Lattice QCD and effective field theories

Akaki Rusetsky, University of Bonn

Tbilisi State University, 14 March 2013

LQCD & EFT @ UB: factsheet

- Senior investigators: U.-G. Meißner, A. Rusetsky
- Postdocs: M. Döring, M. Mai, G. Rios
- Ph.D. Students: A. Agadjanov, D. Agadjanov
- External collaborators: V. Bernard (Orsay), C. Liu (Peking), A. Khelashvili (Tbilisi), T. Nadareishvili (Tbilisi), E. Oset (Valencia), G. Schierholz (DESY), . . .
- 3 Ph.D. theses were defended in 2012
- Grants:
  - CRC 16 (Bonn – Bochum – Gießen)
  - CRC 110 (Bonn – TU Munich – IHEP Beijing – Peking U.)
  - Rustaveli DI/13/02 (Bonn – Tbilisi)
- Related research at UB
  - C. Urbach (lattice QCD)
  - H.-W. Hammer, U.-G. Meißner (nuclear physics on the lattice)

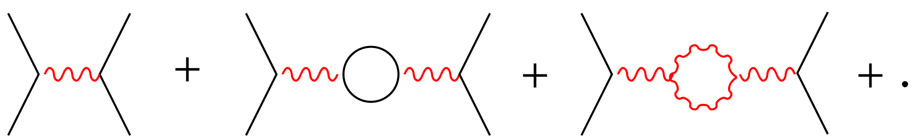
Plan of the presentation

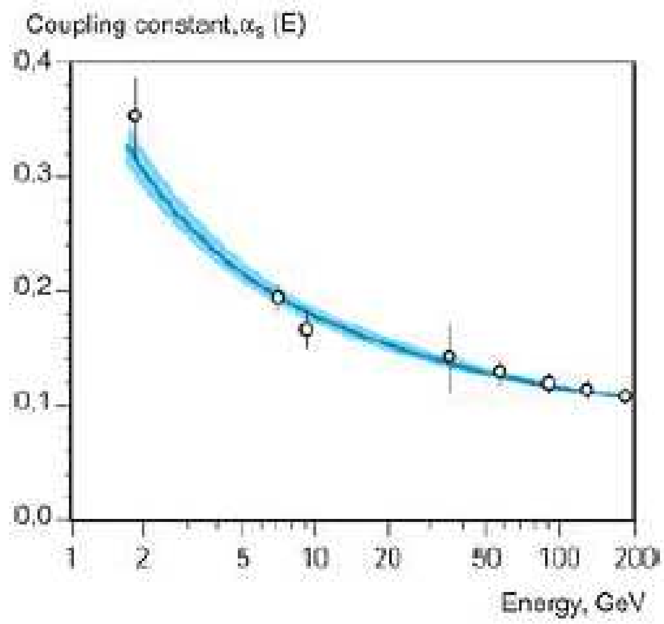
- Introduction: Resonances in lattice QCD
- Effective field theories in a finite volume
- Example 1: scalar resonances and quest for exotica
- Example 2: electromagnetic formfactor
- Example 3: 3-body problem in a finite volume
- Outlook

Confinement

$$\mathcal{L}_{QCD} = \bar{\psi}(i\gamma^\mu(\partial_\mu - ig\,T^a\,A^a_\mu) - \mathcal{M})\psi - \frac{1}{4}\,F^a_{\mu\nu}F^{\mu\nu a}$$

$$\psi = \begin{pmatrix} u \\ d \\ \dots \end{pmatrix}, \qquad \mathcal{M} = \text{diag}\,(m_u, m_d, \cdots)$$





$$\mu\frac{d\alpha_s}{d\mu} = -\frac{\alpha_s}{2\pi}\,(\beta_0\alpha_s + \cdots)\,, \qquad \alpha_s = \frac{g^2}{4\pi}$$

$$\beta_0 = \frac{1}{3}(33 - 2N_f)$$

$$\alpha_s(E) = \frac{12\pi}{(33 - 2N_f)\ln(E^2/\Lambda^2)}$$

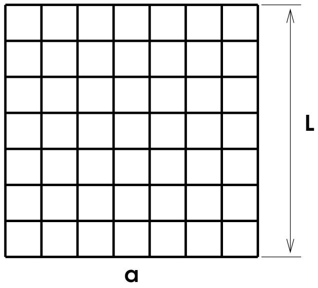
$\hookrightarrow$ Λ is the RG-invariant QCD scale

First-principle calculations in QCD

- In QCD, all physical information is encoded in the Green functions
- Path-integral representation (Euclidean space):

$$\langle 0|\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)|0\rangle = \frac{1}{Z}\int d\psi d\bar\psi dA_\mu\,\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)e^{-S_{QCD}}$$

The gauge-invariant operators $\mathcal{O}_i(x_i)$ are built of the fields $\psi,\bar\psi,A_\mu$



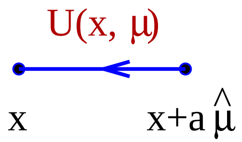
$$L=Na,\quad L_t=N_ta,\quad V_E=L_tL^3$$

- On a finite Euclidean *lattice,* the path integral transforms into a multiple integral

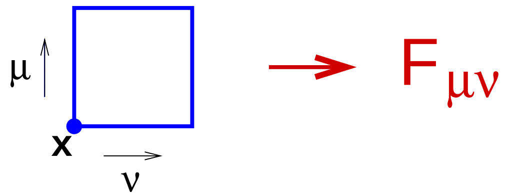
$\hookrightarrow$ can be evaluated by using Monte-Carlo technique

Discretization of the continuum theory

- Derivative: $\partial_\mu \psi(x) = \frac{1}{a}\left(\psi(x+a\hat{\mu}) - \psi(x)\right)$
- Link variable: $U(x,\mu) = exp(-iaA_\mu^a(x)T^a)$



- Covariant derivative: $\nabla_\mu \psi(x) = \frac{1}{a}\left(U(x,\mu)\psi(x+a\hat{\mu}) - \psi(x)\right)$
- Plaquette: $P_{\mu\nu}(x) = U(x,mu)U(x+a\hat{\mu},\nu)U^\dagger(x+a\hat{\nu},\mu)U^\dagger(x,\nu)$



- Wilson action: explicitly gauge-invariant for $a \neq 0$

$$
\begin{aligned}
\mathcal{S}_W &= -a^4 \sum_x \left(\frac{1}{2}(\bar{\psi}\gamma_\mu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma_\mu\psi) + \bar{\psi}\mathcal{M}\psi + \frac{a}{2}\nabla_\mu\bar{\psi}\nabla_\mu\psi \right) \\
&+ \frac{1}{g^2}\sum_{x\mu\nu} \text{Tr}\,(1 - P_{\mu\nu})
\end{aligned}
$$

Spectrum of QCD at low energy

Only colorless asymptotic states (confinement!)

