

- $f(\phi^1, \cdots, \phi^d)$ can be Fourier expanded as

$$f(f^1, \cdots, \phi^i, \cdots, \phi^d) = \sum_{m_1, \cdots, m_d} f_{\vec{m}} e^{i\vec{m}\cdot\vec{\phi}}, \quad \vec{m}\cdot\vec{\phi} = \sum_{i=1}^{d} m_i \phi^i, m_i \in \mathbb{Z}.$$

- Torus is the quotient of $\mathbb{R}^d$ by the $d$ dimensional lattice, $\mathbb{R}^d/\mathbb{Z}^d$.

- $\vec{n} = (n_1, \cdots, n_d),\ n_i \in \mathbb{Z}$ denote vectors on this $d$ dimensional lattice.

- A vector field can hence be expanded as

$$X = \sum_{\vec{m}} X^i_{\vec{m}} \ell^i_{\vec{m}}, \qquad \ell^i_{\vec{m}} = i e^{i\vec{m}\cdot\vec{\phi}} \partial_{\phi^i}$$

  which form the following algebra

$$[\ell^i_{\vec{m}}, \ell^j_{\vec{n}}] = m^j \ell^i_{\vec{m}+\vec{n}} - n^i \ell^j_{\vec{m}+\vec{n}}.$$

- This is the algebra of vector fields (infinitesimal diffeomorphisms) on $T^d$, $diff(T^d)$.

## ■ “anisotropic torus” and generalized Witt algebra

- “Anisotropic torus” as a torus with a preferred direction $\vec{k}$ on it.

- $\vec{k}$ may then be a vector on the dual lattice of the torus or not:

  – the flow (also called leaves) of $\vec{k}$ form closed curves which are then diffeomorphic to $S^1$ or,

  – the leaves are diffeomorphic to $\mathbb{R}$ for which each leaf is dense in a subspace of the torus.

- For the case of NHEG, the SL$(2,\mathbb{R})$ isometry of the background NHEG maps the torus to itself, therefore, we should require that $\vec{k}$ is along the dual lattice of the torus.

- One can then show that the ratios $\frac{k^j}{k^i}$ should be rational numbers.

- Using the SL$(d,\mathbb{Z})$ symmetry of the torus, there is a frame such that $\vec{k}$ is along one of the axis of the dual lattice, $\vec{k} = (1, 0, \cdots, 0)$.

- For the case of NHEGs,

  - tori $T^d$ are coming with a metric $\gamma_{ij}$ which is a function of $\theta$ coordinate;

  - $\vec{k}$ is directly related to the conserved charges associated with the $U(1)^d$ isometries of the NHEG, $\vec{J}$, the NHEG angular momenta.

- $k^i/k^j$ being a rational number means $J^i/J^j$ being rational.

- $\vec{k}$ belonging to the dual lattice of $T^d$ at the level of the NHEG, comes from a semi-classical quantization on the corresponding $U(1)^d$ angular momenta.

- For a given vector field on the torus $\xi$, we define

$$\xi_{||} = \sum_{\vec{m}} \xi_{\vec{m}} e^{i\vec{m}\cdot\vec{\phi}} \vec{k} \cdot \vec{\partial} = \sum_{vm} \xi_{\vec{m}} \ell_{\vec{m}}, \qquad \xi_{\vec{m}} = i\, k_j\, \xi^j_{\vec{m}}, \quad \ell_{\vec{m}} = \vec{k} \cdot \vec{\ell}_{\vec{m}}.$$

- Lie bracket of vector fields along $\vec{k}$ yields *generalized Witt* algebra:

$$[\ell_{\vec{n}}, \ell_{\vec{m}}] = \vec{k} \cdot (\vec{n} - \vec{m}) \ell_{\vec{n}+\vec{m}}.$$

- This is algebra of general diffemorphisms on "anisotropic torus" along the anisotropy direction $\vec{k}$.

- Generalized Witt algebra has a usual Witt algebra as a subalgebra of generators when $\vec{m} = m\vec{k}$.

- The set of generators $\ell_{\vec{m}}$ with $\vec{k} \cdot \vec{m} = 0$ are commuting with each other (like "supertranslations" in the BMS algebra).

- In defining the above we need not introduce a metric on the $T^d$.

- However, it is useful to assume $\gamma_{ij}$ to be the metric on the tours and use it to define dual vectors, e.g.

$$\eta_i \equiv \gamma_{ij} k^j, \qquad \vec{\xi} \cdot \vec{k} = \xi^i \eta_i$$

## Centrally extended generalized Witt algebra, NHEG algebra

- Generalized Witt algebra can be centrally extended to get the generalized Virasoro.

- We call this centrally extended algebra the *NHEG algebra* $\widehat{\mathcal{V}_{\vec{k},c}}$.

- As in the Virasoro case, the Jacobi identity, or in more technical terms the cocycle condition, uniquely fixes the form of the central extension (up to redefinitions of the generators):

$$[L_{\vec{m}}, L_{\vec{n}}] = \vec{k} \cdot (\vec{m} - \vec{n})\ L_{\vec{n}+\vec{m}} + c(\vec{n}_\perp)\ (\vec{k} \cdot \vec{m})^3 \delta_{\vec{n}+\vec{m},0}$$

- The central charge is in general a function of $\vec{n}_\perp = \vec{n} - (\vec{n} \cdot \vec{k}/k^2)\vec{k}$ where $k^2 = \vec{k} \cdot \vec{k}$.

## More on the "*NHEG algebra*" $\widehat{\mathcal{V}_{\vec{k},c}}$

- Its structure constants are given by $\vec{k}$.

- In the $d=1$ case $k=1$, and the algebra is just the Virasoro algebra.

- It has infinitely many Virasoro subalgebras:

$$\vec{n} = n\vec{e}, \qquad \ell_n = \frac{1}{\vec{e}\cdot\vec{k}} L_{\vec{n}} \qquad \vec{e}\cdot\vec{k} \neq 0.$$

$$[\ell_m, \ell_n] = (m-n)\ell_{m+n} + \frac{c}{12}(\vec{e}\cdot\vec{k})\ m^3 \delta_{m+n}.$$

- If $\vec{e}\cdot\vec{k}=0$, we have an infinite dimensional Abelian subalgebra.

- The set of generators $L_{\vec{m}}, \vec{m}\cdot\vec{k}=0$ also form an Abelian subalgebra of NHEG algebra. This subalgebra may be viewed as $\mathrm{Vec}(T^{d-1})$.

- $\widehat{\mathcal{V}_{\vec{k},c}}$ is not semi-direct sum of this Virasoro subalgebra and subalgebra of commuting set of generators:

$$\widehat{\mathrm{Diff}_{\vec{k}}(T^d)} \neq \widehat{\mathrm{Diff}(S^1)} \ltimes \mathrm{Vec}(T^{d-1}).$$

- NOTATION: Witt algebra $= diff(S^1)$,

Virasoro algebra $= \widehat{diff(S^1)}$, Virasoro group $= \widehat{Diff(S^1)}$

NHEG algebra $= \widehat{\mathcal{V}_{\vec{k},c}} = \widehat{diff_{\vec{k}}(T^d)}$, NHEG group $= \widehat{\mathrm{Diff}_{\vec{k}}(T^d)}$

## NHEG algebra as symplectic symmetries of NHEG

- $\widehat{\mathcal{V}_{\vec{k},c}}$ was originally obtained as the algebra of charges associated with diffeo's $\chi^\mu$ on the NHEG [G. Compere, K. Hajian, A. Seraj, M.M. Sh-J, 2015]

$$\chi[\epsilon(\vec{\phi})] = \epsilon\vec{k}\cdot\vec{\partial}_\phi - \epsilon'\,(\frac{1}{r}\partial_t + r\partial_r), \qquad \epsilon' \equiv \vec{k}\cdot\vec{\partial}_\phi\epsilon$$

$\epsilon$ is generic periodic function of all coordinates $\phi^i$.

- Expanding in Fourier modes $\epsilon = e^{i\vec{n}\cdot\vec{\phi}}$:

$$[\chi_{\vec{n}}, \chi_{\vec{m}}]_{\text{Lie bracket}} = \vec{k}\cdot(\vec{n}-\vec{m})\,\chi_{\vec{n}+\vec{m}}$$

- The above implies that the vectors $\chi$ form the adjoint representation of the generalized Witt algebra $\widehat{\mathcal{V}_{\vec{k},c}}$.

