String dynamics in static gauge

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Contents

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In preparation G.J., M. Heinze, J.Plefka

Introduction

Action of a particle/string

Geometric action

$$S = -M \int \mathrm{d}s = -M \int \mathrm{d}\tau \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

Polyakov action [Brink, Diveccia, Howe 76]

$$S = \int \mathrm{d}\tau \left[\frac{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}{2\lambda} - \frac{\lambda\, M^2}{2} \right]$$

Dirac type action

$$S = \int \mathrm{d}\tau \left[p_\mu \dot{x}^\mu - \frac{\lambda}{2} \left(g^{\mu\nu}\, p_\mu p_\nu + M^2 \right) \right]$$

$$\dot{x}^\mu = \lambda g^{\mu\nu}\, p_\nu \qquad\qquad g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \lambda^2\, M^2 = 0$$

String actions

Nambu-Goto action

$$S = -\frac{1}{2\pi} \int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{(\dot{x} \cdot x')^2 - (\dot{x})^2 \, (x')^2}$$

Polyakov action

$$S = -\frac{1}{4\pi} \int \mathrm{d}\tau \mathrm{d}\sigma \sqrt{h} \; h^{\alpha\beta} \left(\partial_a x \cdot \partial_\beta x\right)$$

Dirac type action

$$S = \frac{1}{2\pi} \int \mathrm{d}\tau \int \mathrm{d}\sigma \left[(p \cdot \dot{x}) - \lambda_1 (p^2 + (x')^2) - \lambda_2 (p \cdot x') \right]$$

Particle in a static space-time

Static space-time metric tensor

$$g_{\mu\nu} = \begin{pmatrix} g_{00}(x) & 0 \\ 0 & g_{mn}(x) \end{pmatrix} \qquad\qquad g_{00} < 0$$

Space-time coordinates $x^\mu = (x^0, x)$.

The action of a particle in the first order formulation

$$S = \int \mathrm{d}\tau \Big[p_\mu \, \dot{x}^\mu + \frac{\lambda}{2} \left(g^{\mu\nu} p_\mu p_\nu + M^2 \right) \Big]$$

The static gauge $\boxed{x^0 + p_0 \tau = 0}$

The reduced action

$$S = \int \mathrm{d}\tau \Big[p_n \, \dot{x}^n - \frac{1}{2} \, (p_0)^2 \Big]$$

$-p_0 = E > 0$ is the particle energy.

The mass shell condition

$$g^{\mu\nu} p_\mu p_\nu + M^2 = 0$$

Time component $\quad g_{00} = -e^f, \qquad g^{00} = -e^{-f}$

The squared energy

$$E^2 = h^{mn}(x)\, p_m p_n + M^2\, e^{f(x)}$$

This function is associated with the Hamiltonian of a non-relativistic particle moving in the potential $M^2\, e^{f(x)}$ in a curved background with the metric tensor

$$h_{mn} = e^{-f} g_{mn}$$

$h^{mn}(x)$ is the inverse metric

$$h_{mn'}\, h^{n'n} = \delta_m^n$$

Quantization in the coordinate representation

$$\boxed{\hbar = 1}$$

Scalar product with covariant measure

$$\langle\psi_2|\psi_1\rangle = \int \mathsf{d}^N x \sqrt{h(x)}\; \psi_2^*(x)\, \psi_1(x)$$

$h(x) = \det h_{mn}(x)$.

The momentum operators $p_n = -i\partial_n - \frac{i}{4}\partial_n \log h$

The energy square operator

$$E^2 = -\Delta_h + a\mathcal{R}_h(x) + M^2\, e^{f(x)}\ ,$$

Δ_h is the covariant Laplace operator

$\mathcal{R}_h$ is the scalar curvature.

How to fix a? [DeWitt, Bastianeli,...]

Classical description of AdS particle

AdS_{N+1} is realized as a hyperboloid $X^A X_A = -R^2$

$X^A,\ A = (0', 0, 1, ..., N)$ are coordinates of $\mathbb{R}^{2,\,N}$

Metric tensor on $\mathbb{R}^{2,\,N}$ $\eta_{AB} = \mathrm{diag}(-1, -1, 1, ..., 1)$

Parametrization of the hyperboloid

$$X^{0'} = \frac{R\,\sin\theta}{\sqrt{1-x^2}}\,, \qquad X^0 = \frac{R\,\cos\theta}{\sqrt{1-x^2}}\,, \qquad X^n = \frac{R\,x_n}{\sqrt{1-x^2}}\,,$$

with $x^2 := x_n x_n < 1$.

Induced metric tensor

$$g_{\mu\nu} = \frac{R^2}{1-x^2}\begin{pmatrix} -1 & 0 \\ 0 & \delta_{mn} + \frac{x_m\,x_n}{1-x^2} \end{pmatrix}$$

The polar angle θ is interpreted as a time coordinate $\theta = x^0$.

