AGT correspondence and semiclassical limit of conformal blocks

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Review of Toda field theory

The action of the $sl(n)$ conformal Toda field theory on two-dimensional surface with metric g_{ab} and associated to it scalar curvature R has the form

$$\mathcal{A}=\int\left(\frac{1}{8\pi}g^{ab}(\partial_a\varphi\partial_b\varphi)+\mu\sum_{k=1}^{n-1}e^{b(e_k\varphi)}+\frac{(Q,\varphi)}{4\pi}R\right)\sqrt{g}d^2x \tag{1.1}$$

Here φ is the two-dimensional $(n-1)$ component scalar field $\varphi=(\varphi_1\ldots\varphi_{n-1})$:

$$\varphi=\sum_i^{n-1}\varphi_i e_i \tag{1.2}$$

where vectors e_k are the simple roots of the Lie algebra $sl(n)$, b is the dimensionless coupling constant, μ is the scale parameter called the cosmological constant and (e_k,φ) denotes the scalar product.

Review of Toda field theory

If the background charge Q is related with the parameter b as

$$Q = \left(b + \frac{1}{b}\right)\rho = q\rho \qquad (1.3)$$

where ρ is the Weyl vector , then the theory is conformally invariant. The Weyl vector is

$$\rho = \frac{1}{2}\sum_{e>0} e = \sum_{i}^{n-1} \omega_i \qquad (1.4)$$

where ω_i are fundamental weights, such that $(\omega_i, e_j) = \delta_{ij}$. Conformal Toda field theory possesses higher-spin symmetry: there are $n-1$ holomorphic currents $W^k(z)$ with the spins $k = 2, 3, \ldots n$. The currents $W^k(z)$ form closed W_n algebra, which contains as subalgebra the Virasoro algebra with the central charge

$$c = n - 1 + 12Q^2 = (n-1)(1 + n(n+1)(b + b^{-1})^2) \qquad (1.5)$$

Review of Toda field theory

Primary fields of conformal Toda field theory are the exponential field parameterized by a $(n-1)$ component vector parameter α, $\alpha = \sum_i^{n-1} \alpha_i \omega_i$,

$$V_\alpha = e^{(\alpha,\varphi)} \tag{1.6}$$

They have the simple OPE with the currents $W^k(z)$. Namely,

$$W^k(\xi) V_\alpha(z,\bar{z}) = \frac{w^{(k)}(\alpha) V_\alpha(z,\bar{z})}{(\xi - z)^k} \tag{1.7}$$

The quantum numbers $w^{(k)}(\alpha)$ possess the symmetry under the action of the Weyl group $\mathcal{W}$ of the algebra $sl(n)$:

$$w^{(k)}(\alpha) = w^{(k)}(Q + \hat{s}(\alpha - Q)), \;\; \hat{s} \in \mathcal{W} \tag{1.8}$$

Review of Toda field theory

In particular

$$w^{(2)}(\alpha) = \Delta(\alpha) = \frac{(\alpha, 2Q - \alpha)}{2} \qquad (1.9)$$

is the conformal dimension of the field V_α. Eq. (1.8) implies that the fields related via the action of the Weyl group should coincide up to a multiplicative factor. Indeed we have

$$R_{\hat{s}}(\alpha) V_{Q+\hat{s}(\alpha-Q)} = V_\alpha \qquad (1.10)$$

where $R_{\hat{s}}(\alpha)$ is the reflection amplitude [Fateev, hep-th/0103014].

Light asymptotic limit

Denote $\phi = b\varphi$.

$$S = \frac{1}{8b^2\pi}\int\left[(\partial\phi)^2 + 8\pi\mu b^2\sum_{k=1}^{n-1} e^{(e_k\phi)}\right] d^2x \qquad (1.11)$$

In the light asymptotic limit we again take $b \to 0$. Thus $c \to \infty$. We set $\alpha = b\eta$. In this limit the conformal weight is finite

$$\lim_{b\to 0}\Delta(b\eta) = (\eta, \rho)\,. \qquad (1.12)$$

W-algebra

In components:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(n^3-n)\delta_{n,-m} \qquad (2.1)$$

$$[L_n, W_m] = (2n-m)W_{m+n} \qquad (2.2)$$

$$[W_n, W_m] = \frac{c}{3\cdot 5!}(n^2-1)(n^2-4)n\delta_{n,-m} + \qquad (2.3)$$

$$\frac{16}{22+5c}(n-m)\Lambda_{m+n} + (n-m)\left(\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)\right)L_{n+m}$$

where $\Lambda_n = \sum_{k=-\infty}^{\infty} : L_k L_{n-k} : + \frac{1}{5}x_n L_n$, $x_{2l} = (1+l)(1-l)$ and $x_{2l+1} = (2+l)(1-l)$.

