Quantum Group in 2D Lattice Field Theories

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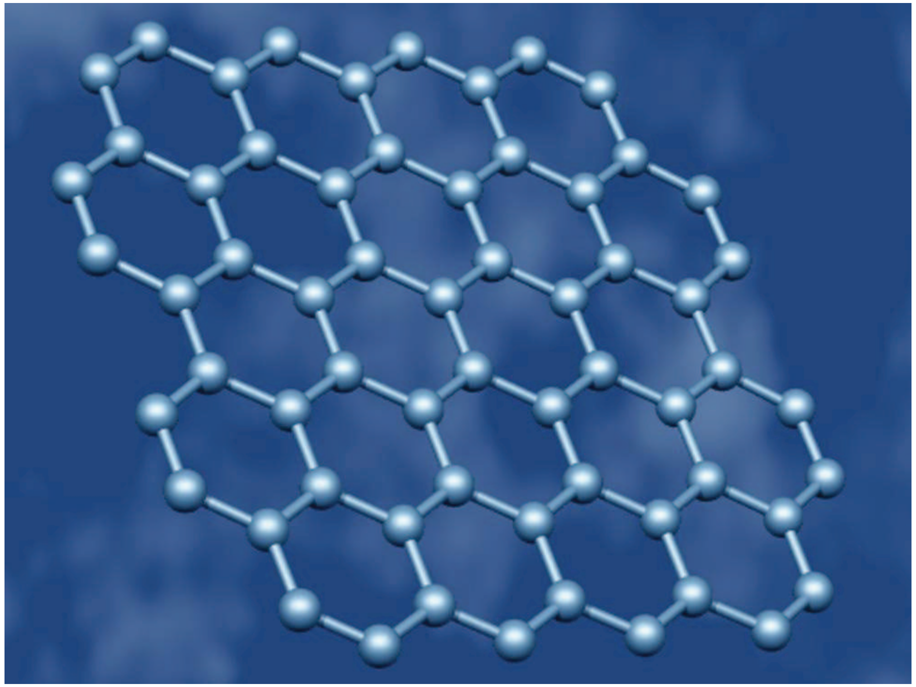
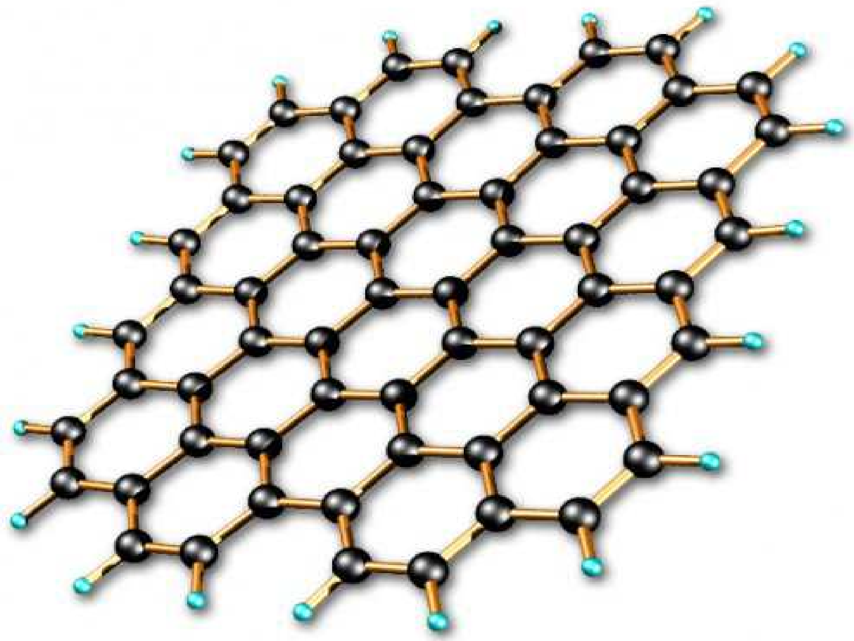