If a vector field $\mathcal{V} = \mathcal{V}(x)^\mu \partial_\mu$ generates space-time isometry transformations

$$\mathcal{L}_\mathcal{V}\, g_{\mu\nu} = 0,$$

then

$$J = \mathcal{V}^\mu\, p_\mu = \mathcal{V}^n(\tau,\, x)\, p_n - \mathcal{V}^0(\tau,\, x)\, E(p, x)$$

is a Noether integral of the system.

The isometries of AdS_{N+1} are the $\mathsf{SO}(2, N)$ transformations

$$\mathcal{V}_{AB}(X^C) = \delta_A^C\, X_B - \delta_B^C\, X_A$$

The corresponding Noether integrals at $\tau = 0$ are

$$E = J_{0'0} = \sqrt{p^2 - (p\cdot x)^2 + \frac{M^2 R^2}{1 - x^2}}\ , \quad J_{mn} = p_m\, x_n - p_n\, x_m\ ,$$

$$J_{n\,0'} = E\, x_n\ , \qquad J_{n\,0} = (p\cdot x)x_n - p_n\ .$$

$\mathfrak{so}(2, N)$ algebra

$$Z_n = J_{n\,0'} - iJ_{n\,0} \ , \qquad Z_n^* = J_{n\,0'} + iJ_{n\,0} \ ,$$

$$\{E,\, Z_n\} = -iZ_n \ , \qquad \{Z_m,\, Z_n^*\} = 2i\delta_{mn}\, E - 2J_{mn} \ .$$

The squared energy in AdS is given by

$$E^2 = h^{mn}(x)\, p_m p_n + \frac{M^2 R^2}{1 - x^2} \ ,$$

with $h^{mn}(x) = \delta_{mn} - x_m\, x_n$.

Its inverse is the metric tensor of the N-dimensional unit semi-sphere

$$h_{mn}(x) = \delta_{mn} + \frac{x_m\, x_n}{1 - x^2}$$

Coordinate representation for the AdS particle

Energy square operator

$$E^2 = -\sqrt{1-x^2}\;\partial_m\;\frac{\delta_{mn} - x_m x_n}{\sqrt{1-x^2}}\;\partial_n + a\,N(N-1) + \frac{M^2\,R^2}{1-x^2}$$

Rotation operators $\qquad J_{mn} = i(x_m\partial_n - x_n\partial_m)$

The first boost generators $\qquad J_{n0'} = \sqrt{E}\;x_n\,\sqrt{E}$

Commutation relation

$$[E^2,\,J_{n0'}] = \sqrt{E}\left(N\,x_n - 2V_n\right)\sqrt{E}\;, \qquad [E^2,\,J_{n0'}] = 2iJ_{n0}\,E + J_{n0'}$$

The second boost generators $\qquad J_{n0} = i\sqrt{E}\;\left(V_n - \frac{N-1}{2}\,x_n\right)\;\frac{1}{\sqrt{E}}$

$$\boxed{a = \frac{N-1}{4N}}$$

The Hilbert space is given by the wave functions $\Psi(x)$ with the scalar product

$$\langle\Psi_2|\Psi_1\rangle \;=\; \int_{x^2<1} \frac{d^N x}{\sqrt{1-x^2}}\; \Psi_2^*(x)\,\Psi_1(x)$$

The lowering operators $\quad Z_n = J_{n0'} - iJ_{n0}.$

The ground state has to be a SO(N) scalar, which is annihilated by $Z_n = J_{n0'} - iJ_{n0}.$

$$\left(x_n\Big(E_0 - \frac{N-1}{2}\Big) + V_n\right)\,\Psi_{E_0}(x^2) = 0\ ,$$

The ground state wave function

$$\Psi_{E_0} \sim (1-x^2)^{\frac{E_0}{2}-\frac{N-1}{4}}$$

Since this function has to be an eigenfunction of E^2 with the eigenvalue E_0^2, we find

$$M^2\,R^2 = \left(E_0 - \frac{N}{2}\right)^2 - \frac{1}{4}$$

String theory in a static gauge

Open string action in the first order formulation

$$S = \int \mathrm{d}\tau \int_0^\pi \frac{\mathrm{d}\sigma}{\pi} \Big(\mathcal{P}_\mu \dot{X}^\mu - \lambda_1 (\mathcal{P}_\mu \mathcal{P}^\mu + X'_\mu X'^\mu) - \lambda_2 (\mathcal{P}_\mu X'^\mu) \Big)$$

Virasoro constraints $\quad \mathcal{P}_\mu \mathcal{P}^\mu + X'_\mu X'^\mu = 0 \ , \qquad \mathcal{P}_\mu X'^\mu = 0$