- Mesons $q\bar{q}$
- Baryons qqq
- Exotica:
  - Tetraquarks $qq\bar{q}\bar{q}$?
  - Pentaquarks $qqqq\bar{q}$?
  - Glueballs gg, ggg, $\cdots$?
  - Hybrids $q\bar{q}g$?
  - etc, etc, etc

Most of the observed particles are resonances!

Calculation of the masses of stable particles

- Field operators, carrying given quantum numbers:

$$\Phi(t) = \sum_x \Phi(\mathbf{x}, t)$$

- Correlator: $G(t) = \langle 0|\Phi(t)\Phi^\dagger(0)|0\rangle \to$ calculate using MC
- For large values of t:

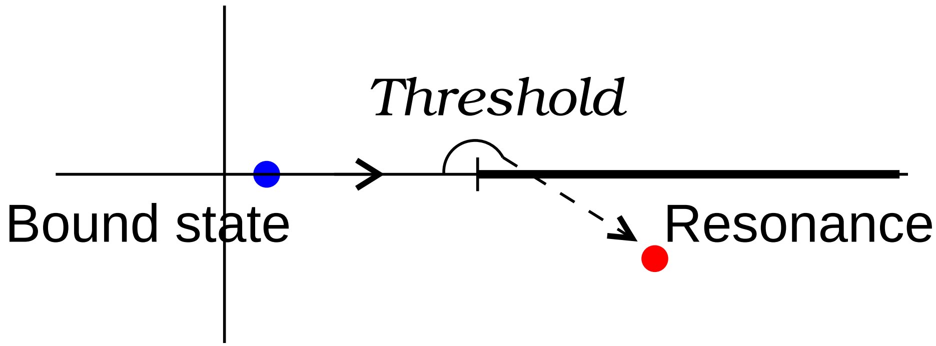
$$G(t) = \sum_n \langle 0|e^{Ht}\Phi(0)e^{-Ht}|n\rangle\langle n|\Phi^\dagger(0)|0\rangle \to e^{-Mt}|\langle 0|\Phi(0)|1\rangle|^2$$

- The mass of a *stable particle*, which corresponds to the lowest energy level, is extracted directly in *Euclidean space*
- Extraction of the excited level spectrum is also possible
- However, *resonance* masses can not be extracted this way!

Resonances in Quantum Field Theory

- Resonances are characterized by their mass, their lifetime, . . .
- These are the *intrinsic* properties of a resonance that should not depend neither on a particular *experiment* nor a particular *theoretical model* which is used to describe the data

  $\hookrightarrow$ Resonances correspond to S-matrix poles on the unphysical Riemann sheets



- Resonances do not show up in the spectrum of the Hamiltonian. Standard procedure on the lattice is not applicable!

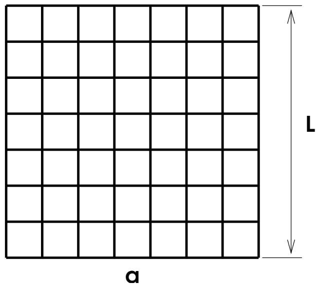
Lüscher's approach

M. Lüscher, lectures given at Les Houches (1988); NPB 364 (1991) 237, $\cdots$

- Lattice simulations are always done at a finite box size L

  It is assumed: $R^{-1}L \simeq ML \gg 1\,.$

  R: the range of interaction



- Momenta are small: $p \simeq 2\pi/L \ll$ the lightest mass
- Finite-volume corrections to the energy levels are only power-suppressed in L
- Studying the dependence of the energy levels on L gives the scattering phase in the infinite volume $\Rightarrow$ Resonances

Non-relativistic effective field theories (NREFT) can be used to study the energy spectrum in a box

Very low energies: non-relativistic EFT

- Very low energies: $p \ll M$, $\sqrt{M^2+p^2} = M + \frac{p^2}{2M} + \cdots$
- Explicit anti-particle degrees of freedom can be omitted

$$\mathcal{L}_{NR} = \Phi^\dagger \left(i\partial_t - M + \frac{\nabla^2}{2M} + \cdots \right) \Phi + d_0 \, \Phi^\dagger \Phi^\dagger \Phi \Phi + \text{derivatives}$$

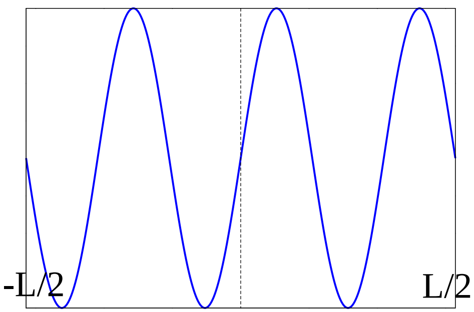
$\Rightarrow$ Lippmann-Schwinger equation: $T = V + VG_0T$

Finite volume

Periodic boundary conditions:

$$\varphi(\mathbf{x} + L\mathbf{e}_i, t) = \varphi(\mathbf{x}, t)$$

$$\mathbf{k} = \frac{2\pi}{L}\,\mathbf{n}, \quad \mathbf{n} \in \mathbb{Z}^3$$



The same Feynman rules, but $\int \frac{d^4k}{(2\pi)^4} \to \int \frac{dk_4}{2\pi} \frac{1}{L^3} \sum_{\mathbf{k}}$

The Lüscher equation (NPB 364 (1991) 237)

Lippmann-Schwinger equation, infinite volume, NREFT (dim.reg.):

$$T = V + VG_0T\,, \qquad G_0 = \frac{ip}{8\pi E}\,, \qquad V(p,p) = \frac{8\pi E}{p}\,\tan\delta(p)$$