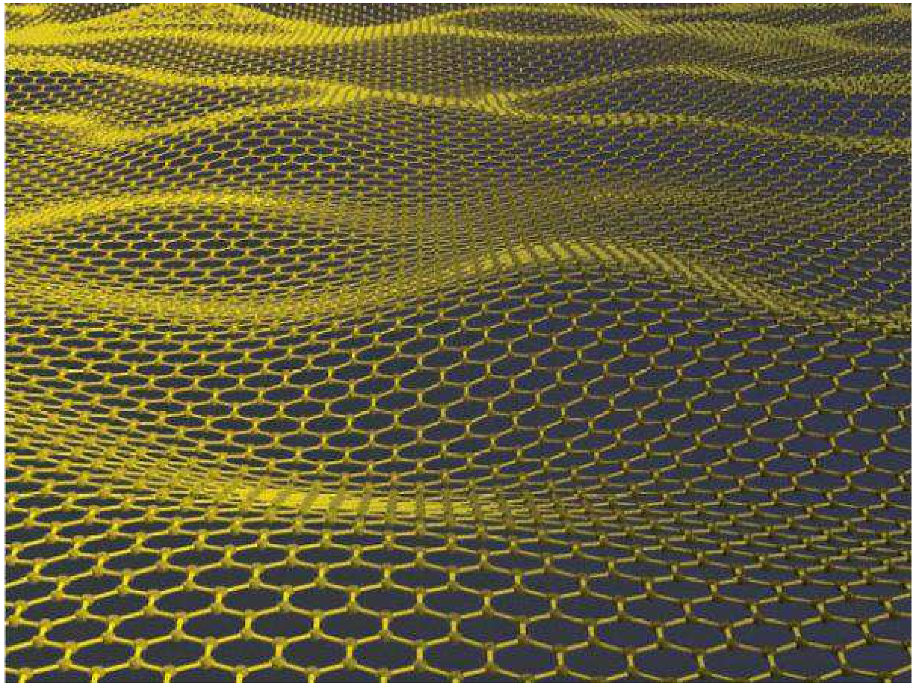
M. Eliashvili, G.I. Japaridze, G. Tsitsishvili, *Journal of Physics* **A45** (2012) 395305

Quantum Group in 2D Lattice Field Theories

Graphene



Ripples in Graphene

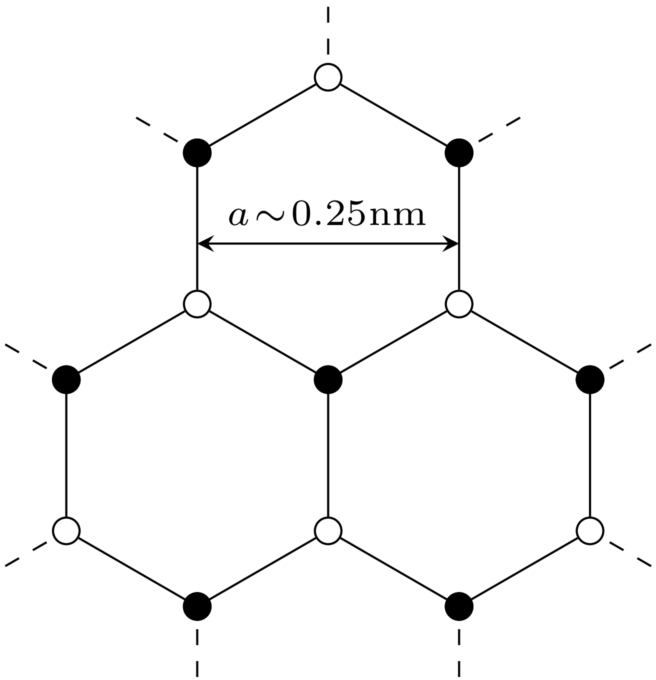


"Ripples arise then due to spontaneous symmetry breaking, following a mechanism similar to that responsible for the condensation of the Higgs field in relativistic field theories"

