SL(2,$\mathbb{R}$) String and Liouville Theory

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Introduction

String dynamics in AdS background is in the focus of research during the last two decades due to its major role in the study of the AdS/CFT correspondence (Maldacena duality conjecture).

The AdS/CFT correspondence states that AdS string dynamics is equivalent to CFT living on the boundary of AdS.

More generally, the duality conjecture can be extended to more realistic models of QFT with corresponding deformations of AdS backgrounds.

SL$(2,\mathbb{R})$ group manifold is an example of AdS$_3$ space.

String dynamics on $\mathrm{SL}(2,\mathbb{R})$ was investigated even before the duality conjecture as a model of string dynamics in a curved background
Balog, O'Raifeartaigh, Forgacs, Wipf (89)

The most important contribution by Maldacena, Ooguri (00)

The model was treated as the $\mathrm{SL}(2,\mathbb{R})$ WZW theory with vanishing energy-momentum tensor, which has a chiral structure.

The main tool was the method of covariant qunatization.
It aims to eliminate negative norm states by the quantum constraints.

An alternative approach to string dynamics is the quantization in physical variables, based on the elimination of nonphysical degrees of freedom on the classical level by Hamiltonian reduction with gauge fixing.

The light-cone type gauge proposed in GJ (09) and Sundborg (13) exhibits a connection between the $\mathrm{SL}(2,\mathbb{R})$ string and Liouville theory.

In this talk I will analyze this connection using the chiral structure of the symplectic form of WZW theory.

I will show that Hamiltonian reduction in the chiral sectors indeed leads to the structure typical to Liouville theory.

The obtained Liouville fields are singular, though they correspond to a regular energy-momentum tensor. I will focus on the elliptic monodromy.

I will start by the chiral reduction of string dynamics in Minkowski space.

String in 3d Minkowski space

Let us consider $3d$ Minkowski space with coordinates X^μ, $\mu = (0, 1, 2)$, and the metric tensor $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1)$.

The dynamics of a closed string in the first order formalism and in the conformal gauge is described by the action

$$S = \int \mathrm{d}\tau \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \Big(\mathcal{P}_\mu \dot{X}^\mu - \frac{1}{2} (\mathcal{P}_\mu \mathcal{P}^\mu + X'_\mu X'^\mu) \ ,$$

where (τ, σ) are worldsheet coordinates, $\mathcal{P}_\mu$ is the canonically conjugated variable to X^μ and one has the Virasoro constraints

$$\mathcal{P}^2 + X'^{\,2} = 0 \ , \qquad \mathcal{P} \cdot X' = 0 \ .$$

Chiral structure

One finds the general solution as the sum of the chiral and antichiral parts

$$X^\mu(\tau,\sigma) = \Phi^\mu(z) + \bar{\Phi}^\mu(\bar{z}) \;, \qquad P_\mu(\tau,\sigma) = \Phi'_\mu(z) + \bar{\Phi}'_\mu(\bar{z}) \;,$$

with $z := \tau + \sigma, \; \bar{z} := \tau - \sigma$.

The Virasoro constraints correspond to the conditions

$$\Phi'^{\,2}(z) = 0 \;, \qquad \bar{\Phi}'^{\,2}(\bar{z}) = 0 \;,$$

The periodicity in σ is equivalent to the monodromy properties

$$\Phi^\mu(z+2\pi) = \Phi^\mu(z) + \pi\, p^\mu \;, \quad \bar{\Phi}^\mu(\bar{z}+2\pi) = \bar{\Phi}^\mu(\bar{z}) + \pi\, p^\mu \;.$$

The mode expansion

$$\Phi^\mu(z) = q^\mu + \frac{1}{2}\, p^\mu z + \frac{i}{\sqrt{2}} \sum_{n\neq 0} \frac{a_n^\mu}{n}\, e^{-inz} \;.$$