Finite volume: Loops modified, Lüscher's zeta-function emerges

$$\int\frac{d^3\mathbf{p}}{(2\pi)^3} \to \frac{1}{L^3}\sum_{\mathbf{p}}\,, \qquad \text{loop} = \frac{ip}{8\pi E} \to \frac{Z_{00}(1;qr)}{4\pi^{3/2}LE} \qquad q = \frac{pL}{2\pi}$$

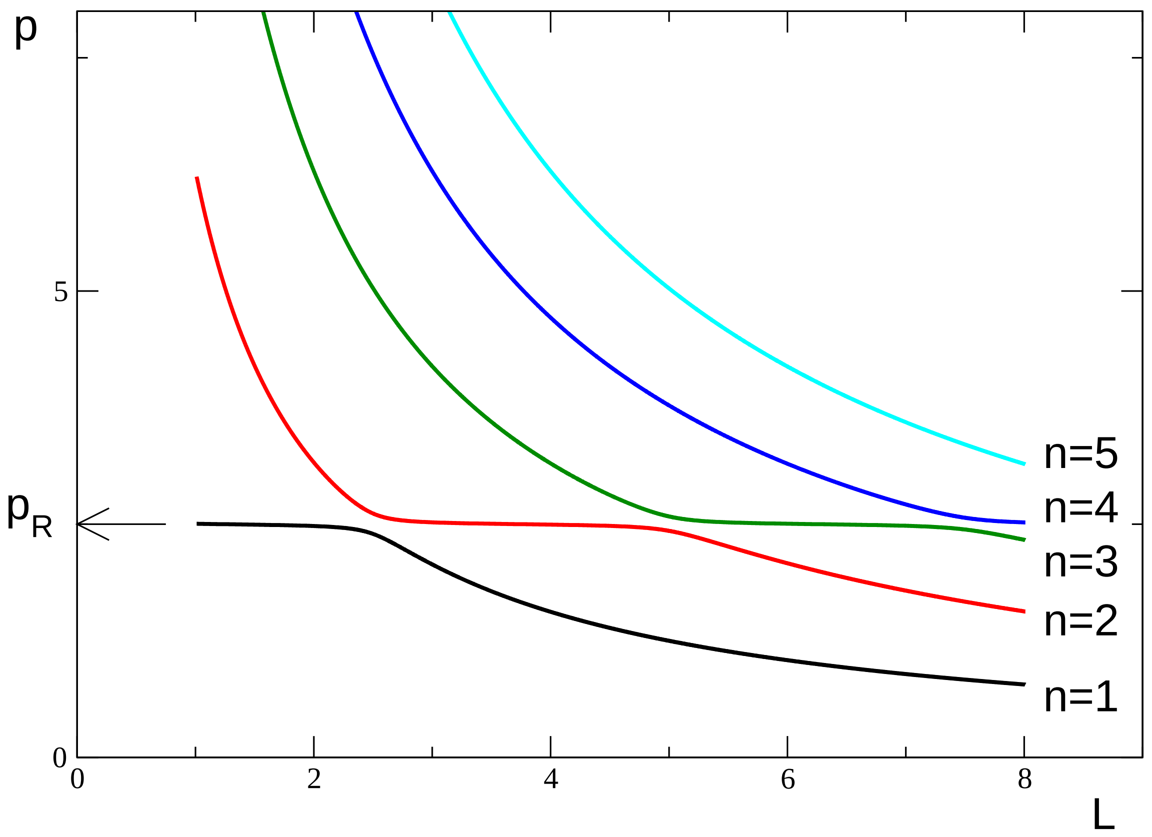
Poles in the LS equation = spectrum of the Hamiltionan

$$\hookrightarrow \text{det}\,(\delta_{ll'}\delta_{mm'} - \tan\delta_l(s)\mathcal{M}_{lm,l'm'}) = 0$$

- $\mathcal{M}_{lm,l'm'}$ is a linear combination of $Z_{lm}(1;q^2)$
- scattering phases $\delta_l(p)$ from the finite-volume energy spectrum
- Beware: partial-wave mixing occurs in a finite volume!

Energy levels in the presence of a narrow resonance

⇒ Lüscher formula predicts an irregular behavior of the levels in the vicinity of a narrow resonance: *avoided level crossing*



Resonances are not described by a single energy level!

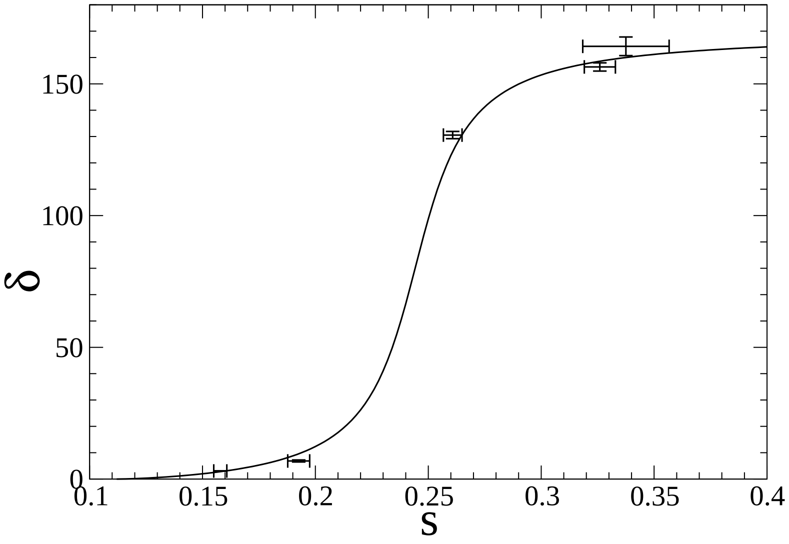
Where are the resonance poles?