P. San-Jose, J. González, F. Guinea, PRL 106 (2011)

2D Field Theory on a Honeycomb Lattice

Tight-binding model with the nearest neighbouring hoppings



Lattice site coordinate $\boldsymbol{r}_{n_1 n_2}$, shortly $\boldsymbol{r}_n$

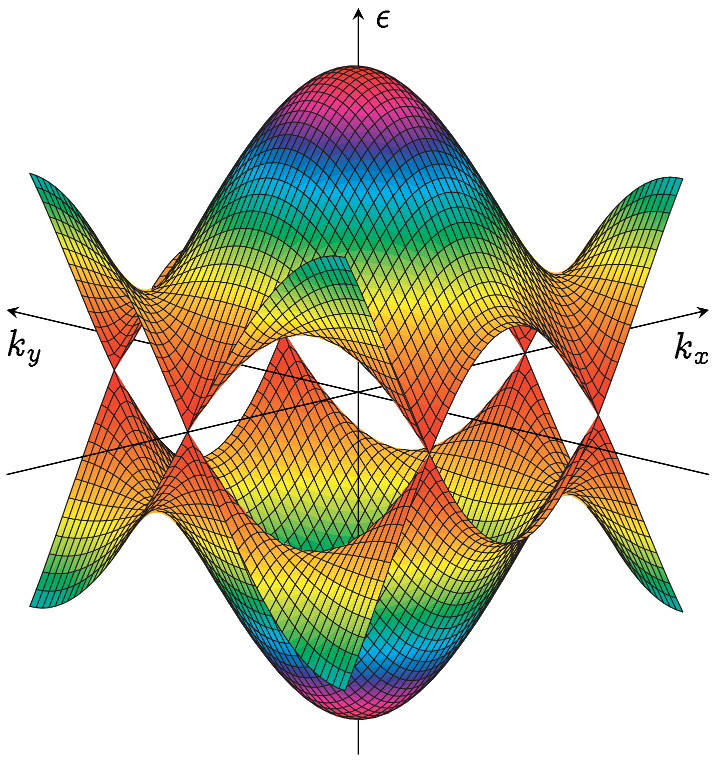
$$H = \sum_{mn} \Big[f_\circ^\dagger(\boldsymbol{r}_n) f_\bullet(\boldsymbol{r}_m) + f_\bullet^\dagger(\boldsymbol{r}_m) f_\circ(\boldsymbol{r}_n) \Big]$$

$$H = \int \epsilon(\boldsymbol{k}) \Big[f_+^\dagger(\boldsymbol{k}) f_+(\boldsymbol{k}) - f_-^\dagger(\boldsymbol{k}) f_-(\boldsymbol{k}) \Big] d\boldsymbol{k}$$

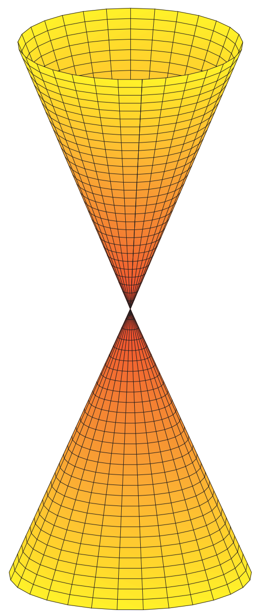
$$\epsilon(\boldsymbol{k}) = \sqrt{1 + 4\cos\left(\tfrac{1}{2}\,k_x a\right)\cos\left(\tfrac{\sqrt{3}}{2}\,k_y a\right) + 4\cos^2\left(\tfrac{1}{2}\,k_x a\right)}$$

Low-Energy Effective Theory

$$\epsilon(\boldsymbol{k}) = \sqrt{1 + 4\cos\left(\tfrac{1}{2}\,k_x a\right)\cos\left(\tfrac{\sqrt{3}}{2}\,k_y a\right) + 4\cos^2\left(\tfrac{1}{2}\,k_x a\right)}$$