W-algebra

Note that modes annihilating the central extension term: L_0, $L_{\pm 1}$, W_0, $W_{\pm 1}$, $W_{\pm 2}$, form the $sl(3)$ algebra. [P.Bowcock and G.M.T.Watts hep-th/9111062]

The corresponding term for the spin s current $W^{(s)}$ is

$$W^{(s)}(z)W^{(s)}(w) = \frac{c}{(z-w)^{2s}} + \cdots \tag{2.4}$$

In components

$$[W_n^{(s)}, W_m^{(s)}] \sim c(n-(s-1))\cdots(n+(s-1))\delta_{n,-m} + \cdots \tag{2.5}$$

what implies that we have $2s-1$ components with the vanishing central term: $W^{(s)}_{-(s-1)}, \ldots W^{(s)}_{(s-1)}$.

The light limit in $sl(3)$

$$ {}^{L}\mathcal{F}_{r,s}\left[\begin{array}{cc} (r_3,0) & (r_2,0) \\ (r_4,s_4) & r_1,s_1) \end{array}\right](z) = \tag{2.6} $$

$$ \sum_{n,i,j}^{\infty} \frac{z^{2n+i+j}}{n!i!j!(r+s-1)_n} \frac{(\beta)_i(-\beta+r)_n(\gamma)_{n+i}}{(r)_{n+i}} \frac{(\alpha)_j(-\alpha+s)_n(\delta)_{n+j}}{(s)_{n+j}} $$

where

$$ \alpha = \frac{1}{3}(s_3+s_4+s-r_4-r)\,, \quad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{2.7} $$

$$ \beta = \frac{1}{3}(s_1+s_2-s-r_1+r) \tag{2.8} $$

$$ \gamma = \alpha - s_4 + r\,, \qquad \delta = \beta - s_1 + s \tag{2.9} $$

The light limit in $sl(3)$

Computed in [Poghosyan, Poghossian, Sarkissian, arXiv:1602.04829]

$$ {}^{L}\mathcal{F}_{r,s}\left[\begin{array}{cc}(r_3,0) & (0,s_2)\\ (r_4,s_4) & r_1,s_1)\end{array}\right](z) = \tag{2.10} $$

$$ \sum_{k,n,m=0}^{\infty}\sum_{l=0}^{m}\frac{(-1)^{k+l}2^{n-m}z^{2k+m+n}}{k!l!n!(m-l)!(r+s-1)_{k+m}} $$

$$ \frac{(A_1)_m(r-A_2)_l(A_2+s-1)_{k-l+m}}{(r)_l}\frac{(B_1)_{k+n}(B_2)_{k+n}}{(s)_{k-l+n}} $$

where

$$ A_1=\frac{1}{3}(r-r_1-s+s_1+s_2) \quad B_1=\frac{1}{3}(r-r_1+2s-2s_1+s_2) \tag{2.11} $$

$$ A_2=\frac{1}{3}(r+r_3+s_4-s-r_4) \quad B_2=\frac{1}{3}(r+r_3-2s_4+2s-r_4) \tag{2.12} $$

The Nekrasov functions

Consider $\mathcal{N}=2$ SYM theory with gauge group $U(n)$ and $2n$ fundamental (more precisely n fundamental plus n anti-fundamental) hypermultiplets in Ω-background. The instanton part of the partition of this theory can be represented as

$$Z_{inst}=\sum_{\vec{Y}} F_{\vec{Y}} z^{|\vec{Y}|}, \qquad (3.1)$$

where $\vec{Y}$ is an array of n Young diagrams, $|\vec{Y}|$ is the total number of boxes and z is the instanton counting parameter related to the gauge coupling in a standard manner. The coefficients $F_{\vec{Y}}$ are given by

$$F_{\vec{Y}}=\prod_{u=1}^{n}\prod_{v=1}^{n}\frac{Z_{bf}(a_u^{(0)},\varnothing\mid a_v^{(1)},Y_v)Z_{bf}(a_u^{(1)},Y_u\mid a_v^{(2)},\varnothing)}{Z_{bf}(a_u^{(1)},Y_u\mid a_v^{(1)},Y_v)}, \qquad (3.2)$$

where

The Nekrasov functions

$$Z_{bf}(a,\lambda \mid b,\mu) = \prod_{s\in\lambda}\left(a-b-\epsilon_1 L_\mu(s)+\epsilon_2(1+A_\lambda(s))\right) \quad (3.3)$$
$$\prod_{s\in\mu}\left(a-b+\epsilon_1(1+L_\lambda(s))-\epsilon_2 A_\mu(s)\right)$$

