Description of tasks

Integrable models, related to gauge theory

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New solutions to Yang-Baxter equation

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Calogero-Sutherland model and Dunkl operator

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Gauge Theories related to Integrable models

It is generally believed that the key to strong coupling problem lies in integrability.
In the most cases the connection between gauge theories and integrable models is implicit. In particular, the hidden symmetries are responsible for the integrability of SYM theories.
In more manifest form the relation to integrable model (Heisenberg spin chain) the gauge theory (QCD) demonstrates in two cases: in Regge limit $s \gg t$ of multicolor QCD $N_c \gg 1$ (the non-planar diagrams are suppressed by factor N_c^{-1}) and in Bjorken limit of the "hard" scattering processes with large momentum transfer.
Among the possible factors:

- Conformal symmetry
- Supersymmetry
- Planar limit

for the integrablity is essential only latter.

Regge limit

The summation of perturbative series for scattering amplitudes in leading logarithmic approximation by $\alpha_s \log \Lambda$ leads to the effective description in terms of new particles (reggeons, carrying quantum numbers of quarks or gluons - collective coordinates, corresponding to contribution of ladder diagrams with an arbitrary number of rungs). As result the four-dimensional space-time splits onto longitudinal and transversal planes.

Behavior in longitudinal plane, containing momenta of scattered particles entirely determined by kinematics of the process, while whole dynamics concentrated in transversal plane, which in turn is factorized on holomorphic and anti-holomorphic parts, corresponding to one-dimensional closed Heisenberg chain with N sites (the process $2 \to N-2$).

Bjorken limit

The evolution of composite Wilson operators by means of renormalization flow in Callan-Symanzik equation in multicolor QCD can be treated in quantum-mechanical sense as the Schrödinger equation with logarithm of the RG scale $\tau = \log \Lambda$ playing the rôle of the evolution time.
The corresponding Hamiltonian can be identified as that of the open and/or closed Heisenberg magnet with the complex spins $\ell = n + i\nu$, where the real number ν corresponds to anomalous dimension of corresponding vertex operator.
So in both cases one deals with the general complex spins, corresponding to infinite-dimensional non-compact representation $SL(2|\mathbb{R})$ group.
The R-operator in this picture appears as a building block of composite hadronic operators.
From the other hand R-operator is also a building block for Baxter Q operator making quantum canonical transformation to representation with separated variables.

Yang-Baxter equation

The integrability of Heisenberg spin chain is based on YBE:
$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v)$,
Defining relation for Universal $\mathbb{R}$-operator:
$\check{\mathbb{R}}_{12}(u-v)L_1(u)L_2(v) = L_1(v)L_2(u)\check{\mathbb{R}}_{12}(u-v)$, $\check{\mathbb{R}}_{12}(u) = \mathbb{P}_{12}\mathbb{R}_{12}(u)$.
R-operator acts on tensor product of two spaces $V_1 \otimes V_2$, $\mathbb{P}_{12}$ is permutation operator: $\mathbb{P}_{12}(a \times b) = (b \times a)$, $a \in V_1,\ b \in V_2$. The Lax operator depends on n parameters: spectral parameter u and $n-1$ quantum numbers ℓ_k, characterizing representation of $s\ell(n)$:
$L = L(u_1, u_2, \ldots u_n)$, $u_k = u + \ell_k$, $\sum_{k=1}^{n} \ell_k = 0$. The $\mathbb{R}$-operator can be represented in factorized form $\mathbb{R} = \mathcal{R}_1\mathcal{R}_2 \ldots \mathcal{R}_n$, or permutation of two sets $(u_1, u_2, \ldots u_n)$ and $(v_1, v_2, \ldots v_n)$ can be can be achieved step by step:

$$\mathcal{R}_k L_1(u_1, \ldots u_k, \ldots u_n) L_1(v_1, \ldots v_k, \ldots u_v) =$$
$$= L_1(u_1, \ldots v_k, \ldots u_n) L_1(v_1, \ldots u_k, \ldots v_n)\mathcal{R}_k .$$