Low-energy excitations → Massless Dirac



$$\epsilon_\pm(\Delta\boldsymbol{k}) = \pm|\Delta\boldsymbol{k}|$$

$$L = i\psi\gamma_n\partial_n\psi$$

$$L = \psi\gamma_n(i\partial_n + A_n)\psi$$

Honeycomb Lattice in Magnetic Field

"Peierls Substitution"

$$f_\bullet^\dagger(\boldsymbol{r}_m) f_\circ(\boldsymbol{r}_n) \rightarrow e^{-i\theta(\boldsymbol{r}_m|\boldsymbol{r}_n)} f_\bullet^\dagger(\boldsymbol{r}_m) f_\circ(\boldsymbol{r}_n)$$

$$\theta(\boldsymbol{r}_m|\boldsymbol{r}_n) = \frac{e}{\hbar}\int_{\boldsymbol{r}_n}^{\boldsymbol{r}_m} \boldsymbol{A} d\boldsymbol{l} \qquad B = \mathrm{rot}\,\boldsymbol{A}$$

Gauge Invariance

$$\begin{cases} A_j(\boldsymbol{r}) \rightarrow A_j(\boldsymbol{r}) + \partial_j \lambda(\boldsymbol{r}) \\ f(\boldsymbol{r}_m) \rightarrow e^{-i\lambda(\boldsymbol{r}_m)} f(\boldsymbol{r}_m) \end{cases}$$

$$e^{-i\theta(\boldsymbol{r}_m|\boldsymbol{r}_n)} f_\bullet^\dagger(\boldsymbol{r}_m) f_\circ(\boldsymbol{r}_n) = \mathrm{inv}$$

$$H = \sum_{mn} \left[e^{-i\theta(\boldsymbol{r}_m|\boldsymbol{r}_n)} f_\bullet^\dagger(\boldsymbol{r}_m) f_\circ(\boldsymbol{r}_n) + e^{+i\theta(\boldsymbol{r}_m|\boldsymbol{r}_n)} f_\circ^\dagger(\boldsymbol{r}_n) f_\bullet(\boldsymbol{r}_m) \right]$$

Honeycomb Lattice in Magnetic Field

$$H = \sum_{mn} \left[e^{-i\theta(\boldsymbol{r}_m|\boldsymbol{r}_n)} f^\dagger_\bullet(\boldsymbol{r}_m) f_\circ(\boldsymbol{r}_n) + e^{+i\theta(\boldsymbol{r}_m|\boldsymbol{r}_n)} f^\dagger_\circ(\boldsymbol{r}_n) f_\bullet(\boldsymbol{r}_m) \right]$$

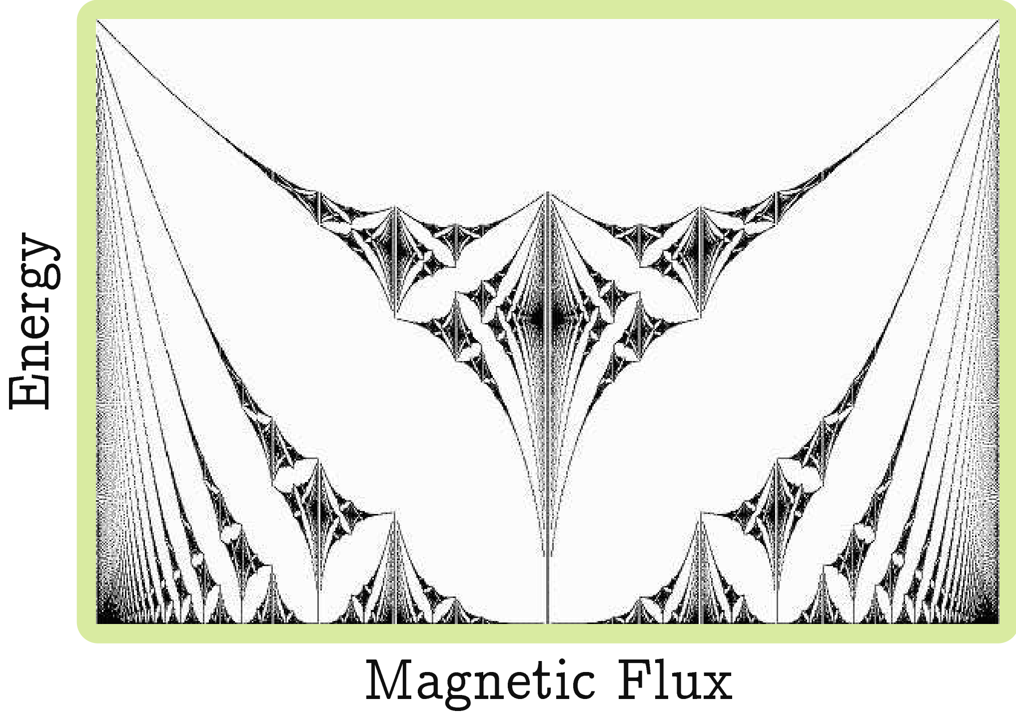
$$\Phi = \frac{\nu}{N} \Phi_0 \qquad \Phi = B \cdot Area_{\hexagon} \qquad \Phi_0 = \frac{2\pi\hbar}{e}$$