- Exponentiating $\chi$ one can obtain form of elements of the NHEG group $\widehat{\text{Diff}_{\vec{k}}(T^d)}$:

$$\bar{\phi}^i = \phi^i + k^i F(\vec{\phi}), \quad \bar{r} = r e^{-\Psi(\vec{\phi})}, \quad \bar{t} = t - \frac{1}{r}(e^{\Psi(\vec{\phi})} - 1), \quad e^{\Psi} \equiv 1 + \vec{k}\cdot\vec{\partial}F,$$

- Accordingly the metric transforms as

$$ds^2 = \Gamma(\theta)\Big[-(\boldsymbol{\sigma} - d\Psi)^2 + \Big(\frac{dr}{r} - d\Psi\Big)^2 + d\theta^2 + \gamma_{ij}(d\tilde{\phi}^i + k^i\boldsymbol{\sigma})(d\tilde{\phi}^j + k^j\boldsymbol{\sigma})\Big],$$

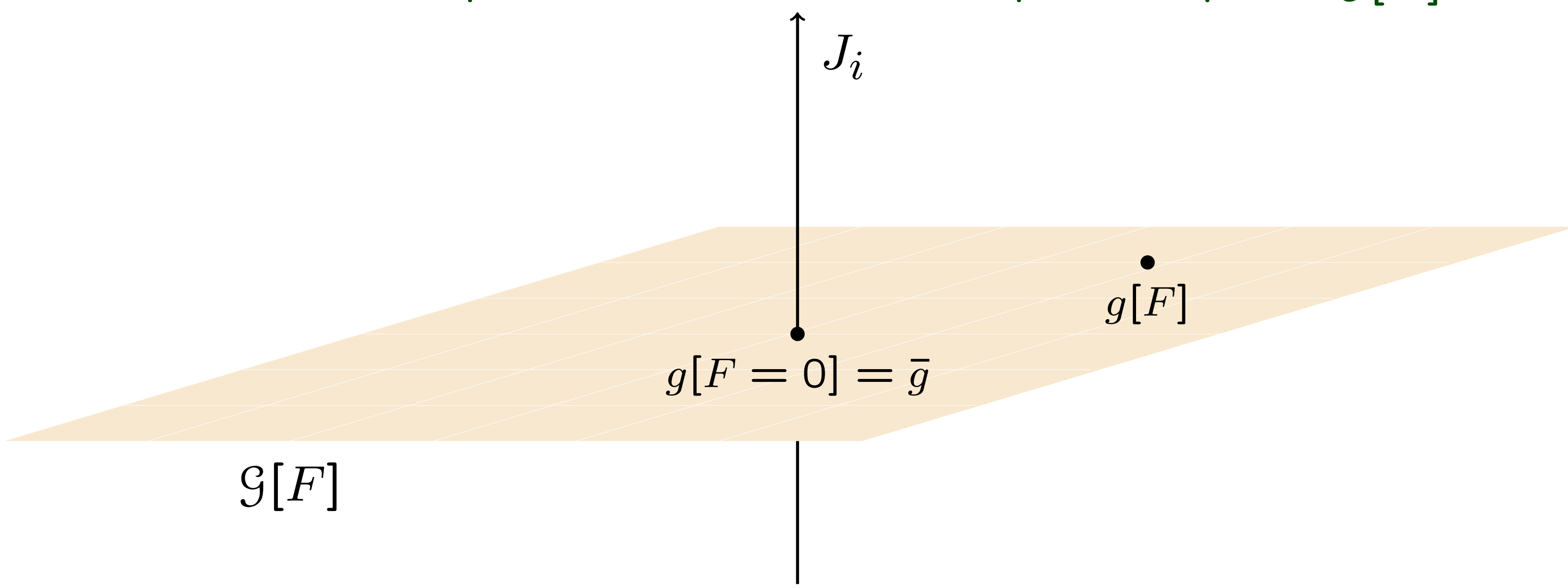
where

$$\boldsymbol{\sigma} = (1 - e^{-\Psi})\frac{dr}{r} + e^{-\Psi} r dt, \qquad \tilde{\phi}^i = \phi^i + k^i(F - \Psi)\,.$$

- The one-function family of metrics form a phase space and

$$\mathcal{L}_\chi g_{\mu\nu}[F] = g_{\mu\nu}[F + \delta_\chi F] - g_{\mu\nu}[F], \qquad \delta_{\chi[\epsilon]}F = e^{\Psi}\epsilon, \qquad \delta_\epsilon\Psi = \epsilon\Psi' + \epsilon'$$

Schematic depiction of the NHEG phase space $\mathcal{G}[F]$.




Vertical axis: Different background NHEG solutions specified by different values of $J_i$ (i.e. different $\vec{k}$ and entropy $S$)

Horizontal plane: The phase space constructed by the action of the finite coordinate transformations. Each point shows a geometry in the phase space identified by a periodic function $F(\vec{\varphi})$.

Moving on the horizontal plane does not change the $J_i$.

## The NHEG group $\widehat{\text{Diff}_{\vec{k}}(T^d)}$

- Coordinate transformations yielding the NHEG algebra are

$$\phi^i \to \phi^i + k^i \xi(\phi^i), \qquad \phi^i \sim \phi^i + 2\pi,$$

$\xi(\phi^i)$ are smooth and periodic for all $\phi^i$.

- As discussed we can always take $\qquad \vec{k} = (1, 0, \cdots, 0).$

- NHEG group $\text{Diff}_{\vec{k}}(T^d)$ is then constructed from finite coordinate transformations:

$$\tilde{\phi}^1 = \phi^1 + F(\phi^i), \qquad \tilde{\phi}^a = \phi^a, \quad a = 2, ..., d,$$

or

$$(\phi^1, \cdots, \phi^d) \mapsto \Big(F(\varphi; \Phi); \Phi\Big), \qquad \phi^1 = \varphi, \qquad \Phi = \{\phi^a\}$$

- Here

$$F(\varphi + 2\pi : \Phi) = F(\varphi; \Phi) + 2\pi \quad \text{and} \quad \partial F/\partial\varphi \neq 0,$$
$$F(\varphi, \phi^2, \cdots, \phi^a + 2\pi, ..., \phi^d) = F(\varphi, ..., \phi^d) + 2\pi N_a, \;\forall\, a = 2, \cdots, d,$$

$N_a \in \mathbb{Z}$ may take different values for different $F$'s.

- Thus the NHEG group $\text{Diff}_{\vec{k}}(T^d)$ is

$$\text{Diff}_{\vec{k}}(T^d) = C^\infty\Big(\text{Diff}(S^1), T^{d-1}\Big).$$

- $\text{Diff}_{\vec{k}}(T^d)$ is the group of smooth maps that send a point $\Phi$ on $T^{d-1}$ to a circle diffeomorphism $F(\varphi; \Phi) \equiv F_\Phi(\varphi)$. Coordinates $\Phi$ are 'spectators' or 'parameters' on which $F$ depends.

- The group operation is $(f, g) \mapsto f \cdot g$ with

$$(f \cdot g)(\varphi; \Phi) = \Big(f\Big(g(\varphi; \Phi); \Phi\Big), \quad \text{i.e.} \quad (f \cdot g)_\Phi = f_\Phi \circ g_\Phi \;\; \forall\, \Phi \in T^{d-1}.$$

## Connected components of the NHEG group.

- Fundamental group of $T^{d-1}$ is $\mathbb{Z}^{d-1}$ and hence the NHEG group $\widehat{\text{Diff}_{\vec{k}}(T^d)}$ has infinitely many connected components for $d > 1$.

- $N_a$ are the winding numbers: the vector $(N_2, ..., N_d)$ represents an element of the fundamental group $\mathbb{Z}^{d-1}$.

- Any two elements of the NHEG group with different winding numbers belong to different connected components of the group.

- There is also orientation changing part of the Diff$(S^1)$.

- Therefore, each connected component of the NHEG group can be labelled by (i) the winding number of its elements, and (ii) a plus or minus sign:

$$f(\varphi, \phi^a) = \pm\varphi + N_a \phi^a.$$

- Any element of the NHEG group with winding numbers $N_a$ can be continuously connected (is homotopic) to such a simple transformation.

- Our general picture is that angles $\phi^a$ play the role of parameters on which our diffeomorphisms $F$ are allowed to depend.

- Here we restrict ourselves to zero winding number, orientation preserving sector.

## NHEG algebra as Virasoro bundles over $T^{d-1}$

- If $\vec{k} = (1, 0, \cdots, 0)$ the NHEG algebra generators may be rewritten as

$$L(\varphi;\Phi) = \sum_{\vec{n}} L_{\vec{n}} e^{i\vec{n}\cdot\vec{\phi}}, \qquad L_n(\Phi) \equiv \frac{1}{2\pi}\int_0^{2\pi} d\varphi L(\varphi;\Phi) e^{-in\varphi},$$

- and NHEG algebra $\widehat{\mathcal{V}_{\vec{k},c}}$ as

$$[L_n(\Phi), L_m(\Phi')] = \left((n-m)L_{n+m}(\frac{\Phi+\Phi'}{2}) + \frac{c(\Phi)}{12}\delta_{n+m,0}\right)\delta^{d-1}(\Phi-\Phi')$$

- The cocycle condition allows for promoting $c$ to $c(\Phi)$.

- The above makes it explicit that the NHEG algebra is a Virasoro algebra where its generators (and also the central charge) are generic smooth functions on $T^{d-1}$.

## Classification of coadjoint orbits of NHEG group $\widehat{\text{Diff}_{\vec{k}}(T^d)}$

- We first review what the coadjoint orbit for Virasoro group is and how one can classify them.

- As discussed the NHEG group is Virasoro bundles of $T^{d-1}$. Therefore, Virasoro coadjoint orbits will have direct relevance to our problem too.

## Virasoro coadjoint orbits and their classiciation

- **Orbit** of each point $m \in \mathbf{M}$ is the set of points which can be reached by the group action $\mathbf{L} : \mathbf{G} \times \mathbf{M} \to \mathbf{M}$:

$$\mathcal{O}_m = \{p \in \mathbf{M} \setminus \exists g \in \mathbf{G},\ p = \mathbf{L}_g m\}.$$

- Stabilizer of a point $m$ is the set $Stab_m \in \mathbf{G}$ whose elements act trivially on $m$:

$$Stab_m = \{g \in \mathbf{G} \setminus m = \mathbf{L}_g m\}.$$

- $Stab_m$ is a subgroup of $\mathbf{G}$ and Stabilizers of all points in an orbit are isomorphic.