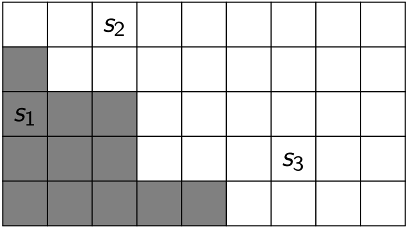

Figure: Arm and leg length with respect to a Young diagram: $A(s_1)=1$, $L(s_1)=2$, $A(s_2)=-2$, $L(s_2)=-3$, $A(s_3)=-2$, $L(s_3)=-4$.

The Nekrasov functions

Here $A_\lambda(s)$ and $L_\lambda(s)$ are correspondingly the arm-length and leg-length of the square s towards the Young tableau λ, defined as oriented vertical and horizontal distances of the square s to outer boundary of the Young tableau λ (see Fig.1).

Let us clarify our conventions on gauge theory parameters $a_u^{(0,1,2)}$, $u = 1, 2, \ldots, n$. The parameters $a_u^{(1)}$ are expectation values of the scalar field in vector multiplet. Without loss of generality we'll assume that the "center of mass" of these expectation values is zero

$$\bar{a}^{(1)} = \frac{1}{n}\sum_{u=1}^{n} a_u^{(1)} = 0\,. \tag{3.4}$$

The Nekrasov functions

In fact this is not a loss of generality since a nonzero center of mass can be absorbed by shifting hypermultiplet masses. Furthermore $a_u^{(0)}$ ($a_u^{(2)}$) are the masses of fundamental (anti-fundamental) hypers. Finally the ϵ_1, ϵ_2 are the Ω-background parameters. Sometimes we will use the notation $\epsilon = \epsilon_1 + \epsilon_2$.

The AGT duality

Due to AGT duality, this partition function is directly related to specific four point conformal block in 2d A_{n-1} Toda field theory [Wyllard, arXiv:0907.2189].

In what follows it would be convenient to represent the roots, weights and Cartan elements of A_{n-1} as n-component vectors with the usual Kronecker scalar product, subject to the condition that sum of components is zero. Of course this is equivalent to more conventional representation of these quantities as diagonal traceless $n \times n$ matrices with the pairing given by trace. In this representation the Weyl vector is given by

$$\rho = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2}\right) \quad \text{or} \quad \rho_u = \frac{n+1}{2} - u \qquad (3.5)$$

The AGT duality

and the highest weight $(\omega_1)_k = \delta_{1,k} - 1/n$.
In what follows a special role is played by the fields $V_{\lambda\omega_1}$ with the dimensions:

$$h_{\lambda\omega_1} = \frac{\lambda(n-1)}{2}\left(q - \frac{\lambda}{n}\right) . \tag{3.6}$$

A four point block:

$$\langle V_{\alpha^{(4)}}(\infty) V_{\lambda^{(3)}\omega_1}(1) V_{\lambda^{(2)}\omega_1}(z) V_{\alpha^{(1)}}(0)\rangle_\alpha = \tag{3.7}$$
$$z^{h_\alpha - h_{\alpha^{(1)}} - h_{\alpha^{(2)}\omega_1}} \mathcal{F}_\alpha \left[\begin{array}{cc} \lambda^{(3)}\omega_1 & \lambda^{(2)}\omega_1 \\ \alpha^{(4)} & \alpha^{(1)} \end{array} \right](z) ,$$

where α specifies the W-family running in s-channel, is closely related to the gauge partition function Z_{inst} see (3.13) (AGT relation). First of all, the instanton counting parameter z gets identified with the cross ratio of insertion points in CFT block.