System splits into $(2N \times 2N)$-dimensional independent blocks $\mathcal{H}(\boldsymbol{k})$

$$H = \int \Psi^\dagger(\boldsymbol{k})\, \mathcal{H}(\boldsymbol{k})\, \Psi(\boldsymbol{k})\, d\boldsymbol{k}$$

Hofstadter Problem: how the flux $\dfrac{\nu}{N}$ affects the spectrum of $\mathcal{H}(\boldsymbol{k})$?

Hofstadter Butterfly for Square Lattice (Hofstadter, 1976)



Harper Equation

$$e^{+ik_xa}\psi_{n+1} + e^{-ik_xa}\psi_{n-1} + t\cos\left(k_ya - 2\pi\frac{\nu}{N}n\right)\psi_n = E\psi_n$$

Hofstadter Butterfly for Honeycomb Lattice

$$\mathcal{H} = \begin{pmatrix} 0 & 1 + X^- \\ 1 + X^+ & 0 \end{pmatrix}_{2N \times 2N}$$

$$X^+ = e^{-ik_x a} \beta^\dagger Q + e^{-ik_y a} Q \beta$$

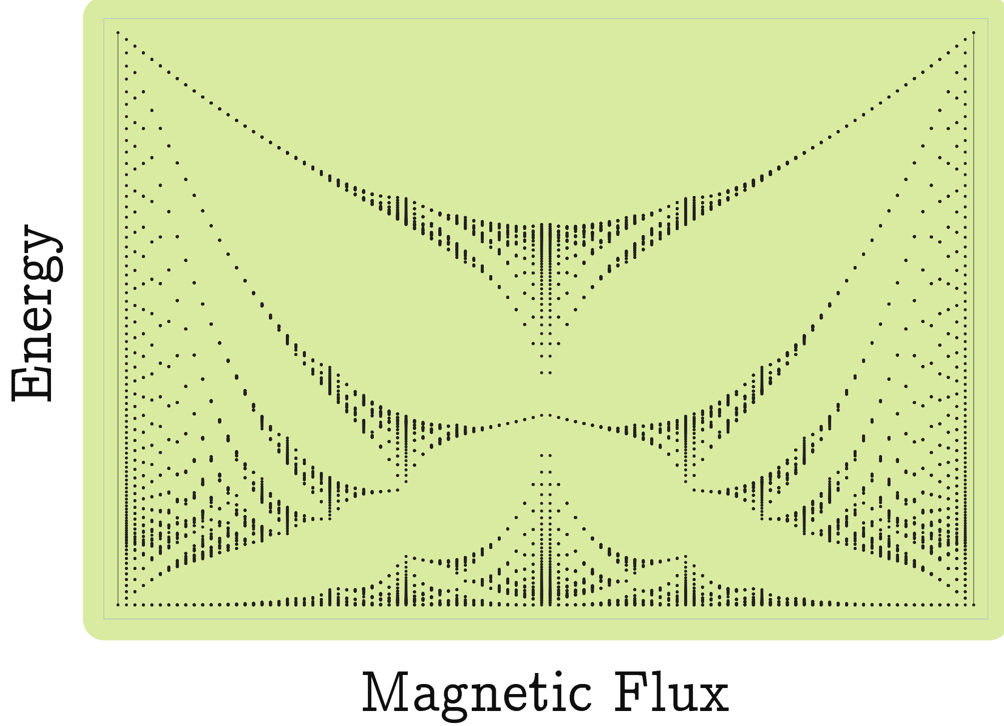
$$Q = \mathrm{diag}(q^1, q^2 \ldots, q^N)$$