- The quotient $G/Stab(m)$ is isomorphic to the orbit $\mathcal{O}_m$:

$$G/Stab(m) \simeq \mathcal{O}_m.$$

## Adjoint Representation.

- Representation of a Lie group **G** on a vector space **V** is a smooth linear action **D** of the group on **V**.

- Adjoint representation of the Lie group **G** is the homomorphism

$$Ad : \mathbf{G} \to Aut(\mathfrak{g}), \qquad g \mapsto Ad_g,$$

  where

$$Ad_g(X) = \frac{d}{d\alpha}(g e^{\alpha X} g^{-1})\Big|_{\alpha=0},$$

  where $e^X$ is the exponential map of $X$.

- Adjoint rep. of a *Lie algebra* is the differential of the adjoint rep. of the group near its identity element:

$$ad_X Y \equiv \frac{d}{dt}(Ad_{e^{tX}} Y)\Big|_{t=0} = [X, Y].$$

## Coadjoint representation.

- Dual to the Lie algebra $\mathfrak{g}$, denoted by $\mathfrak{g}^*$, is defined by the pairing which is a bilinear form between the elements of $\mathfrak{g}, \mathfrak{g}^*$:

$$\langle \alpha, X \rangle \in \mathbb{R}, \qquad \alpha \in \mathfrak{g}^*, \quad X \in \mathfrak{g}.$$

- *Coadjoint representation* on the dual space $\mathfrak{g}^*$ is the homomorphism

$$Ad^* : \mathbf{G} \to Aut(\mathfrak{g}^*), \qquad g \mapsto Ad^*_g,$$

  where

$$\langle Ad^*_g(\alpha), X \rangle \equiv \langle \alpha, Ad_{g^{-1}}(X) \rangle, \qquad \forall X \in g.$$

- Nondegeneracy of the pairing on $\mathfrak{g}, \mathfrak{g}^*$ ensures that the above uniquely fixes the coadjoint action.

- Note: $Aut(M)$ is the group of linear maps from $M$ into itself. For finite dimensional spaces, it is equal to GL$(M)$.

- The above leads to a representation of the Lie algebra $\mathfrak{g}$ on its dual space $\mathfrak{g}^*$ through the action $X \mapsto ad^*_X$:

$$\langle ad^*_X(\alpha), Y\rangle \equiv -\langle \alpha, ad_X Y\rangle, \quad \forall \alpha \in \mathfrak{g}^*, \forall Y \in \mathfrak{g}.$$

- This implies that the pairing is **G** invariant:

$$\delta_Y \langle \alpha, X\rangle = \langle ad^*_Y \alpha, X\rangle + \langle \alpha, ad_Y X\rangle = 0$$

- Existence of a **G**-invariant inner product on $\mathfrak{g}$ induces an isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$, and hence an isomorphism between adjoint and coadjoint representations.

- The coadjoint orbit of a vector $\alpha \in \mathfrak{g}^*$ is defined as

$$\mathcal{O}_\alpha = \{Ad^*_g(\alpha) | g \in \mathbf{G}\}.$$

- This automatically classifies the dual space $\mathfrak{g}^*$ into distinct coadjoint orbits of **G**.

- The group action on each orbit is an irrep of the group.

## Virasoro coadjoint orbits

- Let us start with Witt group $Diff(S^1)$.

- To construct its coadjoint orbits consider vector fields on a circle $Vec(S^1)$ with elements $X = X(\phi)\partial_\phi$.

- This space is an algebra given the Lie bracket between vector fields

$$ad_X(Y) = [X, Y] = (XY' - YX')\partial_\phi.$$

- Expanding in Fourier modes $\ell_n = ie^{in\phi}$, yields the Witt algebra

$$[\ell_n, \ell_m] = (n - m)\ell_{n+m}.$$

- To define dual space $Vec(S^1)^*$, we need an invariant pairing (inner product).

- A tensor density of weight 2, $L = L(\phi)d\phi^2$ can be paired by elements of the algebra:

$$\langle L d\phi^2, X\partial_\phi\rangle = \oint d\phi L(\phi)X(\phi)$$

- Dual space of Witt algebra is hence space of tensor densities of weight 2.

- The coadjoint action of a vector $Y$ on $L$, $ad^*_Y(L)$, is then

$$\delta_Y\langle L, X\rangle = 0 \implies \langle ad^*_Y(L), X\rangle = -\langle L, ad_Y X\rangle = \langle L, [X, Y]\rangle.$$

yielding

$$ad^*_Y(L) = XL' + 2LX',$$

as expected for a tensor density of weight 2.

## Virasoro case, $\widehat{Diff(S^1)}$

- Elements of the algebra: $(X,\alpha) \in \widehat{diff(S^1)}$ and their duals $(L,c) \in \widehat{diff(S^1)}^*$ where $\alpha, c$ are real numbers.

- The algebra is defined as

$$[(X,\alpha),(Y,\beta)] = ([X,Y], C(X,Y)),$$

- The central extension is independent of $\alpha, \beta$ and given by the Gelfand-Fuchs cocycle

$$C(X,Y) = \int d\phi (X'Y'' - Y'X''),$$

and the pairing by

$$\langle (L,c),(X,\alpha)\rangle = \int_{S^1} L(\phi)X(\phi)d\phi + \alpha c.$$

- Invariance of pairing under the Virasoro action, fixes the coadjoint action of Virasoro algebra to

$$ad^*_{(X,\alpha)}(L,c) = -(XL' + 2LX' - \frac{c}{24\pi}X''', 0).$$

- Finite form of the above infinitesimal transformation gives the coadjoint action of the Virasoro group:

$$Ad^*_f(L,c) = (\frac{L}{f'^2} + S[f,\theta], c),$$

where $S[f,\theta]$ is the Schwartz derivative

$$S[f,\theta] = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .$$

## Classification of Virasoro coadjoint orbits

- Stabilizer subgroups are those
$$ad^*_{(X,\alpha)}(L,c) = -(XL' + 2LX' - \frac{c}{24\pi}X''', 0) = 0,$$
which is a linear third order differential equation.

- There are at most three solutions to the above equation for a given coadjoint vector $(L,c)$ for nonvanishing $c$.

- NOTE: We are only interested in periodic solutions $X$.

- Solutions to stabilizer equation can be constructed from solutions to Hill's equation
$$\frac{d^2}{d\phi^2}\psi + L(\phi)\psi = 0.$$

- Denoting two linearly independent, real solutions by $\psi_1, \psi_2$, then

$$\psi_1\psi_2, \psi_1^2, \psi_2^2,$$

provide the three solutions to the stabilizer equation.

- $\psi_1, \psi_2$ are generically non-periodic. However, due to Floquet theorem:

$$\psi_1 = e^{b\phi} P_1(\phi), \qquad \psi_2 = e^{-b\phi} P_2(\phi),$$

  - $b$ is the Floquet index and in general a real or pure imaginary number and $P_1, P_2$ are two periodic functions.

  - Then, $\psi_1\psi_2$ is always periodic.

  - Three periodic solutions happens only when $b = in/2, \; n \in \mathbb{Z}$.

- Covariance of Hill's equation under Diff($S^1$) implies $\psi_1, \psi_2$ should transform as a density of weight $+1/2$.

## Monodromy matrix.

- Normalizing $\psi_1, \psi_2$ as

$$\psi_1 \psi_2' - \psi_2 \psi_1' = -1$$

Most general solutions to Hill's equation are

$$\tilde{\mathbf{\Psi}}(\phi) \equiv \begin{pmatrix} \tilde{\psi}_1(\phi) \\ \tilde{\psi}_2(\phi) \end{pmatrix} = \mathbf{A} \begin{pmatrix} \psi_1(\phi) \\ \psi_2(\phi) \end{pmatrix} , \qquad \mathbf{A} \in Sl(2, \mathbb{R}).$$

- Periodicity of $L(\phi)$ then implies

$$\begin{pmatrix} \psi_1(\phi + 2\pi) \\ \psi_2(\phi + 2\pi) \end{pmatrix} = \mathbf{M} \begin{pmatrix} \psi_1(\phi) \\ \psi_2(\phi) \end{pmatrix}, \quad \mathbf{M} \in Sl(2, \mathbb{R}).$$

- Virasoro coadjoint orbits can be classified knowing in which $Sl(2, \mathbb{R})$ conjugacy classes monodromy matrix **M** is.

## Classicifcation of Virasoro coadjoint orbits

- Circular orbits

$$M = (-1)^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad n \in \mathbb{Z},$$

  - $\psi_1 = \sqrt{2/n}\,\cos(n\phi/2), \qquad \psi_2 = \sqrt{2/n}\,\sin(n\phi/2)$.
  - $L(\phi)$ in this orbit may all be mapped to the *representative value* $L = -\frac{n^2}{4}$.
  - The stabilizer group is $PSL^{(n)}(2, \mathbb{R})$.
  - $n$ is the winding number and shows how many times the generators of the stabilizer group cover the $S^1$.

- Elliptic orbits

$$M = (-1)^n \begin{pmatrix} \cos \pi\nu & -\sin \pi\nu \\ \sin \pi\nu & \cos \pi\nu \end{pmatrix}, \qquad n \in \mathbb{Z}, \nu \in (0,1).$$

  - $\psi_1 = \sqrt{2/n} \cos(n+\nu)\phi/2, \quad \psi_2 = \sqrt{2/n} \sin(n+\nu)\phi/2$
  - $L(\phi)$ may all be mapped to the *representative value* $L = -\frac{(n+\nu)^2}{4}$.
  - If $\nu = 1/K,\ K \in \mathbb{Z}$, then stabilizer group is $PSL^{(n)}(2, \mathbb{R})/Z_K$.