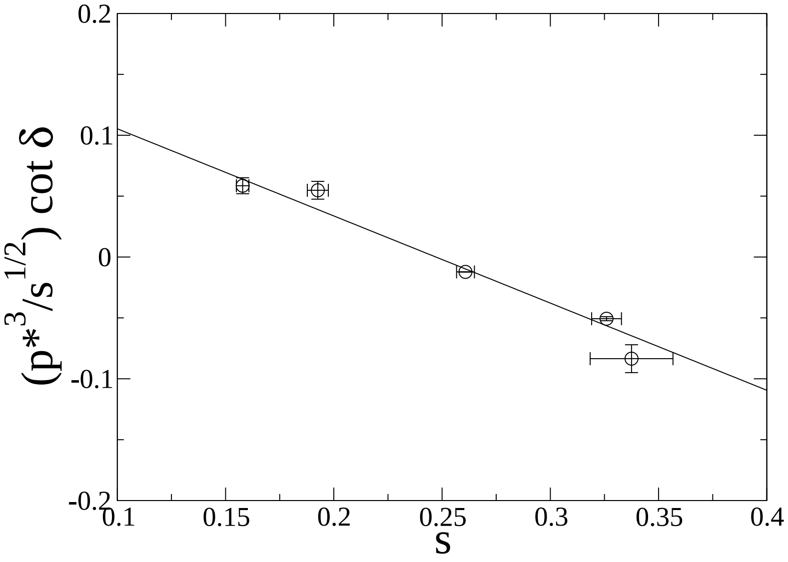
Assume effective range expansion:

$$p \cot \delta(p) = A_0 + A_1 p^2 + \cdots , \qquad \cot \delta(p_R) = -i \quad \surd$$

$\Rightarrow$ $A_0, A_1, \cdots$ are measured on the lattice

$\Rightarrow$ Resonance pole p_R in the complex momentum plane





Phase shift for the ρ-meson: S. Prelovsek *et al.,* arXiv:1111.0409

Ex. 1: scalar mesons

V. Bernard, M. Lage, U.-G. Meißner and AR, JHEP 1101 (2011) 019

$f_0(980)$: Two-channel equation with $\boxed{1} = K\bar{K}$, $\boxed{2} = \pi\pi$

$$\begin{aligned} T_{11} &= V_{11} + V_{11} ip_1 T_{11} + V_{12} ip_2 T_{21} \\ T_{21} &= V_{21} + V_{21} ip_1 T_{11} + V_{22} ip_2 T_{21} \end{aligned}$$

Resonance pole(s) are determined from the secular equation:

$$1 - ip_1 V_{11} - ip_2 V_{22} - p_1 p_2 (V_{11} V_{22} - V_{12}^2) = 0$$

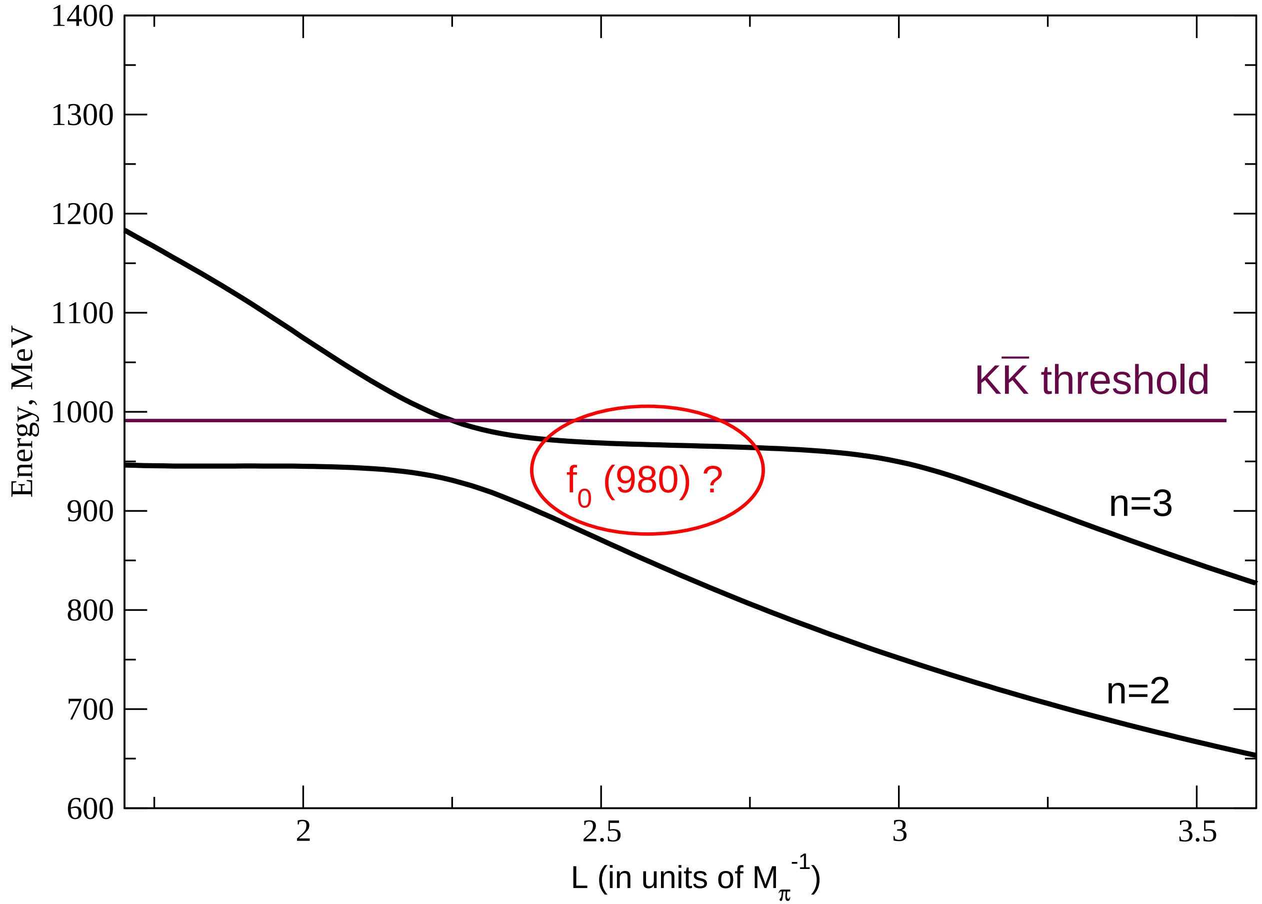
Finite volume: multi-channel Lüscher equation: $ip_i \to \frac{2}{\sqrt{\pi} L} Z_{00}(1, q_i^2)$

$\Rightarrow$ Find $V_{ij}(s)$ from the lattice data: one equation for V_{11}, V_{12}, V_{22}

$\Rightarrow$ Find the position of the pole(s)

$\Rightarrow$ Nature of a resonance: molecule or quark compound?