$$q = e^{i\pi(\nu/N)}$$

$$\beta = \begin{pmatrix} 0 & \mathbf{1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathbf{1} & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Harper Equation

$$\begin{pmatrix} 0 & 1 + X^- \\ 1 + X^+ & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} = E \begin{pmatrix} \xi \\ \zeta \end{pmatrix}$$



The Quantum Group $U_q(sl_2)$

$$\mathcal{H} = \begin{pmatrix} 0 & 1 + X^- \\ 1 + X^+ & 0 \end{pmatrix}_{2N \times 2N}$$

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The Quantum Group $U_q(sl_2)$

$$[X^+, X^-] = i^2 (q - q^{-1})(K - K^{-1})$$

$$K X^\pm K^{-1} = q^{\pm 2} X^\pm$$

$$K = q e^{+i\boldsymbol{k}(\boldsymbol{a}_1 - \boldsymbol{a}_2)} Q \beta Q^\dagger \beta$$

$U_q(sl_2)$: deformation of usual sl_2

$$q \to 1 \quad \Rightarrow \quad \begin{cases} \dfrac{X^\pm}{i(q - q^{-1})} \to S^\pm \\[2ex] \dfrac{K - K^{-1}}{q - q^{-1}} \to S_z \end{cases}$$

The Quantum Group $U_q(sl_2)$

$$\mathcal{H} = \begin{pmatrix} 0 & 1 + X^- \\ 1 + X^+ & 0 \end{pmatrix}_{2N \times 2N}$$

$$X^+ = e^{-ik_x a}\beta^\dagger Q + e^{-ik_y a} Q\beta$$

$$Q = \mathrm{diag}(q^1, q^2 \dots, q^N)$$

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The Quantum Group $U_q(sl_2)$

$$[X^+, X^-] = i^2(q - q^{-1})(K - K^{-1})$$

$$KX^\pm K^{-1} = q^{\pm 2} X^\pm$$

$$K = q e^{+i\boldsymbol{k}(\boldsymbol{a}_1 - \boldsymbol{a}_2)} Q\beta Q^\dagger \beta$$

Wiegmann & Zabrodin (1994)

Square Lattice $\mathcal{H}_\Box = X^+ + X^-$

Emerging quantum group
allows to employ
the functional representation

Harper Equation in Functional Representation

$$\mathcal{H}\Psi = E\Psi \qquad \mathcal{H} = \begin{pmatrix} 0 & 1+X^- \\ 1+X^+ & 0 \end{pmatrix} \qquad \Psi = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}$$

$$\left.\begin{aligned} (1+X^+)\xi &= E\zeta \\ (1+X^-)\zeta &= E\xi \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \xi(z) + iqz\xi(qz) - iq^{-1}z\xi(q^{-1}z) &= E\zeta(z) \\ \zeta(z) + iz^{-1}\zeta(qz) - iz^{-1}\zeta(q^{-1}z) &= E\xi(z) \end{aligned}\right.$$

Harper Equation in Functional Representation

$$\mathcal{H}\Psi = E\Psi \qquad \mathcal{H} = \begin{pmatrix} 0 & 1+X^- \\ 1+X^+ & 0 \end{pmatrix} \qquad \Psi = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}$$