Particular $\mathcal{R}$-operator

Note that $\mathcal{R}_{12}^{(k)}$ changes "spins" of the representation space:
$\mathcal{R}_{12}^{(k)}: V_{\ell_1\ldots\ell_n}\otimes V_{\rho_1\ldots\rho_n}\to V_{\ell_1\ldots\ell_k+\xi_k\ldots\ell_n}\otimes V_{\rho_1\ldots\rho_k-\xi_k\ldots\rho_n},\ \xi_k=\frac{u_k-v_k}{2}$.
It satisfies to relations:
$\mathcal{R}_{12}^{(k)}(0)=\mathbb{I}$,
$\mathcal{R}_{12}^{(k)}(\lambda)\mathcal{R}_{12}^{(k)}(\mu)=\mathcal{R}_{12}^{(k)}(\lambda+\mu)$,
$\mathcal{R}_{12}^{(k)}(\lambda)\mathcal{R}_{23}^{(k)}(\lambda+\mu)\mathcal{R}_{12}^{(k)}(\mu)=\mathcal{R}_{23}^{(k)}(\mu)\mathcal{R}_{12}^{(k)}(\lambda+\mu)\mathcal{R}_{23}^{(k)}(\lambda)$,
$\mathcal{R}_{12}^{(k)}(\lambda)\mathcal{R}_{23}^{(j)}(\mu)=\mathcal{R}_{23}^{(j)}(\mu)\mathcal{R}_{12}^{(k)}(\mu)$,
In simplest case of $s\ell(2)$ one has:
$\mathcal{R}_{12}^{(2)}(u_1,u_2|v_2)=\frac{\Gamma(u_1-u_2)}{\Gamma(u_1-v_2)}\frac{\Gamma(x_{12}\partial_1+u_1-v_2)}{\Gamma(x_{12}\partial_1+u_1-u_2)}$, and another one is:
$\mathcal{R}^{(2)}\leftrightarrow\mathcal{R}^{(1)}$, $1\leftrightarrow 2,\ u_1\leftrightarrow v_2,\ u_2\leftrightarrow v_1$.
The particular Q-operator is given by:
$Q^{(2)}(u)=tr_{V_0}(\mathbb{P}_{10}\mathcal{R}_{10}^{(2)}(u_1,u_2|0)\ldots\mathbb{P}_{L0}\mathcal{R}_{L0}^{(2)}(u_1,u_2|0))=$
$=\left(\frac{\Gamma(1-\ell-u)}{\Gamma(-2\ell)}\right)^L tr_{V_0}(\mathbb{P}_{10}\frac{\Gamma(x_{01}\partial_0-2\ell)}{\Gamma(x_{01}\partial_0+1-\ell-u)}\ldots\mathbb{P}_{L0}\frac{\Gamma(x_{0L}\partial_0-2\ell)}{\Gamma(x_{0L}\partial_0+1-\ell-u)})$

New solutions to YBE

q-deformation: $[H, e] = -e$, $[H, f] = f$, $[e, f] = \frac{k-k^{-1}}{q-q^{-1}} = [2H]_q$,.

Co-product: $k = q^{2H_1+2H_2} = k_1 k_2$, $e = e_1 + k_1 e_2$, $f = f_1 k_2^{-1} + f_2$.

$\Delta(k) = k \otimes k, \quad \Delta(e) = e \otimes 1 + k \otimes e, \quad \Delta(f) = f \otimes k^{-1} + 1 \otimes f.$

Representations of new kinds: cyclic, semi-cyclic, nilpotent, indecomposable appear.

Invariance of the $\check{R}$-matrix: $\check{R} \cdot \Delta = \Delta \cdot \check{R}$, $\Rightarrow$ $\check{R}(u) = \sum r_i(u) P_i$,

where $c = \sum c_i P_i$ is Casimir operator.

At exceptional values of $q = e^{2\pi i \frac{k}{n}}$ the new Casimir operators e^n, f^n and k^n appear.

At exceptional values of q fusion rules also become degenerate, the number of projection operators grows.

At $q = \pm i$ the fusion $\mathcal{I}_4 \otimes \mathcal{I}_4$ contains 32 terms.

Calogero model and Dunkl operator

Calogero model is (super)integrable with conformal long-range interaction: $H_c = -\frac{1}{2}\Delta + \sum_{i<j}^{N} \frac{c(c-1)}{x_i - x_j}$,
It allows trigonometric and elliptic extensions.
Dunkl operator: $\nabla_k = \partial_k - c\sum_{i\neq k} \frac{1}{x_i - x_k} P_{ik}$ (P_{ik} is permutation operator) appears naturally in quantum theory: the principle of identity of particles.
By means of restriction on symmetric functions:
$Res(\sum_{i=1}^{N} \nabla_i^2) = -2H_c$,.
$[\nabla_i; \nabla_k] = 0, \Rightarrow \nabla_k = U^{-1}\partial_k U$.
U has sense of the S-matrix in the representation of the interaction: $\psi = U \cdot \psi_0$.