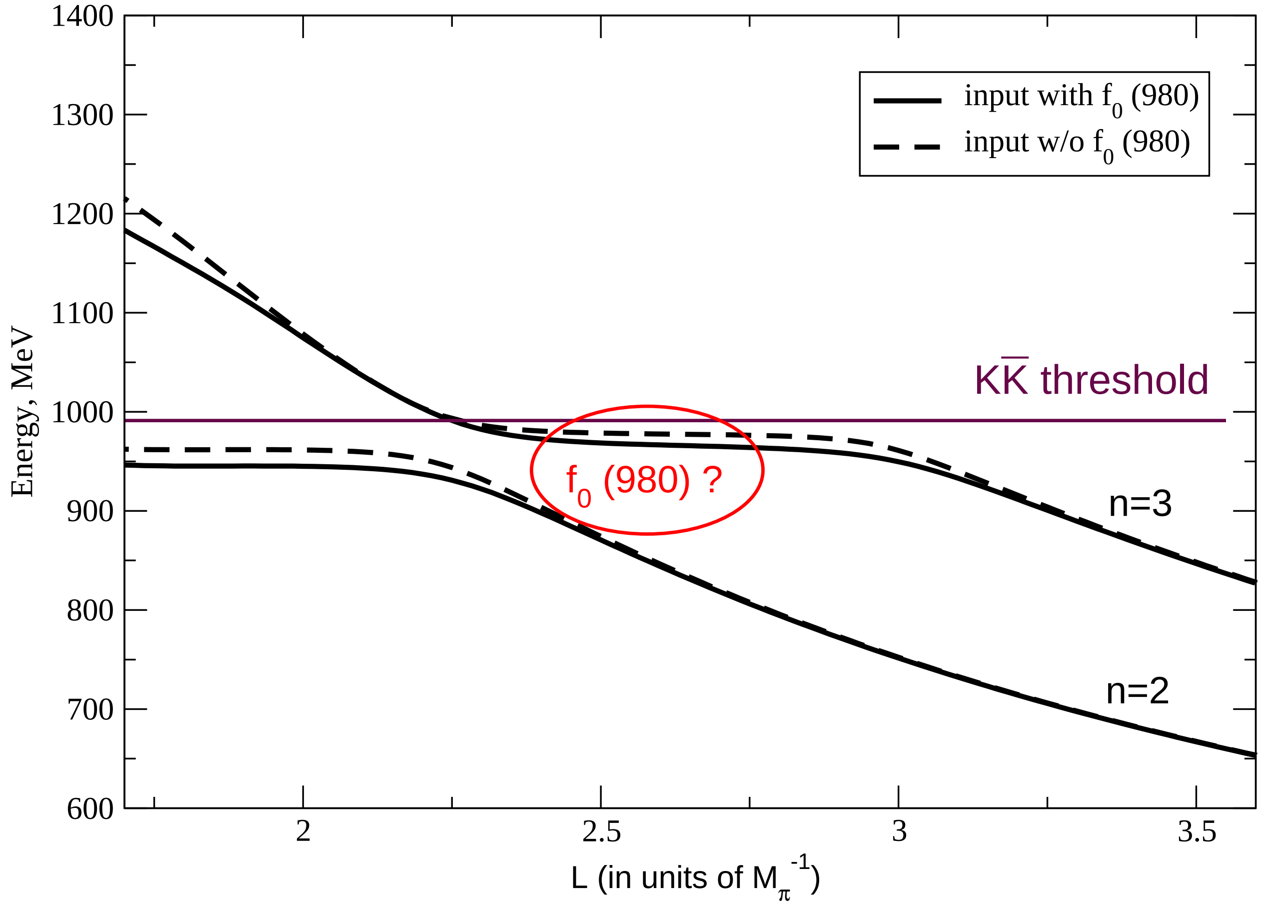
Molecular states: resonance or threshold?



Using Unitarized Chiral Perturbation Theory to produce energy levels

M. Döring, U.-G. Meißner, E. Oset and AR, EPJA 47 (2011) 139

Changing the input . . .



- Weaker coupling to the $K\bar{K}$ channel, $f_0(980)$ disappears
- Avoided level crossing still occurs at the same place

Unitary ChPT in a finite volume

M. Döring, U.-G. Meißner, E. Oset and AR, EPJA 47 (2011) 139, arXiv:1205.4838

- Use dynamical input from unitary ChPT at lowest order: approximate potentials by the polynomials in s:

$$V(s) = V_0 + V_1(s - s_t) + \cdots$$

- Fit the parameters of the potential to the lattice data
- Predict the position of the poles by using scattering equations
- Useful: twisted boundary conditions $\Phi(x + L) = e^{i\theta}\Phi(x)$

Some further developments:

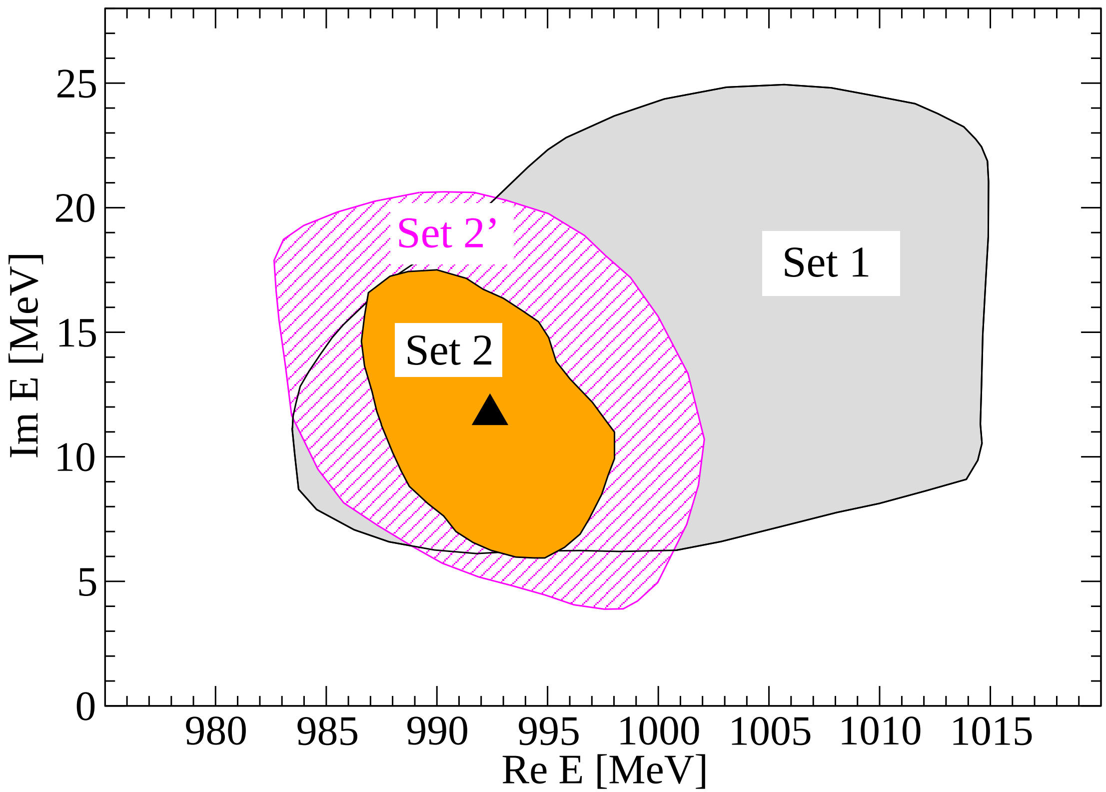
M. Döring, J. Haidenbauer, U.-G. Meißner and AR, EPJA 47 (2011) 163