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Kohmoto & Sedrakyan (2006): $\mathcal{H}^2\Psi = E^2\Psi$

$$(1+X^-)(1+X^+)\xi = E^2\xi$$

$$(1+X^+)(1+X^-)\zeta = E^2\zeta$$

Conformal Invariance

$$\mathcal{H}\Psi = E\Psi \qquad \mathcal{H} = \begin{pmatrix} 0 & 1+X^- \\ 1+X^+ & 0 \end{pmatrix} \qquad \Psi = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}$$

$$\left.\begin{aligned} (1+X^+)\xi &= E\zeta \\ (1+X^-)\zeta &= E\xi \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &\xi(z) + iqz\xi(qz) - iq^{-1}z\xi(q^{-1}z) = E\zeta(z) \\ &\zeta(z) + iz^{-1}\zeta(qz) - iz^{-1}\zeta(q^{-1}z) = E\xi(z) \end{aligned}\right.$$

The system is invariant under special conformal transformation

$$\xi(z) \to z^{N-1}\zeta(-z^{-1})$$

$$\zeta(z) \to z^{N-1}\xi(-z^{-1})$$

Functional Eigenvalue Equation

$$\mathcal{H}\Psi = E\Psi \qquad \mathcal{H} = \begin{pmatrix} 0 & 1+X^- \\ 1+X^+ & 0 \end{pmatrix} \qquad \Psi = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}$$

$$\left.\begin{aligned} (1+X^+)\xi &= E\zeta \\ (1+X^-)\zeta &= E\xi \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &\xi(z) + iqz\xi(qz) - iq^{-1}z\xi(q^{-1}z) = E\zeta(z) \\ &\zeta(z) + iz^{-1}\zeta(qz) - iz^{-1}\zeta(q^{-1}z) = E\xi(z) \end{aligned}\right.$$

$$\zeta(z) + \frac{i}{z}\zeta(qz) - \frac{i}{z}\zeta(q^{-1}z) = E\,z^{N-1}\zeta\Big(-\frac{1}{z}\Big)$$

Quantum Fields and Low Dimensional Physical Systems

(Lecture Course)

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Contents

8. **Basics of planar field theory** (2 hr)
    8.1. CS coupled to matter fields – Anyons
    8.2. Fermions in 2+1 dimensions
    8.3. Discrete symmetries : P,C,T
    8.4. Poincare algebra in 2+1 dimensions
    8.5. Non-Abelian CS theory

9. **Canonical quantization of CS theories** (2 hr)
    9.1. Canonical structure of CS theories
    9.2. CS Quantum Mechanics
    9.3. Canonical quantization of Abelian CS theories
    9.4. CS theories with boundaries

10. **Composite-particle mean-field theory** (2 hr)
    10.1. Berry phase and statistics of quasiparticles
    10.2. Composite boson/fermion theories
    10.3. CS Ginsburg-Landau theory
    10.4. Mean field theory

11. **Non Commutative (NC) geometry** (2 hr)
    11.1. NC coordinates
    11.2. Weyl operators and symbols
    11.3. Magnetic translations
    11.4. NC C-S theory

12. **Relativistic Landau levels** (2 hr)
    12.1. Eigenstates of the 2D Dirac Hamiltonian
    12.2. Symmetries and Lorentz transformations

13. **Integral Quantum Hall Effect** (2 hr)
    13.1. Laughlin arguments
    13.2. Fermi level and currents
    13.3. Edge currents vs. bulk currents
    13.4. Ground state wave function

14. **Fractional Quantum Hall Effect** (2 hr)
    14.1. Limit of strong magnetis field
    14.2. Hamiltonian and ground state
    14.3. Laughlin wave function

15. **Fractional Quantization** (2 hr)
    15.1. Quasiparticles and collective excitations
    15.2. Fractional quantization and fractional charge

References

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2. J.Jain; *Composite Fermions* (Cambridge University Press; 2007)

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5. D.Dunne; *Aspect of Chern-Simons Theory* (arXiv: hep-th/9902115v1)
6. A.Lerda; *Anyons* (Springer; 1992)
7. E.Fradkin; *Field Theories of Condensed Matter Systems,*(Addison-Wesley Publ. Company; 1991)

Problems and exercises will be proposed.