- Hyperbolic orbits

$$M = (-1)^n \begin{pmatrix} e^{2\pi b} & 0 \\ 0 & e^{-2\pi b} \end{pmatrix}, \qquad n \in \mathbb{Z}, b \in \mathbb{R}^+.$$

  - In this case

$$\psi_1 = \frac{e^{b\phi}}{\sqrt{F_{n,b}(\phi)}}\sqrt{\frac{2}{n}}\left(\frac{2b}{n^2}\cos\frac{n\phi}{2} + \frac{2}{n}\sin\frac{n\phi}{2}\right), \quad \psi_2 = \frac{e^{b\phi}}{\sqrt{F_{n,b}(\phi)}}\sqrt{\frac{2}{n}}\cos\frac{n\phi}{2},$$

    where

$$F_{n,b}(\phi) = (\frac{2b}{n^2}\cos\frac{n\phi}{2} + \frac{2}{n}\sin\frac{n\phi}{2})^2 + \cos^2\frac{n\phi}{2}$$

  - $L(\phi)$ may be mapped to the *representative value*

$$L_{n,b} = b^2 + \frac{n^2 + 4b^2}{F_{n,b}(\phi)} - \frac{3n^2}{4F^2_{n,b}(\phi)}.$$

  - For $n = 0$ case we have a constant representative orbit with positive definite value for $L$.

  - Stabilizer group is $U(1)$.

- Parabolic orbits

$$M = (-1)^n \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}, \qquad n \in \mathbb{Z},\ q = \pm 1.$$

  - In this case for $n = 0$ case $q = +1$ and $\psi_1 = \frac{\phi}{\sqrt{2\pi}},\ \psi_2 = \frac{1}{\sqrt{2\pi}},$
  - The representative value is $L = 0$.
  - For $n \in \mathbb{N}$ case, both $q = \pm 1$ values are possible and,

$$\psi_1 = \frac{1}{\sqrt{H_{n,q}(\phi)}} \left( \frac{q\phi}{2\pi} \sin \frac{n\phi}{2} - \frac{2}{n} \cos \frac{n\phi}{2} \right), \quad \psi_2 = \frac{1}{\sqrt{H_{n,q}(\phi)}} \sin \frac{n\phi}{2},$$

and

$$H_{n,q}(\phi) = 1 + \frac{q}{2\pi} \sin^2 \frac{n\phi}{2}, \quad L_{n,q} = \frac{n^2}{2H_{n,q}(\phi)} - \frac{3n^2(1 + \frac{q}{2\pi})}{4H^2_{n,q}(\phi)}.$$

  - Stabilizer group consists of lower-triangle matrices of the form

$$(-1)^n \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}, \ r \in \mathbb{R}.$$

## ■ Classification of coadjoint orbits of $\widehat{\text{Diff}_{\vec{k}}(T^d)}$

We can classify the orbits in two different ways,

- Through studying the stabilizer equation and monodromy matrix

- Making direct use of the fact that NHEG coadjoint orbits are bundles of Virasoro orbits over $T^{d-1}$ (up to the "winding numbers" $N_a$ discussed above).

## Calssification of orbits using monodromy map

Invariant pairing & dual elements.

- NHEG coadjoint orbits are built on the dual space to NHEG algebra.

- To construct this dual space, consider a vector field $V_{\vec{k}}$ on $T^d$,

$$V_{\vec{k}} = v(\vec{\phi})\vec{k} \cdot \partial, \qquad \vec{\phi} \text{ is a point on } T^d.$$

- These are are generators of finite coordinate transformations,

$$\tilde{\phi}^i = \phi^i + F(\vec{\phi})k^i, \quad \frac{\partial \tilde{\phi}^j}{\partial \phi^i} = \delta_i^j + \frac{\partial F(\vec{\phi})}{\partial \phi^i} k^j.$$

- Next consider matrix $M$:

$$M_i^j = \delta_i^j + t^i u_j,$$

where $u$,$t$ are arbitrary $d$-vectors.

- One can then easily verify that

$$\text{det}(M) = 1 + t^i u_i, \quad [M^{-1}]_j^i = \delta_j^i - \frac{t^i u_j}{\text{det}(M)}, \quad u^i M_i^j = \text{det}(M) u^j.$$

- If $V_{\vec{k}}$ is a vector field under the coordinate transformation,

$$v(\vec{\phi})k^i \cdot \frac{\partial \tilde{\phi}^j}{\partial \phi^i} = \tilde{v}(\vec{\tilde{\phi}})k^j, \quad \tilde{v}(\vec{\tilde{\phi}}) = v(\vec{\phi})(1 + k^i \frac{\partial F(\vec{\phi})}{\partial \phi^i}),$$

where we assumed $\vec{k}$ is invariant under coordinate transformations.

Next, we define invariant pairing and appropriate tensor densities:

- consider the vector $l_i$ and denote the volume element on $T^d$ by $\Omega$ and consider the "$d+1$ density"

$$W_{\vec{l}} = w(\vec{\phi})(l_i d\phi^i)\Omega,$$

- $\Omega, l_i$ transform as

$$\Omega \to (1 + k^i \frac{\partial F(\vec{\phi})}{\partial \phi^i})\Omega, \quad l_i \to \tilde{l}_i$$

  such that $k \cdot l$ remains invariant: $k^i l_i = k^i \tilde{l}_i$. Then,

$$w(\phi) \to \tilde{w}(\vec{\tilde{\phi}})\ k^i \tilde{l}_i = \frac{w(\vec{\phi})\ k^i l_i}{(1 + k^i \frac{\partial F(\vec{\phi})}{\partial \phi^i})^2},$$

- The invariant pairing is

$$\langle W_{\vec{l}}, V_{\vec{k}} \rangle = \int_{\mathbf{T}^d} \Omega\ v(\vec{\varphi}) w(\vec{\phi})(\vec{k} \cdot \vec{l})$$

**Stabilizer equation.**

- Stabilizer equation for $\widehat{\mathrm{Diff}_{\vec{k}}(T^d)}$ for $\vec{k} = (1, 0, ..., 0)$ is

$$\xi'''(\varphi; \Phi) - 4L(\varphi; \Phi)\xi'(\varphi; \Phi) - 2L'(\varphi; \Phi)\xi(\varphi; \Phi) = 0,$$

  where prime denotes derivative w.r.t. $\varphi$.

- Solutions to this equation describe the generators of the stabilizer subgroup.

- NHEG Hill's equation:

$$\psi''(\varphi; \Phi) - L(\varphi; \Phi)\psi(\varphi; \Phi) = 0.$$

- Solutions to the Hill's equation $\psi_1, \psi_2$ may be put in a doublet

$$\Psi(\varphi;\Phi) \equiv \begin{pmatrix} \psi_1(\varphi;\Phi) \\ \psi_2(\varphi;\Phi) \end{pmatrix}, \qquad \psi_1'\psi_2 - \psi_2'\psi_1 = -1$$

- Since $L(\varphi;\Phi)$ is periodic:

$$\Psi(\phi^1, \cdots, \phi^i + 2\pi, \cdots, \phi^d) = \mathbf{A}_i \Psi(\vec{\phi}), \;\; \mathbf{A}_i \in \mathsf{SL}(2,\mathbb{R}), \;\; \forall i = 1, \cdots, d.$$

- Since NHEG Hill's equation only involves derivative w.r.t. $\phi^1$, the monodromy matrices $\mathbf{A}_i$ are in general function of $\Phi = (\phi^2, \cdots, \phi^d)$:

$$\mathbf{A}_i = \mathbf{A}_i(\phi^2, \cdots, \phi^d)$$

- If $\mathbf{T}^d$ is a commutative torus, the monodromies associated with shift of coordinates $i, j$ by $2\pi$ should locally commute:

$$\Psi(\phi^1, \cdots, \phi^i + 2\pi, \cdots, \phi^j + 2\pi, \cdots) = \mathbf{A}_i \mathbf{A}_j \Psi(\vec{\phi}) = \mathbf{A}_j \mathbf{A}_i \Psi(\vec{\phi})$$

equivalently,

$$[\mathbf{A}_i(\Phi), \mathbf{A}_j(\Phi)] = 0, \qquad \forall i, j = 1, \cdots, d,$$

where $\Phi$ is a generic point on $T^{d-1}$.

- Classification of NHEG coadjoint orbits is then given through $d$ commuting monodromy matrices as functions on $T^{d-1}$ each belonging to a conjugacy class of SL$(2, \mathbb{R})$.

- Either $\mathbf{A}_i$ belong to the same conjugacy class or they are in special circular orbit, for which monodromy is identity matrix.

- These conjugacy classes can also have different winding numbers.

- Since we only have derivatives w.r.t $\varphi$, requiring that $\psi$ are smooth functions over $T^{d-1}$ implies all the monodromies $\mathbf{A}_a,\ a = 2, \cdots, d$ should be trivial.

- Therefore, the only possibility is

$$\mathbf{A}_1 = \mathbf{A}(\Phi), \qquad \mathbf{A}_a = \mathbf{1},$$

  where $\mathbf{A}(\Phi)$ is one of the standard SL$(2, \mathbb{R})$ conjugacy classes.

- NHEG coadjoint orbits are exactly the same as those of Virasoro with the new feature that the continuous label on the orbit (relevant to the elliptic and hyperbolic orbits) is now a function on $T^{d-1}$.

## Backup slide: More on $\mathbf{A}_a = 1$:

- classification of orbits is classification of all distinct functions $L(\varphi;\Phi)$.