M. Döring and U.-G. Meißner, JHEP 1201 (2012) 009

A. M. Torres, L. R. Dai, C. Koren, D. Jido and E. Oset, PRD 85 (2012) 014027

M. Albaladejo, J.A. Oller, E. Oset, G. Rios, L. Roca, arXiv:1205.3582

Extraction of the $f_0(980)$ pole position, CM frame



13 (synthetic) data points in each set:

Set 1: Energy levels 2+3, periodic b.c.

Set 2: Energy level 2, periodic + antiperiodic b.c.

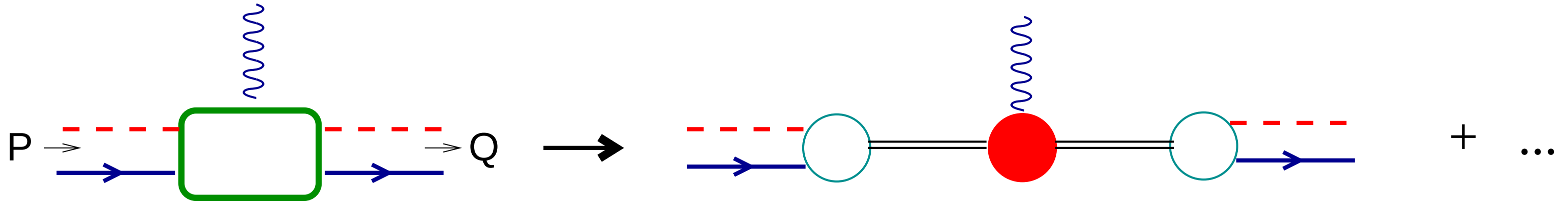
Set 2': Set 2 + statistical error

Ex. 2: Matrix elements with the resonances

D. Hoja, U.-G. Meißner and AR, JHEP 1004 (2010) 050

V. Bernard, D. Hoja, U.-G. Meißner and AR, JHEP 1209 (2012) 023

A consistent definition of a formfactor of an unstable particle in QFT



$P^2, Q^2 \to s_R$

Example: electromagnetic formfactor of the Δ-resonance:

- Gauge independent
- Invariant under field redefinitions

Note: Definitions which do not imply analytic continuation, do not have the above properties

How does one perform analytic continuation of the lattice data?

Resonance matrix elements on the lattice

Field operators with resonance quantum numbers:

$$O_{\mathbf{P}}(t) = \sum_{\mathbf{x}} e^{-i\mathbf{P}\mathbf{x}} O(\mathbf{x}, t)\,,$$

Three- and two-point functions on the lattice:

$$\tilde{V}_\mu(\mathbf{P}, t'; \mathbf{Q}, t) = \langle 0|T O_{\mathbf{P}}(t') J_\mu(0) O^\dagger_{\mathbf{Q}}(t)|0\rangle\,,$$

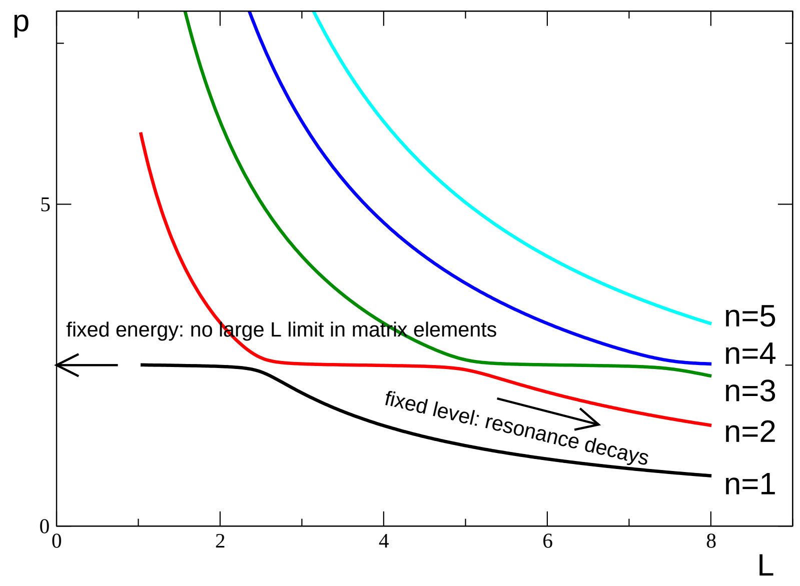
$$D(\mathbf{P}, t) = \langle 0|T O_{\mathbf{P}}(t) O^\dagger_{\mathbf{P}}(0)|0\rangle\,.$$

Extraction of the formfactor (ground-state):

$$\langle \mathbf{P}|J_\mu(0)|\mathbf{Q}\rangle_0 = \lim_{\substack{t'\to\infty \\ t\to-\infty}} \tilde{V}_\mu(\mathbf{P}, t'; \mathbf{Q}, t) \sqrt{\frac{D(\mathbf{Q}, t')D(\mathbf{P}, t)}{D(\mathbf{Q}, t)D(\mathbf{Q}, t'-t)D(\mathbf{P}, t-t')D(\mathbf{P}, t')}}$$