The canonical symplectic form has the chiral structure

$$\int_0^{2\pi} \frac{\mathsf{d}\sigma}{2\pi} \;\; \delta P_\mu(\tau,\sigma) \wedge \delta X^\mu(\tau,\sigma) = \omega + \bar{\omega} \ ,$$

with

$$\omega = \int_\tau^{\tau+2\pi} \frac{\mathsf{d}z}{2\pi} \, \delta\Phi'_\mu(z) \wedge \delta\Phi^\mu(z) + \frac{1}{2}\,\delta p_\mu \wedge \delta\Phi^\mu(\tau) \ ,$$

The light-cone gauge

$$\Phi^+(z) = \frac{1}{2}\,p^+ z \ , \qquad \bar{\Phi}^+(\bar{z}) = \frac{1}{2}\,p^+ \bar{z} \ ,$$

is defined by the light-cone coordinates

$$X := X^1 \ , \qquad X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^2) \ .$$

Hamiltonian reduction

On the constraint surface $\Phi'^2(z) = 0,\ \Phi^+(z) = \frac{1}{2}\, p^+ z$
the chiral symplectic form

$$\omega = \int_\tau^{\tau+2\pi} \frac{\mathrm{d}z}{2\pi}\ \delta\Phi'_\mu(z) \wedge \delta\Phi^\mu(z) + \frac{1}{2}\,\delta p_\mu \wedge \delta\Phi^\mu(\tau)\ ,$$

reduces to

$$\omega = \delta q^- \wedge \delta p^+ + \int_\tau^{\tau+2\pi} \frac{\mathrm{d}z}{2\pi}\ \delta\Phi'(z) \wedge \delta\Phi(z) + \frac{1}{2}\,\delta p \wedge \delta\Phi(\tau)\ .$$

One obtains the canonical Poisson brackets

$$\{q^-, p^+\} = 1\ , \quad \{p, q\} = 1\ , \quad \{a_m, a_n\} = \frac{im}{2}\,\delta_{m,-n}\ .$$

and similarly for the anti-chiral part.

The Noether charges related to the Poincare symmetry

$$P^\mu = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\, \mathcal{P}^\mu(\sigma)\ , \quad J^{\mu\nu} = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \Big(\mathcal{P}^\mu(\sigma)\, X^\nu(\sigma) - \mathcal{P}^\nu(\sigma)\, X^\mu(\sigma) \Big)\ ,$$

after the Hamiltonian reduction become

$$P^+ = p^+, \qquad P^1 = p\,, \qquad P^- = p^- = \frac{L_0 + \bar{L}_0}{p^+}\ ,$$

$$J^{+-} = p^+\, q^-, \qquad J^{+1} = p^+\, q\ ,$$

$$J^{1-} = p\, q^- - p^-\, q - \frac{2i}{p^+} \sum_{n\neq 0} \frac{L_{-n}\, a_n + \bar{L}_{-n}\, \bar{a}_n}{n}\ .$$

Poincare algebra

The canonical Poisson brackets realize the 3d Poincare algebra

$$\{P^\mu, P^\nu\} = 0 \ , \quad \{J^\mu, P^\nu\} = \epsilon^{\mu\nu\rho} P_\rho \ , \quad \{J^\mu, J^\nu\} = \epsilon^{\mu\nu\rho} J_\rho \ .$$

Here, $J^\mu = \frac{1}{2}\,\epsilon^{\mu\nu\rho} J_{\nu\rho}$ is dual to $J^{\nu\rho}$, with $\epsilon_{012} = 1$.

In the light-cone coordinates one has

$$\eta^{++} = \eta^{--} = 0, \quad \eta^{+-} = -1 \ , \quad \epsilon^{+1-} = 1.$$

$$J^+ = J^{+1} \ , \quad J^1 = J^{+-} \ , \quad J^- = J^{1-}.$$

The Casimir numbers of 3d Poincare algebra

$$-P_\mu P^\mu = 4 \sum_{n\geq 1} (a_{-n} a_n + \bar{a}_{-n} \bar{a}_n) \ , \quad P_\mu J^\mu = 2i \sum_{n\neq 0} \frac{L_{-n}\, a_n + \bar{L}_{-n}\, \bar{a}_n}{n}.$$

Note that the dependence on the zero mode p is canceled in $P_\mu J^\mu$.