- Consider "constant representative orbits" where $L = L(\Phi)$. In this case the solutions to Hill's equation are of the form

$$\psi_1 = \frac{\chi(\Phi)}{2\sqrt{K(\Phi)}} e^{K(\Phi)\varphi}, \quad \psi_2 = \frac{1}{2\chi(\Phi)\sqrt{K(\Phi)}} e^{-K(\Phi)\varphi}, \quad L(\Phi) = K^2(\Phi).$$

  where $\chi(\Phi)$ is an arbitrary smooth function on $T^{d-1}$.

- Function $\chi$, while may lead to non-trivial $\mathbf{A}_a$, has no appearance either in $L$ or in the solution to stabilizer equation, $\xi = \psi_1\psi_2$.

- $\mathbf{A}_a$ monodromies resulting from $\chi$ does not have any effect on the classification of $L$ functions and all non-equivalent orbits are only labeled by $\mathbf{A}$ monomdromy.

## NHEG group as a Virasoro bundle over $T^{d-1}$

- Recall that the NHEG group $\text{Diff}_{\vec{k}}(T^d)$ is

$$\text{Diff}_{\vec{k}}(T^d) = C^\infty\Big(\text{Diff}(S^1), T^{d-1}\Big).$$

- $\text{Diff}_{\vec{k}}(T^d)$ is the group of smooth maps that send a point $\Phi$ on $T^{d-1}$ to a circle diffeomorphism $F(\varphi;\Phi) \equiv F_\Phi(\varphi)$.

- Then the NHEG coadjoint orbits should follow suit:
  - At any point on $T^{d-1}$ we should have a Virasoro coadjoint orbit.
  - While the value of the representative $L$ can depend on $\Phi$, the winding number cannot change as we move on $T^{d-1}$.

## NHEG algebra from Abelian current algebra

- Consider a set of currents $J_i(\vec{\phi}),\ i \in \{1, \cdots, d\}$ and assume that their Fourier modes,

$$J_{i,\vec{n}} = \oint_{T^d} e^{-i\vec{n}\cdot\vec{\phi}} J_i(\vec{\phi}),$$

satisfy the algebra

$$[J_{i,\vec{n}}, J_{j,\vec{m}}] = i(\vec{k}\cdot\vec{n})\,\gamma_{ij}\,\delta_{\vec{n}+\vec{m},0},$$

where $\gamma_{ij}$ is the metric on the torus $T^d$.

- For the case where $\vec{k}$ is a vector on the dual lattice of the $T^d$, there are two sets of generators:
  - parallel modes $J_{i,\vec{n}}$ with $\vec{n} = n\vec{k}$, denoted by by $J_{i,n}$
  - perpendicular modes $J_{i,\vec{n}}$ with $\vec{n}\cdot\vec{k} = 0$, denoted by $J_{i,\vec{n}_\perp}$.

- Parallel modes satisfy a usual current algebra while perpendicular modes commute with themselves and with the parallel modes:

$$[J_{i,n}, J_{j,m}] = in\,\gamma_{ij}\,\delta_{n+m,0}, \qquad [J_{i,n_\perp}, J_{j,\vec{m}}] = 0.$$

- Perpendicular modes are central elements in the algebra of charges.

- Set of parallel and perpendicular modes overlap on $J_{i,0}$: If we choose the basis on $T^d$ where $\vec{k}$ is along the $\varphi$ direction then,

$$J_{i,n}(\Phi) = \int d\varphi J_i(\varphi, \Phi) e^{in\varphi}, \quad [J_{i,n}(\Phi), J_{j,m}(\Phi')] = in\,\gamma_{ij}\,\delta_{n+m,0}\;\delta^{d-1}(\Phi - \Phi').$$

- $J_{i,0}(\Phi)$ is the center element of the algebra for all $\Phi \in T^{d-1}$.

## NHEG algebra from twisted Sugawara construction

- Consider the twisted Sugawara construction

$$L(\varphi;\Phi) \equiv \beta(\Phi)\, k^i J_i'(\varphi;\Phi) + \frac{1}{2}\gamma^{ij} J_i(\varphi;\Phi) J_j(\varphi;\Phi)$$

  $\beta(\Phi)$ is an arbitrary function of $\Phi$ and *prime* is derivative w.r.t. $\varphi$.

- $L_n(\Phi)$ satisfies the NHEG algebra with central charge

$$\frac{c(\Phi)}{12} = \beta^2(\Phi)\, k^i \gamma_{ij} k^j .$$

- $L, J$ algebra

$$[L_{\vec{n}}, J_{i,\vec{m}}] = -(\vec{k}\cdot\vec{m}) J_{i,\vec{n}+\vec{m}} + i\beta(\vec{k}\cdot\vec{m})^2 k_i\, \delta_{\vec{n}+\vec{m}}$$

- $J_{i,0}$ form the center of the algebra:

$$[J_{i,0}, L_{\vec{m}}] = 0 = [J_{i,0}, J_{j,\vec{m}}], \qquad \forall \vec{m}, j.$$

- Representations of $\widehat{\mathcal{V}_{\vec{k},c}}$ can hence be labelled by eigenvalues of $J_{i,0}$.

## Unitary irreducible representations, NHEG modules

- Currents $J_{i,\vec{n}}$ and their algebra can be used to construct a unitary representation and the Hilbert space associated with the NHEG algebra (and its coadjoint orbits).

- We should first define a vacuum state and a hermitian conjugation which may then be used to define positive norm and unitarity.

- NOTATION: we use boldface symbols for operator-valued objects and normal ones for their eigenvalues.

- Hermitian conjugation is a $Z_2$ operation which is an inner automorphism of the algebra of currents:

$$\boldsymbol{J}_{i,\vec{n}}{}^{\dagger} = \boldsymbol{J}_{i,-\vec{n}},$$

- The vacuum state $|J_i^0;0\rangle$

  $$\boldsymbol{J}_{i,n}(\Phi)|J_i^0(\Phi);0\rangle = 0 \ \ \forall n>0, \quad \boldsymbol{J}_{i,0}(\Phi)|J_i^0(\Phi);0\rangle = J_i^0(\Phi)|J_i^0(\Phi);0\rangle.$$

- $|J_i^0;0\rangle$ are primary states of weight $\frac{1}{2}(J_i^0)^2$.

- Descendants of these vacuum states, their modules:

  $$|\{n_1,n_2\cdots\};J_i^0(\Phi)\rangle = \boldsymbol{J}_{-n_1}(\Phi_1)\boldsymbol{J}_{-n_2}(\Phi_2)\cdots|J_i^0(\Phi);0\rangle, \quad \forall n_l>0$$

- These modules are in one-to-one correspondence with the coadjoint orbit of the NHEG algbera labeled by $\boldsymbol{L}_{n=0}(\Phi) = \frac{1}{2}(J_i^0(\Phi))^2$:

$$\{|\{n_1,n_2\cdots\};J_i^0(\Phi)\rangle,\ \forall n_l>0\} = \{L_{-p_1}(\Phi_1)L_{-p_2}(\Phi_2)\cdots|J_i^0(\Phi)\rangle,\ \forall p_l>0\}.$$

- The above representations are unitary if $\frac{1}{2}(J_i^0(\Phi))^2 > -\frac{c(\Phi)}{24}$.

# Concluding Remarks and Outlook

⊛ NHEG algebra $\widehat{\mathcal{V}_{\vec{k},c}}$: $\quad [L_{\vec{m}}, L_{\vec{n}}] = \vec{k}\cdot(\vec{m}-\vec{n})\ L_{\vec{n}+\vec{m}} + c(\vec{n}_\perp)\ (\vec{k}\cdot\vec{m})^3\delta_{\vec{n}+\vec{m},0}$

⊛ $\widehat{\mathcal{V}_{\vec{k},c}}$ has many Virasoro subalgebras, when $\vec{n}$ is restricted to a given direction on $T^d$ lattice.

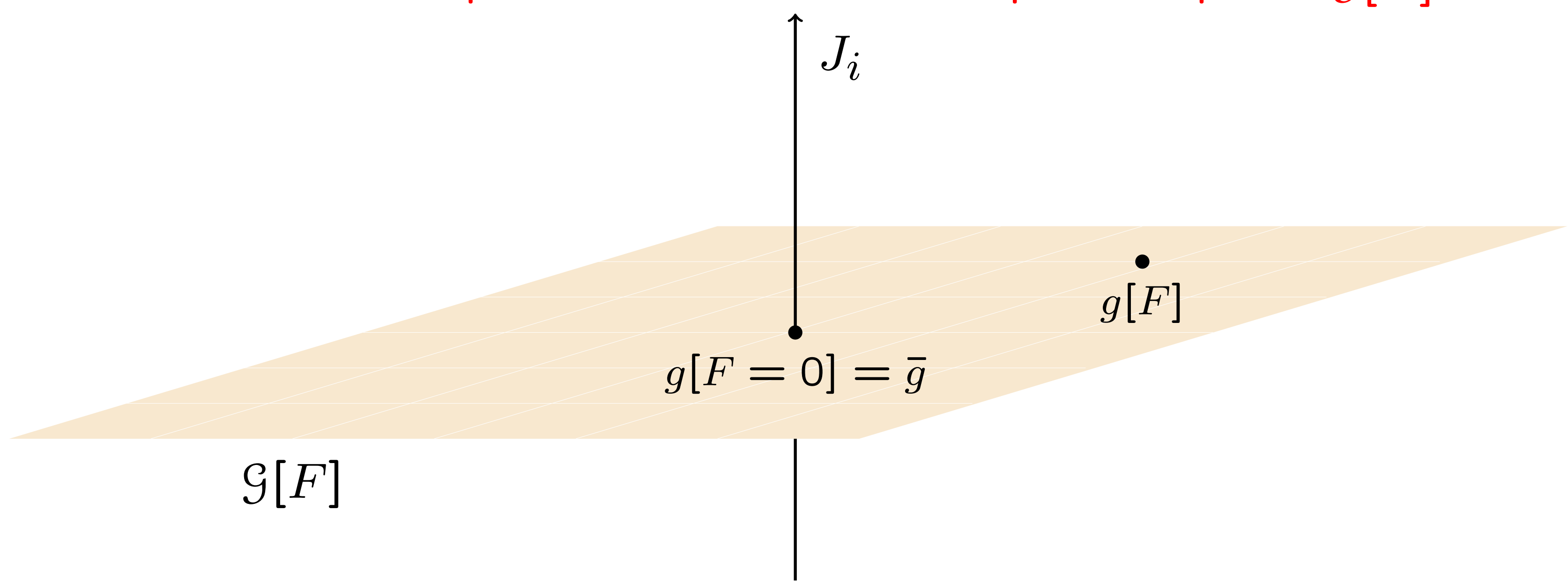
⊛ NHEG group $\text{Diff}_{\vec{k}}(T^d) = C^\infty\Big(\text{Diff}(S^1), T^{d-1}\Big)$.