The AGT duality

and the Toda parameter b is related to Ω-background parameters via

$$b = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\,. \qquad (3.8)$$

The map between the gauge parameters in (3.1) and conformal block parameters in (3.7) should be established from the following rules. To formulate them we define the rescaled gauge parameters

$$A_u^{(0)} = \frac{a_u^{(0)}}{\sqrt{\epsilon_1\epsilon_2}}\,; \qquad A_u^{(1)} = \frac{a_u^{(1)}}{\sqrt{\epsilon_1\epsilon_2}}\,; \qquad A_u^{(2)} = \frac{a_u^{(2)}}{\sqrt{\epsilon_1\epsilon_2}}\,. \qquad (3.9)$$

The AGT duality

- The differences between the "centers of mases" of the successive rescaled gauge parameters (3.9) give the charges of the "vertical" entries of the conformal block:

$$\bar{A}^{(1)} - \bar{A}^{(0)} = \frac{\lambda^{(3)}}{n}\,; \qquad \bar{A}^{(2)} - \bar{A}^{(1)} = \frac{\lambda^{(2)}}{n}\,. \tag{3.10}$$

- The rescaled gauge parameters with the subtracted centers of masses give the momenta of the "horizontal" entries of the conformal block:

$$\begin{aligned} A_u^{(0)} - \bar{A}^{(0)} &= Q_u - \alpha_u^{(4)}\,; \\ A_u^{(1)} - \bar{A}^{(1)} &= Q_u - \alpha_u\,; \\ A_u^{(2)} - \bar{A}^{(2)} &= Q_u - \alpha_u^{(1)}\,. \end{aligned} \tag{3.11}$$

The AGT duality

Using (3.4), (3.5) and (3.9)-(3.11) we obtain the relation between the gauge and conformal parameters:

$$\begin{aligned}
\frac{a_u^{(0)}}{\sqrt{\epsilon_1\epsilon_2}} &= -\alpha_u^{(4)} - \frac{\lambda^{(3)}}{n} + q\left(\frac{n+1}{2} - u\right); \\
\frac{a_u^{(1)}}{\sqrt{\epsilon_1\epsilon_2}} &= -\alpha_u + q\left(\frac{n+1}{2} - u\right); \\
\frac{a_u^{(2)}}{\sqrt{\epsilon_1\epsilon_2}} &= -\alpha_u^{(1)} + \frac{\lambda^{(2)}}{n} + q\left(\frac{n+1}{2} - u\right).
\end{aligned} \quad (3.12)$$

With all these preparations one can write the AGT correspondence between the Nekrasov function defined in (3.1) and the conformal block in (3.7):

The AGT duality

$$Z_{inst} = (1-z)^{\lambda^{(3)}\left(q-\frac{\lambda^{(2)}}{n}\right)} \mathcal{F}_{\alpha}\left[\begin{array}{cc} \lambda^{(3)}\omega_1 & \lambda^{(2)}\omega_1 \\ \alpha^{(4)} & \alpha^{(1)} \end{array}\right](z)\,. \quad (3.13)$$

Light asymptotic limit of the Nekrasov functions

Now we compute [Poghosyan, Poghossian, Sarkissian, arXiv:1602.04829] the light asymptotic limit of the Nekrasov functions. For the horizontal entries we simply put

$$\alpha_u^{(1)} = b\eta_u^{(1)}; \qquad \alpha_u^{(4)} = b\eta_u^{(4)}; \qquad \alpha_u = b\eta_u \quad (3.14)$$

keeping all the parameters η finite. As for the parameters λ of the special fields $V_{\lambda\omega_1}$, there are two inequivalent alternatives:

(i) $\lambda = b\eta$

or

(ii) $nq - \lambda = b\eta$.

Though in both cases the conformal dimension takes the same value (see eq. (3.6))

$$h = \frac{\eta(n-1)}{2} \; , \quad (3.15)$$

Light asymptotic limit of the Nekrasov functions

these fields are not identical, which can be seen e.g. from the fact that the zero mode eigenvalues of odd W-currents for these fields have the same absolute values but opposite signs. It is easy to check that $V_{(nq-b\eta)\omega_1}$ is equivalent to $V_{b\eta\omega_{n-1}}$ (ω_{n-1} is the highest weight of the anti-fundamental representation) since the corresponding momentum parameters $Q-b\eta\omega_1$ and $Q-(nq-b\eta)\omega_{n-1}$ are related by a Weyl reflection.
Here we consider in detail the case when $V_{\lambda_3\omega_1}$ is a light field of type (i) while $V_{\lambda_2\omega_1}$ is of type (ii). In other words we set

$$\lambda^{(3)} = b\eta^{(3)}; \qquad nq - \lambda^{(2)} = b\eta^{(2)}\,. \tag{3.16}$$

Light asymptotic limit of the Nekrasov functions

For such choice we will see below, that the corresponding instanton sum simplifies drastically and leads to a simple explicit expression for the conformal block. Note that this choice is very convenient since the prefactor in front of conformal block in (3.13) now goes to 1 in the light asymptotic limit. The opposite case when two special fields are of the same type, has been investigated in [V. Fateev and S. Ribault, arXiv:1109.6764]. in particular case of A_2 Toda. In the case considered in [V. Fateev and S. Ribault, arXiv:1109.6764]. the above mentioned prefactor survives.