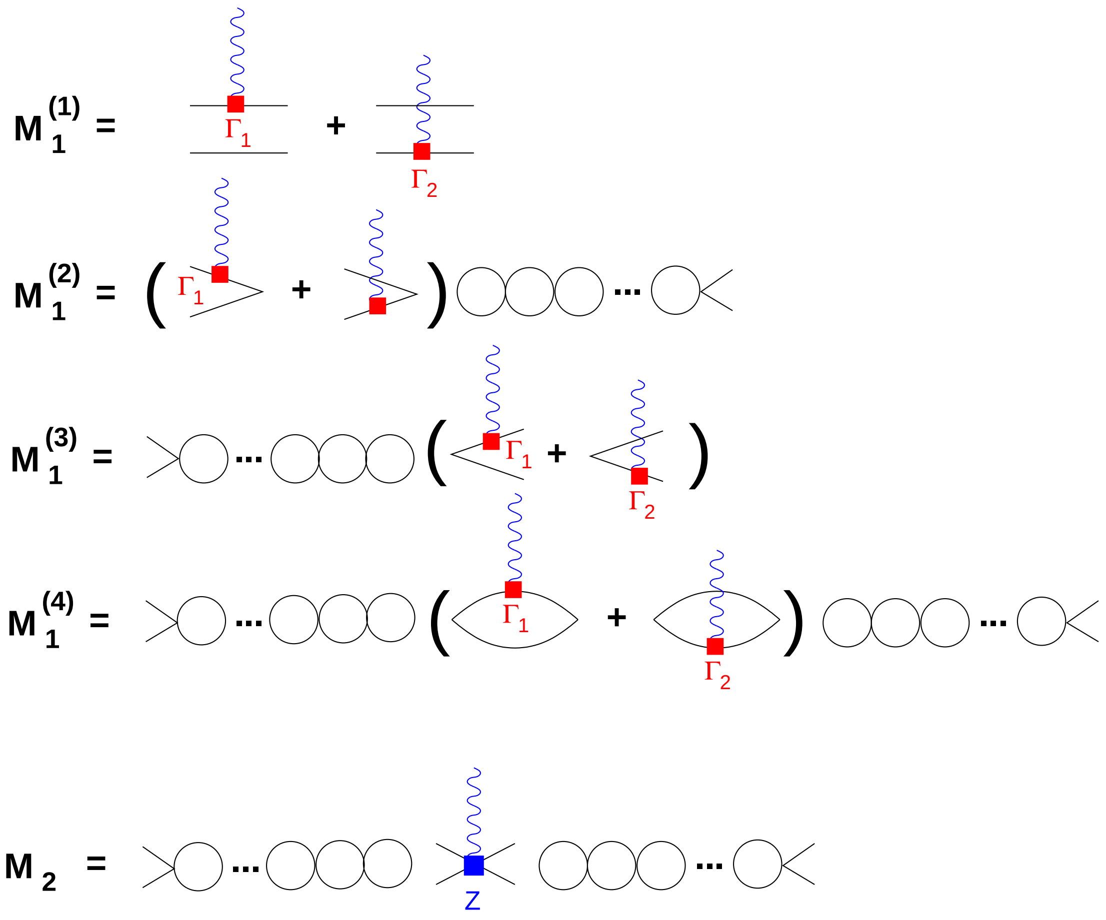
Infinite-volume limit of the matrix elements

- For stable particles, the limit $L \to \infty$ exists
- Both methods give the matrix element sandwiched by the eigenvectors of the Hamiltonian. The resonances, however, do not correspond to a single energy level. How does one calculate the infinite-volume limit for these matrix elements?



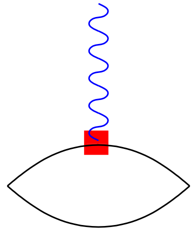
- Fixed energy levels decay in the limit $L \to \infty$
- The matrix elements at fixed energy oscillate in the limit $L \to \infty$

Framework: non-relativistic EFT with the external fields



- Use NREFT in a finite volume to calculate the matrix element
- Extract the matrix element in the infinite volume

Loop graph: analytic continuation (rest system)



$$= \underbrace{\frac{m^2-p^2}{8\pi E^3 p^2}\, p\cot\delta(p)}_{(\text{polynomial in } p^2)/p^2} + \underbrace{\frac{1}{32\pi E p}\,(1+\cot^2\delta(p))\eta\phi'(\eta)}_{\text{culprit}}$$

$$p = p_n = \sqrt{\frac{E_n^2}{4} - m^2}\,, \qquad \tan\phi(\eta) = \frac{\pi^{3/2}\eta}{Z_{00}(1;\eta^2)}\,, \qquad \eta = \frac{pL}{2\pi}$$

- A polynomial in p^2, can be analytically continued $p^2 \to p_R^2$
- An analytic continuation of $\eta\phi'(\eta)$ is ambiguous

EFT framework for extracting resonance formfactors

- Measure the quantities $\langle \mathbf{P}|J_\mu(0)|-\mathbf{P}\rangle_n$ on the lattice, Breit frame

- $V_{nn}(p) = \underbrace{\frac{\delta'(p)+\phi'(q)}{4\sin^2\delta(p)}\,\frac{L^3E_n}{2\pi\sqrt{E_n^2-\mathbf{P}^2}}}_{\text{Lüscher-Lellouch factor}}\,\langle \mathbf{P}|J_\mu(0)|-\mathbf{P}\rangle_n$

$\hookrightarrow$ Form the linear combination: $b_{n,m}(p,\mathbf{P})$ are universal factors

$$\bar V(p) = \frac{b_m(p,\mathbf{P})V_{nn}(p) - b_n(p,\mathbf{P})V_{mm}(p)}{b_n(p,\mathbf{P}) - b_m(p,\mathbf{P})}$$

$\hookrightarrow$ Effective-range expansion for $\bar V(p)$ holds

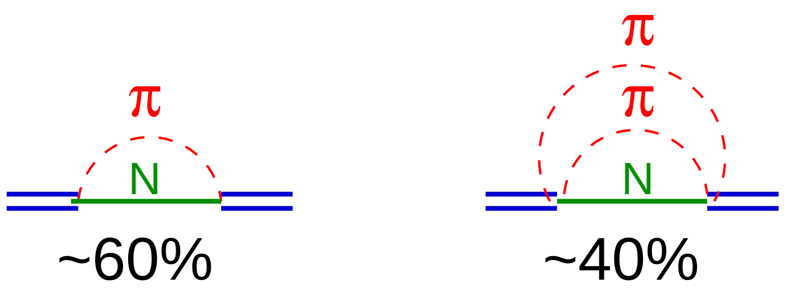
$$\bar V(p) = \frac{V_{-1}}{p^2} + V_0 + V_1p^2 + \cdots \to \frac{V_{-1}}{p_R^2} + V_0 + V_1p_R^2 + \cdots$$