⊛ We constructed coadjoint orbits of $\widehat{\mathcal{V}_{\vec{k},c}}$.

⊛ We showed NHEG coadjoint orbits are described by the same Virasoro coadjoint orbits bundled over $T^{d-1}$.

⊛ We introduced the one-function family of NHEG metrics which form a phase space and $\widehat{\mathcal{V}_{\vec{k},c}}$ is its symplectic symmetry:

Schematic depiction of the NHEG phase space $\mathcal{G}[F]$.




The NHEG phase space is then a coadjoint orbit of $\widehat{\mathcal{V}_{\vec{k},c}}$.

⊛ **NHEG algebra** can be obtained from $d$ $U(1)$ current algebras via tiwsted Sugawara construction.

⊛ $L_{\vec{n}}$ can be expressed in a Liouville-type stress-tensor:

$$L_{\vec{n}} = \oint_{T^d} \boldsymbol{\epsilon}_{\mathcal{H}} \; T[\Psi] e^{i\vec{n}\cdot\vec{\varphi}}, \qquad T[\Psi] = \frac{1}{16\pi G}\Big( (\partial\Psi)^2 - 2\partial^2\Psi + 2e^{2\Psi} \Big).$$

- $\partial$ denotes the directional derivative $\partial \equiv \vec{k}\cdot\vec{\partial}$

- $e^{\Psi}$ is a primary of weight one.

*Very interesting to explore this theory and its connection to $d$ dim. Einstein gravity.*

## ⊛Relevance for extremal black hole microstates.

- We have put forward the **horizon fluff** proposal [arXiv:1607.00009].
- It states that black hole microstate are labeled by the *soft hair* sector of the $U(1)$ current algebra appearing the twisted Sugawara construction.
- This proposal has been applied to 4d extremal Kerr [arXiv:1708.06378].
- The NHEG modules and unitary reps play a crucial role in carrying out this proposal.
- The “Liouville-type field theory” should then arise as the long string sector (or low energy effective theory) of the horizon fluff proposal.

It is desirable to study the points discussed here and complete the bottom-up black hole microstate construction, and in particular the horizon fluff proposal, outlined in the beginning.

*Thank You For Your Attention*

## *Backup Slides*

## Semiclassical and Quantum Aspects of Black Holes

- Works of last forty years, most notably by Bekenstein and Hawking, have established black holes as
  - thermodynamical systems,
  - generically specified by temperature and other (chemical, rotational or electrical) horizon potentials, as well as
  - the Noether-Wald charges, and the entropy.

- Black holes
  - Laws of Thermodynamics,
  - generically shed their charges through, a blackbody radiation, the Hawking radiation.

- Within Einstein GR + semiclassical coupling to other fields,
  - black holes can form (gravitational collapse), and
  - nothing prevents them from the Hawking evaporation.
- Process of formation and evaporation of black holes is not unitary.
- Resolution may lie in identifying an underlying stat.mech. description, *black hole microstates.*
- Lore: Black hole microstates are amssed around the horizon.

- Extremal black holes have
  - zero Hawking temperature and do not Hawking radiate;
  - lowest mass for given set of charges,
  - generically nonzero entropy.
  - They can be a good model for some observed real black holes.
  - There are uniqueness theorems for extremal black hole solutions [arXiv:0906.2367].
- Supersymmetric (BPS) black holes are necessarily extremal (the reverse is not true).
- Microstate counting has been carried out for special class of BPS black holes [Strominger-Vafa'95; A. Sen-since 2004].

# Microstate counting:

- *Top-down approach*, embedding into Quantum Gravity;
- *Bottom-up approach*, using semiclassical gravity picture and relying on AdS/CFT ideas.

▶ Examples of top-down approach:

- Original Strominger-Vafa idea,
- D1-D5 system (and its variants),
- Quantum Entropy Function program proposed and pushed by Ashoke Sen and collaborators have studied in last 15 years,
- and some $AdS_5$ black holes;

▶ Examples of bottom-up approach,

- Kerr/CFT [M. Guica, Monica, T. Hartman, W. Song, A. Strominger], [G. Compere, arXiv:1203.3561].

- Carlip's idea [S. Carlip, PRL, 1999].

The above relies on Cardy formula to perform microstate counting and reproduce the entropy.

▶▶ Horizon Fluff idea M.M.Sh-J, D. Grumiller, H, Afshar, H. Yavartanoo, K. Hajian, 2016-2017.

Here, I discuss the first steps of applying the **horizon fluff idea** for Near Horizon Extremal Geometries.

## ▶ *De tour to Wald-Noether conserved charges*

- For diffeomorphism invariant theory with a Lagrangian density $\mathcal{L}$:

$$I[\Phi_\alpha] = \int d^D x \sqrt{-\det g}\,\mathcal{L}(\Phi_\alpha; x) \equiv \int \boldsymbol{L}$$

$\boldsymbol{L}$ is a $D$-form.

- For any generic diffeo generated by one-form $\boldsymbol{\xi}$

$$\delta_\xi \boldsymbol{L} = \boldsymbol{E}_\alpha \delta_\xi \Phi_\alpha + d\boldsymbol{\Theta}(\Phi, \delta_\xi \Phi)$$

where $\boldsymbol{E}_\alpha = 0$ are e.o.m and $\boldsymbol{\Theta}$ is a $(D-1)$-form linear in $\delta_\xi \Phi$.

- Recalling the identity

$$\delta_\xi \boldsymbol{X} = d(\xi \cdot \boldsymbol{X}) + \xi \cdot d\boldsymbol{X}$$

and that $d\boldsymbol{L} = 0$, then

$$d\boldsymbol{\Theta} - d(\xi \cdot \boldsymbol{L}) = -\boldsymbol{E}_\alpha \delta_\xi \Phi_\alpha$$

and hence the Noether current (or Noether $(D-1)$-form)

$$\boldsymbol{J} \equiv \boldsymbol{\Theta} - \xi \cdot \boldsymbol{L}$$

is conserved on-shell.

- Using *Poincaré Lemma*, we locally have

$$\boldsymbol{J} = d\boldsymbol{Q}$$

where $\boldsymbol{Q}$ is the $(D-2)$-form Noether charge density.

## ■ Ambiguities in the definition of Noether charge density $\boldsymbol{Q}$

Noting the above derivation, there are three kind of ambiguities appear in the definition of $\boldsymbol{Q}$:

$$\boldsymbol{Q} = \boldsymbol{E}^{\mu\nu}\nabla_\mu \xi_\nu + \mathbf{W}_\mu(\Phi)\xi^\mu + \mathbf{Y}(\Phi, \delta_\xi \Phi) + d\mathbf{Z}(\Phi, \xi)$$

where

$$\boldsymbol{E}^{\mu\nu}{}_{\alpha_3\cdots\alpha_D} = (E^{\mu\nu\rho\beta})\epsilon_{\rho\beta\alpha_3\cdots\alpha_D}\,, \qquad E^{\mu\nu\rho\beta} \equiv \frac{\delta\mathcal{L}}{\delta R_{\mu\nu\rho\beta}}\,,$$

$\mathbf{W}(\Phi)$ is a $(D-1)$-form,
$\mathbf{Y}(\Phi, \delta_\xi \Phi)$ is a $(D-2)$ form linear in $\delta_\xi \Phi$, and
$d\mathbf{Z}(\Phi, \xi)$ is a $(D-3)$-form linear in $\xi$.

$\mathbf{W}$ and $\mathbf{Y}$ come from the fact that Lagrangian $\mathcal{L}$ is generically defined up to a total derivative.

- Noether-Wald conserved charges are then defined as integrals of $\mathbf{Q}$ over a codimension two surface.

- This surface can be either asymptotic space or bifurcation surface of a black hole horizon.

- In our case, this codimension two surface is in fact $\mathcal{H}$ itself.

- $\mathcal{H}$ is defined as constant $t = t_H, r = r_H$ surfaces (for arbitrary $t_H, r_H$) on NHEG.

- Although $\mathcal{H}$ is defined at given $t, r$, its volume form, is Hodge dual to the AdS$_2$ volume form

$$\epsilon_{\mathbf{2}} \equiv \xi^a \wedge dn_a = \Gamma dt \wedge dt$$

is independent of $t, r$ and is $SL(2,R) \times U(1)^N$ invariant.