Light asymptotic limit of the Nekrasov functions

Coming back to our case of interest using (3.16), (3.14) we can rewrite the AGT map (3.12) as

$$\begin{aligned} a_u^{(0)} &= -\epsilon_1\left(\eta_u^{(4)}+\frac{\eta^{(3)}}{n}\right)+\epsilon\left(\frac{n+1}{2}-u\right) ; \\ a_u^{(1)} &= -\epsilon_1\eta_u+\epsilon\left(\frac{n+1}{2}-u\right) ; \\ a_u^{(2)} &= -\epsilon_1\left(\eta_u^{(1)}+\frac{\eta^{(2)}}{n}\right)+\epsilon\left(\frac{n+3}{2}-u\right) . \end{aligned} \tag{3.17}$$

In view of (3.8) the small b limit is equivalent to $\epsilon_1 \to 0$. Hence we are interested in the $\epsilon_1 \to 0$ limit of (3.2).

Light asymptotic limit of the Nekrasov functions

We found that total degree of ϵ_1 in $F_{\vec{Y}}$ is

$$N = \sum_{u=1}^{n} Y_{u,u} \,. \tag{3.18}$$

where $Y_{v,i}$ is the number of boxes in the i'th row of diagram Y_v. Each term here is non-negative and in order to get a vanishing total degree $N = 0$, the array of Young diagrams should satisfy the conditions $Y_{1,1} = Y_{2,2} = \cdots Y_{n,n} = 0$, which means that each Young diagram Y_u consists of at most $u - 1$ rows.

Light asymptotic limit of the Nekrasov functions

For $F_{\vec{Y}}$ in the light asymptotic limit we finally get

$$F_{\vec{Y}} = \prod_{u=2}^{n} \prod_{v=2}^{u} \left(\frac{u-v+1}{n-u+v-1} \right)^{Y_{u,v-1}} \tag{3.19}$$

$$\frac{\prod_{i=0}^{Y_{u,u-v+1}-1} \left(-\eta_u + \eta_v^{(4)} + \frac{\eta^{(3)}}{n} - u + v + i \right) \left(\eta_u - \eta_v^{(1)} - \frac{\eta^{(2)}}{n} + u - v - i \right)}{\prod_{k=u-v+1}^{u-1} \prod_{i=Y_{u,k+1}}^{Y_{u,k}-1} (\eta_u - \eta_{v-1} + u - v + Y_{v-1,k+v-u} - i)(\eta_u - \eta_v + u - v + Y_{v,k+v-u} - i)} .$$

where $Y_{v,i}$ is the number of boxes in the i'th row of diagram Y_v. As we have mentioned already, with the prescription (3.16) of the field, in the light asymptotic limit the prefactor in (3.13) becomes 1, and hence, remembering also that the field $V_{(nq-b\eta^{(2)})\omega_1}$ is equivalent to $V_{b\eta^{(2)}\omega_{n-1}}$, we can write

Light asymptotic limit of the Nekrasov functions

$$ {}^L\mathcal{F}_\eta \left[\begin{array}{cc} \eta^{(3)}\omega_1 & \eta^{(2)}\omega_{n-1} \\ \eta^{(4)} & \eta^{(1)} \end{array} \right](z) \equiv \tag{3.20} $$

$$ \lim_{b\to 0}\mathcal{F}_{b\eta} \left[\begin{array}{cc} b\eta^{(3)}\omega_1 & b\eta^{(2)}\omega_{n-1} \\ b\eta^{(4)} & b\eta^{(1)} \end{array} \right](z) = \sum_{\vec{Y}} F_{\vec{Y}} z^{|\vec{Y}|} \; . $$

The sum is taken over all Young diagrams Y_u, $u = 2, \ldots, n$, with at most $u-1$ rows, i.e. over all allowed row lengths $Y_{u,1} \geq Y_{u,2} \geq \cdots \geq Y_{u,u-1} \geq 0$.
Let us consider the particular cases when $n = 2$ (Liouville) and $n = 3$ separately.