Quantization of 3d String

Momentum dependent Fock space with vacuum states $|p^+, p; \ \rangle$

In the momentum representation: $q^- = -i\partial_{p^+}$, $q = i\partial_p$,
a_n and a_{-n} (for $n > 1$) are the annihilation and creation operators.

P^+ and P become multiplications by p^+ and p, respectively.

Other symmetry generators

$$P^- = \frac{1}{p^+}\left(\frac{p^2}{2} + \sum_{n\geq 1}(a_{-n}a_n + \bar{a}_{-n}\bar{a}_n)\right) \ ,$$

$$J^+ = ip^+\,\partial_p \ , \qquad J^1 = -i(p^+\,\partial_{p^+} + 1/2) \ ,$$

$$J^- = ip\,\partial_{p^+} + \frac{ip}{2p^+} + iP^-\,\partial_p + \frac{i}{p^+}\sum_{n\geq 1}\frac{L_{-n}\,a_n - a_{-n}L_n + \bar{L}_{-n}\,\bar{a}_n - \bar{a}_{-n}\bar{L}_n}{n} \ .$$

They realize the 3d Poincare algebra.

SL(2,$\mathbb{R}$) geometry

Using the Poincare coordinates for $g \in \mathsf{SL}(2, \mathbb{R})$

$$g = \frac{1}{X} \begin{pmatrix} 1 & X_+ \\ X_- & X^2 + X_+ X_- \end{pmatrix}$$

one finds the metric tensor

$$\langle (g^{-1} \mathrm{d}g)^2 \rangle = \frac{(\mathrm{d}X)^2 + \mathrm{d}X_+ \mathrm{d}X_-}{X^2}$$

SL(2,$\mathbb{R}$) is a group manifold with Lorentzian metric tensor.

It has a negative constant curvature.

SL(2,$\mathbb{R}$) can be treated as a space-time.

In fact, SL(2,$\mathbb{R}$) is AdS_3, like $\mathsf{SU}(2) \sim \mathsf{S}^3$

SL(2,$\mathbb{R}$) WZW model

The standard string action in the conformal gauge

$$S_0 = \int \mathrm{d}\tau \int \mathrm{d}\sigma \, \langle \, g^{-1}\bar{\partial} g \, g^{-1} \partial g \, \rangle$$

leads to the equation of motion $\partial \left(g^{-1} \, \bar{\partial} g \right) + \bar{\partial} \left(g^{-1} \, \partial g \right) = 0$.

If one adds the WZ-term, which is the integral of the 2-form F, with $\mathrm{d}F = \frac{1}{3} \langle g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \rangle$, one gets the equations of WZW theory

$$\bar{\partial}(\partial g \, g^{-1}) = 0 = \partial(g^{-1} \, \bar{\partial} g) \ ,$$

One can choose

$$F = \frac{\langle t_0 \ g^{-1} \ \mathrm{d}g \rangle \wedge \langle t_0 \ \mathrm{d}g \ g^{-1} \rangle}{1 + \langle t_0 \ g \ t_0 \ g^{-1} \rangle} \ .$$

SL(2,$\mathbb{R}$) WZW model

The WZW field $\mathrm{g}(z,\bar{z}) = g(z)\,\bar{g}(\bar{z})$.

Monodromy properties $g(z+2\pi) = g(z)\,M$, $\bar{g}(\bar{z}+2\pi) = M\,\bar{g}(\bar{z})$.

Kac-Moody currents $J(z) = \dot{g}(z)\,g^{-1}(z)$, $\bar{J}(\bar{z}) = \bar{g}^{-1}(\bar{z})\,\dot{\bar{g}}(\bar{z})$.