- Note that

$$Vol_{NHEG} = \epsilon_{\mathbf{2}} \wedge Vol_{\mathcal{H}}$$

- Then a generic NHEG conserved charges is defined as

$$Q_\xi = \oint_{\mathcal{H}} \mathbf{Q}_\xi = \oint_{\mathcal{H}} \sqrt{-g}\ (*\mathbf{Q}_\xi)^{tr}$$

*End of De tour* ◀

## Proof cont'd

- Angular momentum and electric (Noether-Wald) $U(1)^N$ charges:

$$\boldsymbol{q}_r = -\oint_{\mathcal{H}} \mathrm{d}\Sigma_{\mu\nu} \frac{\partial \mathcal{L}}{\partial F^r{}_{\mu\nu}}, \qquad \boldsymbol{J}_i = -\oint_{\mathcal{H}} \mathrm{d}\Sigma^{\mu\nu} \mathbf{E}_{\mu\nu\alpha\beta} \nabla^\alpha \mathbf{m}_i^\beta$$

  where

$$\mathbf{E}_{\mu\nu\alpha\beta} = \frac{\delta \mathcal{L}}{\delta R^{\mu\nu\alpha\beta}}$$

- Both of the above charges could be defined at any $\mathcal{H}$ defined at any arbitrary $t, r$.

- In particular, one can defined $\boldsymbol{q}_r$, $\boldsymbol{J}_i$ are $r = \infty$.

- The $\boldsymbol{J}_i$ is essentially the Komar integral.

- The $SL(2,R)$ charges:

$$\mathbf{Q}^{\mu\nu}_{\xi_a} = \frac{2}{C_2}\mathcal{L}\ f_a{}^{bc}\xi^{\mu}_b\xi^{\nu}_c + \sum_r \Lambda^r \frac{\partial \mathcal{L}}{\partial F^r_{\mu\nu}}\,,$$

$f^a{}_{bc}$ is the structure constant and $C_2$ is the second rank Casimir for $SL(2,R)$ ($C_2 = 2$), and $\Lambda^r = \Lambda^r(\xi_a)$ is determined such that

$$\delta_{\xi_a} A^r_\mu = \partial_\mu \Lambda^r .$$

- For our case,

$$\delta_{\xi_1} A^s = \delta_{\xi_2} A^s = 0\,, \qquad \delta_{\xi_3} A^s = -\frac{e^s}{r^2} dr = d(\frac{e^s}{r})\,,$$

and hence

$$\boldsymbol{Q}_a \equiv \oint_{\mathcal{H}} d\Sigma_{\mu\nu}\mathbf{Q}^{\mu\nu}_a = n_a \oint_{\mathcal{H}} d\Sigma_{tr}\mathcal{L} - \delta_{a3}\sum_p \frac{e^p}{r}\boldsymbol{q}_p$$

$$\boxed{n^a \boldsymbol{Q}_a = \sum_p e^p \boldsymbol{q}_p - \oint_{\mathcal{H}} d\Sigma_{tr}\mathcal{L}\,.}$$

## Proof of Entropy Perturbation Law

We start with Neother current associated with an arbitrary diffeo $\zeta$

$$\mathbf{J}_\zeta \equiv \mathbf{\Theta}(\Phi, \delta_\zeta \Phi) - \zeta \cdot \mathbf{L}\,,$$

Under the variation $\Phi_0 \to \Phi_0 + \delta\Phi$, imposing l.e.o.m for $\delta\Phi$ and some straightforward algebra we arrive at

$$\delta \mathbf{J}_\zeta = \boldsymbol{\omega}(\Phi_0, \delta\Phi, \delta_\zeta \Phi) + \mathsf{d}(\zeta \cdot \mathbf{\Theta}(\Phi_0, \delta\Phi))\,,$$

where

$$\boldsymbol{\omega}(\Phi_0, \delta_1\Phi, \delta_2\Phi) \equiv \delta_1 \mathbf{\Theta}(\Phi_0, \delta_2\Phi) - \delta_2 \mathbf{\Theta}(\Phi_0, \delta_1\Phi)$$

is the *symplectic current*, the $(D-1)$-form associated with variations $\delta_1, \delta_2$, and is bilinear in its arguments [Wald, 1993].

NOTE: $\delta\Phi$ are (necessarily) not generated by diffeo's & $\zeta$ is not necessarily a Killing.

- *If $\zeta$ is a Killing,* $\delta_\zeta \Phi = 0$ (possibly up to gauge transformations), $\omega = 0$, and therefore,

$$\delta \mathbf{J}_\zeta = \mathrm{d}(\zeta \cdot \boldsymbol{\Theta}(\Phi_0, \delta\Phi))\,.$$

- That is $\delta \mathbf{J}_\zeta$ is conserved. Note that this is despite the fact that perturbations $\delta\Phi$ are NOT invariant under $\zeta$, $\delta_\zeta(\delta\Phi) \neq 0$.

- Using Poincaré Lemma, we can write $\delta \mathbf{J}_\zeta = d\delta \boldsymbol{Q}_\zeta$, where

$$\delta \boldsymbol{Q}_\zeta = \zeta \cdot \boldsymbol{\Theta}(\Phi_0, \delta\Phi)\,.$$

- When we have other internal gauge symmetries there are other extra terms added to the above, but the conservation result still remains.

- One can now compute the charge perturbations integrating over the corresponding $\delta \boldsymbol{Q}_\zeta$'s.

- Following similar steps as in the derivation of Entropy Law,

- taking care of ambiguities in Noether-Wald current-density ,

- taking care of technicalities associated with integrals over closed surfaces over NHEG,

- taking care of the fact that we have internal gauge symmetries,

- using $\boldsymbol{\zeta}_H = n_H^a \boldsymbol{\xi}_a - k^i \boldsymbol{m}_i$, and that $\zeta_H$ vanishes on surface $\mathcal{H}$ defined at $r = r_H, t = t_H$, we finally arrive at

$$\frac{\delta \mathbf{S}}{2\pi} = k^i \delta \boldsymbol{J}_i + e^r \delta \boldsymbol{q}_r + n^a_H \delta \boldsymbol{\mathcal{E}}_a \,,$$

where

$$\delta \boldsymbol{J}_i \equiv - \oint_\infty \delta \mathbf{Q}_{m_i} = \oint_{\mathcal{H}} \delta \mathbf{Q}_{m_i} \,, \qquad \delta \boldsymbol{q}_r \equiv - \oint_\infty \delta \left( \frac{\delta \mathcal{L}}{\delta F_{\mu\nu}} \right) ,$$

$$\frac{\delta \mathbf{S}}{2\pi} = -2\delta \oint_{\mathcal{H}} d\boldsymbol{\Sigma}_{\mu\nu} \mathbf{E}^{\mu\nu}{}_{\alpha\beta} \nabla^\alpha \zeta^\beta_H \,, \qquad \delta \boldsymbol{\mathcal{E}}_a \equiv \oint_\infty \left( \delta \mathbf{Q}_{\xi_a} - \xi_a . \boldsymbol{\Theta} \right) ,$$

$\delta \boldsymbol{\mathcal{E}}_a$ is the canonical generator of the symmetry $\xi_a$ in the covariant phase space [arXiv:gr-qc/9503052].

- Consider a generic black hole with parameters $(q_i, m)$. $m$ is associated with mass and $q_i$ to all the other charges.

- Recall the 1st law

$$TdS = dE - \sum_i \Omega_i dJ_i$$

  $\Omega_i$ denote horizon angular velocities and/or electric potentials and, $J_i$ black hole angular momenta and/or electric charges.

- All the thermodynamic quantities are functions of $(q_i, m)$.

- Consider extremal black hole $T = 0$.

- On the extremal surface,

$$m = m(q_i), \qquad \Omega_i^{ext} = \frac{\partial E}{\partial J_i}$$

and one can *integrate* the first law over the extremal surface to obtain the BPS condition

$$E = E(J_i)$$

- One may study a near extremal black hole, by moving slightly away from a given point on the extremal surface.

- In general this can be
  - either along the extremal surface, parameterized by $\delta^i_{||}$,

$$dm = \partial_i m(q_i)\delta^i_{||}$$

  - or transverse to the extremal surface, parameterized by $\delta^i_{\perp}$.

- We then note that

$$T = \partial_{\perp} T\ \delta_{\perp}, \qquad \Omega_i = \Omega^{ext}_{i\ ||} + \partial_{\perp}\Omega_i\ \delta_{\perp}$$

NOTE: $\Omega^{ext}_{i\ ||}$ includes changes of $\Omega_i$ caused by moving along the extremal surface.

NOTE: $T$ does not have a term proportional to $\delta^i_{||}$.

- One can show that

$$dE - \Omega_{i\,||}^{ext} dJ_i = \mathcal{O}(\epsilon^2) \qquad (\divideontimes)$$

where $\delta_\perp \sim \delta_{||} \sim \epsilon$.

▶ *De tour to Proof:*

- Consider a black hole with $g^{rr} = f(r)$, where

$$f = f(m, q_i; r)$$

- For the extremal case, $m = m_0$,

$$m_0 = m_0(q_i)$$

$$f = C(r - r_h)^2 + D(r - r_h)^3 + \cdots$$

where $C, D$ and $r_h$ are functions of $q_i$.

- Now consider the near extremal case:

$$m = m_0 + \delta m, \qquad r_\pm = r_h \pm \Delta\epsilon.$$

while $q_i$ are fixed.