- Resonance formfactor: $\langle \mathbf{P}|J_\mu(0)|-\mathbf{P}\rangle = \underbrace{B_R}_{\text{w.f. norm.}}\;\bar V(p_R)$ $\surd$

Ex. 3: Three particles in a finite volume

K. Polejaeva and AR, EPJA 48 (2012) 67

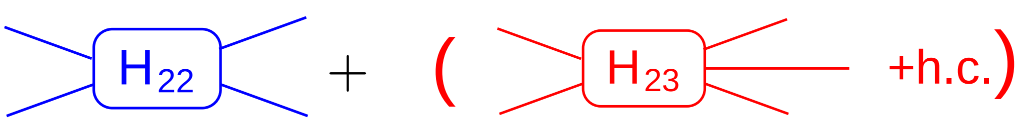
The problem:
finite-volume effects in the spectrum of the Roper resonance



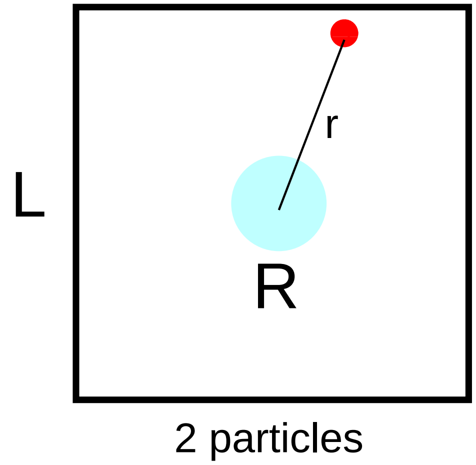
Consider the problem first in the NR Quantum Mechanics:

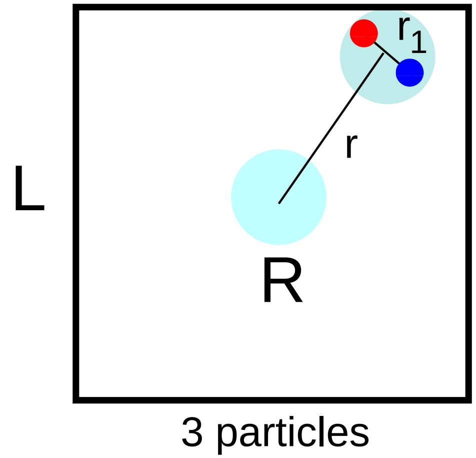
- No Lorentz-invariance
- No 4- and more particle states
- No 2- and 3-particle bound states

$$H = \sum_{i=1}^{3} H_0^{(i)} +$$



The Central problem in the 3-body scattering





- In case of 2 particles: $r \gg R$, when particles are near the walls
- In case of 3 particles: it may happen that $r \gg R, \quad r_1 \simeq R$, when the particles are near the walls

The problem with the disconnected contributions: is the finite-volume spectrum in the 3-particle case determined solely through the on-shell scattering matrix?

Naive analog of Faddeev equations in a finite volume

$$
\begin{aligned}
\mathbf{R}_{4\beta} &= \boldsymbol{\theta}_4 \mathbf{G}_{\mathsf{F}} \left(\boldsymbol{\theta}_\beta + \sum_{\gamma=1}^{3} \mathbf{R}_{\gamma\beta} \right) \\
\mathbf{R}_{\alpha\beta} &= \boldsymbol{\theta}_\alpha \mathbf{G}_{\mathsf{F}} \boldsymbol{\theta}_\beta + \boldsymbol{\theta}_\alpha \mathbf{G}_{\mathsf{F}} \left(\sum_{\gamma=1}^{3} (1-\delta_{\alpha\gamma}) \mathbf{R}_{\gamma\beta} + \mathbf{R}_{4\beta} \right) \\
\mathbf{R}_{\alpha 4} &= \boldsymbol{\theta}_\alpha \mathbf{G}_{\mathsf{F}} \sum_{\gamma=1}^{3} (1-\delta_{\alpha\gamma}) \mathbf{R}_{\gamma 4} + \boldsymbol{\theta}_\alpha \mathbf{G}_{\mathsf{F}} \mathbf{R}_{44} \\
\mathbf{R}_{44} &= \boldsymbol{\theta}_4 + \boldsymbol{\theta}_4 \mathbf{G}_{\mathsf{F}} \sum_{\gamma=1}^{3} \mathbf{R}_{\gamma 4}
\end{aligned}
$$

$$
\boldsymbol{\theta}_\alpha = \mathbf{K}_\alpha + \mathbf{K}_\alpha \mathbf{G}_{\mathsf{F}} \boldsymbol{\theta}_\alpha \, , \qquad \boldsymbol{\theta}_4 = \mathbf{K}_4 + \mathbf{K}_4 \mathbf{G}_{\mathsf{F}} \boldsymbol{\theta}_4
$$

Disconnected contributions

Naive Faddeev equations in a finite volume incorrect due to the presence of the disconnected contributions:

- One iteration gives a tree diagram: no finite-volume effects
- The term $\boldsymbol{\theta}_\alpha G^{\mathsf{F}} \boldsymbol{\theta}_\beta$ in the naive Faddeev equations superfluous
- Dropping this term, the Born series of the Faddeev equations in a finite volume are shown to coinside order by order with that of the original Lippmann-Schwinger equation

$\hookrightarrow$ Despite the presence of the disconnected contributions, the energy spectrum of the 3-particle system in a finite box is still determined by the on-shell scattering matrix elements in the infinite volume

Outlook

- Exotic scalar mesons with the partially twisted boundary conditions (D. Agadjanov)
- The decay $\Delta \to N\gamma$ on the lattice (A. Agadjanov)
- Exotic molecular states of D, D_s and π, K mesons (G. Rios)
- Three-particle problem in a finite volume (M. Döring)
- Quark mass dependence of the resonance parameters (in coll. with J. Nebreda and J. J. Pelaez; G. Schierholz)
- ...