Constraints $\langle J(z)\,J(z)\rangle = 0 = \langle \bar{J}(\bar{z})\,\bar{J}(\bar{z})\rangle$.

Gauge fixing conditions $J^{+}(z) = \rho$, $\bar{J}^{+}(\bar{z}) = \bar{\rho}$.

sl(2,$\mathbb{R}$) algebra

The basis of sl$(2,\mathbb{R})$

$$\mathbf{t} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{t}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{t}_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

The commutators of the basis elements

$$[\mathbf{t}, \mathbf{t}_\pm] = \pm 2\mathbf{t}_\pm \;, \qquad [\mathbf{t}_-, \mathbf{t}_+] = \mathbf{t} \;.$$

Expansion in the basis $\mathbf{a} = a\mathbf{t} + a^+\mathbf{t}_+ + a^-\mathbf{t}_-$.

The inner product in sl$(2,\mathbb{R})$

$$\langle \mathbf{a}\,\mathbf{b} \rangle = \frac{1}{2}\mathrm{Tr}(\mathbf{a}\,\mathbf{b}) = ab - \frac{1}{2}(a^+b^- + a^-b^+).$$

Chiral structure

The structure of the currents

$$J(z) = \begin{pmatrix} f(z) & \rho \\ -\frac{f^2(z)}{\rho} & -f(z) \end{pmatrix}.$$

The structure of the chiral WZW fields

$$g(z) = \begin{pmatrix} \psi(z) & \chi(z) \\ \frac{\dot\psi(z) - f(z)\psi(z)}{\rho} & \frac{\dot\chi(z) - f(z)\chi(z)}{\rho} \end{pmatrix}.$$

Hill equations and the Wronskians

$$\ddot\psi(z) = \dot f(z)\psi(z), \quad \ddot\chi(z) = \dot f(z)\chi(z); \qquad \psi(z)\,\dot\chi(z) - \dot\psi(z)\,\chi(z) = \rho \ ,$$

Particle solution

$$f(z) = v, \qquad \bar{f}(\bar{z}) = \bar{v},$$
$$\psi(z) = \sqrt{\rho}, \qquad \bar{\psi}(\bar{z}) = \sqrt{\bar{\rho}}\,\bar{z},$$
$$\chi(z) = \sqrt{\rho}\,z, \qquad \chi(\bar{z}) = \sqrt{\bar{\rho}}.$$

$$\mathrm{g}(\tau) = \begin{pmatrix} \sqrt{\frac{\rho}{\bar{\rho}}}\,(1 - 2\bar{v}\tau) & 2\sqrt{\rho\,\bar{\rho}}\,\tau \\ \frac{1}{\sqrt{\rho\,\bar{\rho}}}(2v\bar{v}\tau - v - \bar{v}) & \sqrt{\frac{\bar{\rho}}{\rho}}(1 - 2v\tau) \end{pmatrix}.$$

The corresponding monodromy matrix is

$$M = e^{2\pi \mathbf{t}_+} = \begin{pmatrix} 1 & 2\pi \\ 0 & 1 \end{pmatrix}.$$

Chiral symplectic form

The symplectic form of the chiral sector is given by

$$\omega = \int_0^{2\pi} \frac{dz}{2\pi} \langle \dot{\theta}(z) \wedge \theta(z) \rangle + \frac{1}{2\pi} \langle \delta M \, M^{-1} \wedge \theta(0) \rangle \ ,$$

where θ is the Maurer-Cartan form

$$\theta(z) = g^{-1}(z) \delta g(z) \ ,$$

and M is the monodromy matrix.