- For the near extremal case

$$\begin{aligned} f &= f(m_0 + \delta m, q_i; r) \\ &= \delta A + \delta B(r - r_h) + (C + \delta C)(r - r_h)^2 \\ &\quad + (D + \delta D)(r - r_h)^3 + \cdots \end{aligned}$$

$\delta A, \delta B, \delta C, \delta D$ are linear in $\delta m$.

- Demanding $f$ two have two roots $r_\pm$, yields

$$\delta A = -C\Delta\epsilon^2, \qquad \delta B \sim \mathcal{O}(\epsilon^2), \cdots$$

- The above is an outcome of regularity of metric at the horizon.

- Therefore, $\delta m \sim \epsilon^2$. ■

- Finally we note that $dE - \sum_i \Omega_i^{ext} dJ_i \propto \delta m$

*End of De tour* ◀

- Constant $t, r$ surface $\mathcal{H}$ (for arbitrary $t_H, r_H$) are birfucation surface of the horizon, generated by

$$\zeta_H = n_H^a \xi_a + k^i \boldsymbol{m}_i$$

$$n^1 = -\frac{t^2 r^2 - 1}{2r}, \qquad n^2 = t\, r\,, \qquad n^3 = -r.$$

- $\zeta_H$ vanishes at $t = t_H,\ r = r_H$.

- All with the same surface gravity $\kappa$ (Unruh temperature),

$$\kappa^2 = -\frac{1}{2}(\nabla \zeta_H)^2 = 1.$$

# Laws of NHEG Mechanics

[A. Seraj, K. Hajian, M.M. Sh-J. 2013]

- Zeroth Law of NHEG Mechanics

*All codimension two surfaces $\mathcal{H}$ have surface gravity equal to one*

- NHEG Entropy and the Entropy Law:

  - NHEG has an entropy $S$ as the conserved Noether-Wald charge associated with $\zeta_H$:

$$\frac{S}{2\pi} = -\oint_{\mathcal{H}} \boldsymbol{\epsilon}_{\mathcal{H}} \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\alpha\beta}} \epsilon^{\perp}_{\mu\nu} \epsilon^{\perp}_{\alpha\beta}$$

where $\boldsymbol{\epsilon}_{\mathcal{H}} = \Gamma^{\frac{d-2}{2}} \sqrt{\gamma}\, d\theta\, d\vec{\varphi}$ is the volume form of $\mathcal{H}$ and $\epsilon^{\perp}_{\mu\nu} dx^{\mu} dx^{\nu} = \Gamma dt \wedge dr$.

▶ NHEG entropy law (for the family we consider here):

$$\frac{\boldsymbol{S}}{2\pi} = k^i \boldsymbol{J}_i$$

## ■ Entropy Perturbation Law

- Generic perturbation $\delta\Phi$ satisfying the linearized field equations,
- with charge perturbations $\delta J_i$ and $\delta S$ and
- assuming that perturbations are invariant under $\xi_1, \xi_2$ Killing vectors; $\mathcal{L}_{\xi_1}\delta\Phi = \mathcal{L}_{\xi_2}\delta\Phi = 0$:

$$\frac{\delta S}{2\pi} = k^i \delta J_i$$

## ▶ Properties Fixing $\chi$

1. $\xi_1, \xi_2$-invariance: $[\xi_1, \chi] = [\xi_2, \chi] = 0$.

2. Volume preserving: $\nabla_\mu \chi^\mu = 0$.

3. Lagrangian preserving: $\delta_\chi \boldsymbol{L} = 0$, where $\boldsymbol{L}$ is the Lagrangian density computed over the NHEG anstaz metric, a functional of $\Gamma(\theta)$, $\gamma_{ij}(\theta)$.

4. Null direction-preserving: $\delta_\chi v = 0$, $v = t + \frac{1}{r}$.

5. Smoothness of metric perturbations implies $\theta$-independence of $\chi$ components and $\chi^\theta = 0$.

6. Having *conserved and integrable* symplectic structure.

## Finite transformations:

$$\bar{x}^\mu \quad \to x^\mu = x^\mu(\bar{x}), \qquad \chi^\mu[\epsilon(\varphi)] = \frac{\partial x^\mu}{\partial \bar{x}^\alpha} \bar{\chi}^\alpha[\bar{\epsilon}(\bar{\varphi})].$$

Solving the above yields:

$$\boxed{\bar{t} = t - \frac{1}{r}(e^\Psi - 1), \quad \bar{r} = r e^\Psi, \quad \bar{\varphi}^i = \varphi^i + k^i F(\vec{\varphi})}$$

where

$$e^\Psi = 1 + \vec{k} \cdot \vec{\partial} F = 1 + \partial F$$

Notes:

$$\bar{t} + \frac{1}{\bar{r}} = t + \frac{1}{r}, \quad \bar{\epsilon}(\bar{\varphi}) = e^{-\Psi}\ \epsilon(\varphi)$$

## The symplectic structure

- Symplectic structure $\boldsymbol{\omega}$ is a *finite*, *closed*, *nondegenerate* two-form over tangent space and a $d-1$ form in space time:

$$\boldsymbol{\omega} = \boldsymbol{\omega}[\delta_1\Phi, \delta_2\Phi; \Phi]$$

- $\delta$ denotes exterior derivative on the phase space and $d$ is the exterior derivative over the spacetime.

- We build $\boldsymbol{\omega}$ within the *covariant phase space method*, constructed in [Lee-Wald '1990, Wald '1993] and refined in [Barnich-Brandt '2002, Barnich-Comperé '2008].

- Presymplectic potential $\boldsymbol{\theta}[\delta\Phi;\Phi]$: $\boldsymbol{\omega} = \delta\boldsymbol{\theta}$, or

$$\boldsymbol{\omega}[\delta_1\Phi, \delta_2\Phi; \Phi] = \delta_1\boldsymbol{\theta}[\delta_2\Phi;\Phi] - \delta_2\boldsymbol{\theta}[\delta_1\Phi;\Phi]$$

## Construction of the symplectic structure

- The presympelctic structure $\boldsymbol{\theta}$ is a spacetime $d-1$ form and a one-form over the phase space.

- The Lee-Wald constribution to $\boldsymbol{\theta}$:

$$\delta \boldsymbol{L}\big|_{on-shell} = d\boldsymbol{\theta}_{(LW)}.$$

- Consistency of symplectic usually requires addition of *boundary terms* $\mathbf{Y}$:

$$\boldsymbol{\theta} = \boldsymbol{\theta}_{(LW)} + d\mathbf{Y}.$$

$\boldsymbol{Y}$ is a $d-2$ form on spacetime and one-form on phase space.

# Consistency of symplectic structure means its

- Conservations: $\boldsymbol{\omega}[\delta_1\Phi, \delta_2\Phi; \Phi] \approx 0$ (i.e. for $\Phi = \bar{\Phi}$, $\forall \delta\Phi$).

  Covariance of $\boldsymbol{\omega}$ and that the whole phase space consists of diffeomorphic metric then implies

$$\boldsymbol{\omega}[\delta_1\Phi, \delta_2\Phi; \Phi] \approx 0 \quad \forall \Phi, \delta\Phi$$

- Integrability [Lee-Wald '1991]:

$$\oint_{\mathcal{H}} \chi \cdot \boldsymbol{\omega}[\delta_1\Phi, \delta_2\Phi; \Phi] = 0, \quad \forall \chi, \delta\Phi$$

- There exists $\boldsymbol{Y}$ terms which guarantee the above [arXiv:1506.nnnnn].

# Representation of charges on the phase space

- By construction, any geometry $g[F]$ in our phase space $\mathcal{G}[F]$ is uniquely specified by the set of charges $H_{\vec{n}}$ associated with it.

- Conversely, given $g[F]$ one can give an explicit expression of $H_{\vec{n}}$ in terms of $F$.

- It is obtained that $H_{\vec{n}}$ are the Fourier modes of the tensor $T[\Psi]$:

$$H_{\vec{n}} = \oint_{\mathcal{H}} \boldsymbol{\epsilon}_{\mathcal{H}} \; T[\Psi] \; e^{i\vec{n}\cdot\vec{\varphi}},$$

where $\boldsymbol{\epsilon}_{\mathcal{H}}$ is the volume form on $\mathcal{H}$ and

$$T[\Psi] = \frac{1}{16\pi G}\Big((\partial\Psi)^2 - 2\partial^2\Psi + 2e^{2\Psi}\Big).$$

$\partial$ denotes the directional derivative $\partial \equiv \vec{k}\cdot\vec{\partial}$.

$T[\Psi]$ resembles a Liouville theory stress tensor:

- Consider two metrics $g[F]$ and $g[F + \delta_\epsilon F]$ related by a diffeomorphism generated by $\chi[\epsilon]$:

$$\mathcal{L}_{\chi[\epsilon]}(g_{\mu\nu}[F]) = g_{\mu\nu}[F + \delta_\epsilon F] - g_{\mu\nu}[F].$$

- Then we learn that $\delta_\epsilon F = (1 + \vec{k} \cdot \vec{\partial}_\varphi F)\epsilon = e^\Psi \epsilon$, or

$$\delta_\epsilon \Psi = \epsilon\, \partial \Psi + \partial \epsilon.$$

- $e^\Psi$ is a "primary field of weight one".

- $T[\Psi]$ then transforms as

$$\delta_\epsilon T = \epsilon \partial T + 2 \partial \epsilon T - \frac{1}{8\pi G} \partial^3 \epsilon.$$