Light asymptotic limit of the Nekrasov functions

When $n = 2$ we have a single sum

$$
{}^{L}\mathcal{F}_{\eta}^{Liouv}\left[\begin{array}{cc}\eta^{(3)} & \eta^{(2)}\\ \eta^{(4)} & \eta^{(1)}\end{array}\right](z) = \sum_{i=0}^{\infty}\frac{\left(-\eta^{(4)}+\eta+\frac{\eta^{(3)}}{2}\right)_i\left(-\eta^{(1)}+\eta+\frac{\eta^{(2)}}{2}\right)_i}{i!(2\eta)_i}\; z^i
$$

$$
= {}_2F_1\left(-\eta^{(1)}+\eta+\frac{\eta^{(2)}}{2}, -\eta^{(4)}+\eta+\frac{\eta^{(3)}}{2}, 2\eta;\; z\right), \qquad (3.21)
$$

where ${}_2F_1(a,b;c;x)$ is the Gauss hyper-geometric function. This result was first obtained in [A. Mironov and A. Morozov, arXiv:0909.3531].

Light asymptotic limit of the Nekrasov functions

When $n = 3$ we get

$$ {}^{L}\mathcal{F}_{\eta}^{W_3}\left[\begin{array}{cc} \eta^{(3)}\omega_1 & \eta^{(2)}\omega_2 \\ \eta^{(4)} & \eta^{(1)} \end{array}\right](z) = \sum_{i,j,l=0}^{\infty}(-)^l 2^{j-i}\, z^{2l+i+j} \quad (3.22) $$

$$ \times\left(\frac{\eta^{(3)}}{3}-\eta_2+\eta_2^{(4)}\right)_i\left(\frac{\eta^{(3)}}{3}-\eta_3+\eta_2^{(4)}-1\right)_l\left(\frac{\eta^{(3)}}{3}-\eta_3+\eta_3^{(4)}\right)_{j+l} $$

$$ \times\frac{\left(\frac{\eta^{(2)}}{3}-\eta_2+\eta_2^{(1)}\right)_i\left(\frac{\eta^{(2)}}{3}-\eta_3+\eta_2^{(1)}-1\right)_l\left(\frac{\eta^{(2)}}{3}-\eta_3+\eta_3^{(1)}\right)_{j+l}}{i!j!l!(\eta_1-\eta_2)_i(\eta_1-\eta_3-1)_l(\eta_2-\eta_3)_l(\eta_2-\eta_3-i-1)_l(\eta_2-\eta_3+l-i)_j}\;, $$

Light asymptotic limit of the Nekrasov functions

$$r=\eta_1-\eta_2\,;\qquad s=\eta_2-\eta_3\,;\qquad \eta_1+\eta_2+\eta_3=0 \qquad (3.23)$$

Due to (3.23) and (3.14), (3.16) for our case we have

$$\begin{aligned}
r=\eta_1-\eta_2\,;&\qquad s=\eta_2-\eta_3\,;\\
r_1=\eta_1^{(1)}-\eta_2^{(1)}\,;&\qquad s_1=\eta_2^{(1)}-\eta_3^{(1)}\,;\\
r_4=\eta_1^{(4)}-\eta_2^{(4)}\,;&\qquad s_4=\eta_2^{(4)}-\eta_3^{(4)}\,;\\
s_2=\eta^{(2)}\,;&\qquad r_2=0\,;\\
r_3=\eta^{(3)}\,;&\qquad s_3=0\,.
\end{aligned} \qquad (3.24)$$

Light asymptotic limit of the Nekrasov functions

$$
{}^{L}\mathcal{F}_{r,s}\left[\begin{array}{cc}(r_3,0) & (0,s_2)\\ (r_4,s_4) & r_1,s_1)\end{array}\right](z)=\sum_{k,n,m=0}^{\infty}\frac{(-1)^k 2^{m-n} z^{2k+m+n}}{k!m!n!(r+s-1)_k}\times \quad (3.2
$$

$$
\frac{(A_1)_n(A_2)_n(A_1+s-1)_k(A_2+s-1)_k}{(r)_n}\frac{(B_1)_{k+m}(B_2)_{k+m}}{(s)_k(-n+s-1)_k(k-n+s)_m}
$$