Let's consider the elliptic monodromy

$$M = \begin{pmatrix} \cos(2\pi\lambda) & \sin(2\pi\lambda) \\ -\sin(2\pi\lambda) & \cos(2\pi\lambda) \end{pmatrix} \ .$$

Hamiltonian reduction

Iwasawa decomposition

$$g(z) = e^{\gamma(z)\mathbf{t}_-} \cdot e^{\phi(z)\mathbf{t}} \cdot e^{\alpha(z)\mathbf{t_0}}$$

Periodicity conditions

$$\gamma(z+2\pi) = \gamma(z) \ , \quad \phi(z+2\pi) = \phi(z) \ , \quad \alpha(z+2\pi) = \alpha(z) + 2\pi\lambda \ .$$

The Kac-Moody current

$$J = \left(\dot{\phi} - \gamma\dot{\alpha}\, e^{2\phi}\right)\mathbf{t} + \dot{\alpha}\, e^{2\phi}\mathbf{t}_+ + \left(\dot{\gamma} + 2\gamma\dot{\phi} - \dot{\alpha}(e^{-2\phi} + \gamma^2\, e^{2\phi})\right)\mathbf{t}_- .$$

Hamiltonian reduction

Constraints

$$\langle\, J^2(z)\,\rangle = 0 \ , \qquad\qquad J^+(z) = \rho$$

$$\dot\phi^2 - \dot\alpha^2 + \dot\alpha\,\dot\gamma\, e^{2\phi} = 0 \ , \quad \dot\alpha\, e^{2\phi} = \rho$$

$$-\rho\,\dot\gamma = \dot\phi^2 - \rho^2\, e^{-4\phi} \ .$$

The potential in the Hill's equation in terms of $\phi(z)$

$$\dot f(z) = \ddot\phi + \dot\phi^2 - \rho^2\, e^{-4\phi}.$$

In terms of the diffeomorphism $\alpha(z) = \lambda\,\zeta(z)$

$$\dot f(z) = -\lambda^2 \dot\zeta^2(z) + \mathsf{S}[\zeta(z)]$$

where $\mathsf{S}[\zeta(z)]$ is the Schwarzian derivative.

Hamiltonian reduction

Reduction of the chiral symplectic form

$$\omega = \int_0^{2\pi} \frac{\mathsf{d}z}{2\pi}\Big(\delta\dot{\phi}\wedge\delta\phi - \delta\dot{\alpha}\wedge\delta\alpha\Big) + \delta\rho\wedge\delta\gamma_0 - \delta\lambda\wedge\delta\alpha(0)\ ,$$

where γ_0 is the zero mode

$$\gamma_0 = \int_0^{2\pi}\frac{\mathsf{d}z}{2\pi}\gamma(z)\ .$$

Canonical structure in zero modes

$$\rho = e^q\ ,\quad p = \rho\gamma_0\ ,$$

$$\omega = \delta\partial\wedge\delta q + \int_0^{2\pi}\frac{\mathsf{d}z}{2\pi}\Big(\delta\dot{\phi}\wedge\delta\phi - \delta\dot{\alpha}\wedge\delta\alpha\Big) - \delta\lambda\wedge\delta\alpha(0)\ .$$

Gervais, Neveu type qunatization of the elliptic sector.

Noether charges

Dynamical integrals are defined by the components of the Kac-Moody current

$$I = \int_0^{2\pi} \frac{\mathrm{d}z}{2\pi} f(z) = p \ , \quad I^+ = e^q \ ,$$

$$I^- = \int_0^{2\pi} J^-(z) \ .$$

The component I^- is expressed through the Virasoro generators L_n.

It takes the form similar to J^- in 3d Minkowski space.

Note that L_n in Liouville theory contain 'improved' terms.

As a result, one finds the quantum realization of the sl(2,$\mathbb{R}$) algebra.

Conclusions

- We have developed Hamiltonian reduction scheme in chiral sectors.
- One can prove complete integrability of $T(z) = 0$ gauged WZW models.
- By AdS$_{N+1}$=SO(2,N)/SO(1,N) one can generalize the scheme.
- Supersymmetry for AdS particle dynamics.
- One can apply it for AdS$_5\times$S^5 superstring with PSU$(2,2|4)$.