Static gauge $\quad X^0 + \mathcal{P}_0 \tau = 0 \ , \qquad \mathcal{P}'_0 = 0$

The reduced action

$$S = \int \mathrm{d}\tau \int_0^\pi \frac{\mathrm{d}\sigma}{\pi} \left(\mathcal{P}_k \dot{X}^k - \frac{1}{2} \mathcal{P}_0^2 \right)$$

Free-field Hamiltonian

$$H = \frac{1}{2} \int_0^\pi \frac{\mathrm{d}\sigma}{\pi} \left(\vec{\mathcal{P}}^2 + \vec{X}'^{\,2} \right)$$

The free fields on the (τ, σ) strip $X^k(\tau, \sigma) = \frac{1}{2}\phi^k(\tau + \sigma) + \frac{1}{2}\phi^k(\tau - \sigma)$

Mode expansion $\qquad \phi^k(z) = q^k + p^k z + \mathrm{i} \sum_{n \neq 0} \frac{a_n^k}{n} \, \mathrm{e}^{-\mathrm{i}nz}$

Canonical Poisson brackets

$$\{p^k, q^l\} = \delta^{kl} \qquad\qquad \{a_m^k, a_n^l\} = \mathrm{i}m \, \delta^{kl} \, \delta_{m+n}$$

The generators of conformal transformations

$$L_m = \frac{1}{2} \int_0^{2\pi} \frac{\mathrm{d}z}{2\pi} \, \mathrm{e}^{\mathrm{i}mz} \, \vec{\phi}'(z)^2 = \frac{1}{2} \sum_{n=-\infty}^{\infty} \vec{a}_{m-n} \vec{a}_n$$

The Witt algebra $\qquad \{L_m, L_n\} = \mathrm{i} \, (m - n) \, L_{m+n}$

Constraints in modes

$$L_m = 0, \qquad m \neq 0$$

The dynamical integrals

$$P^\mu = \int_0^\pi \frac{\mathrm{d}\sigma}{\pi}\, \mathcal{P}^\mu \ , \qquad J^{\mu\nu} = \int_0^\pi \frac{\mathrm{d}\sigma}{\pi}\ (\mathcal{P}^\mu\, X^\nu - \mathcal{P}^\nu\, X^\mu)$$

Reduced dynamical integrals

$$P^k = p^k \ , \qquad J^{kl} = p^k q^l - p^k q^l + \mathrm{i} \sum_{n\neq 0} \frac{a^k_{-n} a^l_n}{n} \ ,$$

$$P^0 = p^0 = \sqrt{2L_0} \ , \qquad J^{0k} = p^0 q^k + \frac{\mathrm{i}}{p^0} \sum_{n\neq 0} \frac{a^k_n}{n}\ L_{-n}$$

Deformed boosts

$$\mathcal{J}^{0k} = J^{0k} + \frac{\mathrm{i}}{p^0} \sum_{j\geq 2} \left(\sum_{n_1,\ldots,n_j} f_j(p^0)\ \frac{a^k_n}{n}\ L_{-n_1} \ldots L_{-n_j} \right)$$

Poisson brackets of the form

$$\{\mathcal{J}^{0k},\, L_m\} = \mathcal{A}^k_m\, L_m$$

Quantization in flat space-time

Ground state

$$a_0^k|\vec{p}\rangle = p^k|\vec{p}\rangle \ , \qquad a_n^k|\vec{p}\rangle = 0 \ , \qquad n > 0$$

Virasoro generators $L_0 = \frac{1}{2}\,\vec{p}^2 + N,\qquad N = \sum_{n>0} \vec{a}_{-n}\,\vec{a}_n$

The Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{D-1}{12}(m^3 - m)\delta_{m+n}$$

Physical states $\qquad L_1|\Psi\rangle = 0 \ , \qquad L_2|\Psi\rangle = 0$

Energy operator $\qquad p^0 = \sqrt{\vec{p}^{\,2} + 2(N-a)}$

Boost operators

$$\mathcal{J}^{0k}|\vec{p}, N\rangle = \left(p^0 q^k + \frac{\mathrm{i}}{p^0} \sum_{n=1}^{N} \sum_{(n_1,\ldots,n_j)} f^{(n_1,..,n_j)}(p^0) L_{-n_1} \cdots L_{-n_j} \frac{a_n^k}{n} \right) |\vec{p}, N\rangle$$

Action of boosts on the first excited level

$$\mathcal{J}^{0k}|\vec{p},\,1\rangle = \left(q^k p^0 - \frac{\mathrm{i}\,p^k}{2p^0} + \frac{\mathrm{i}\,f^{(1)}}{p^0}\,L_{-1}a_1^k\right)|\vec{p},\,1\rangle$$

Phys. conditions $L_1\mathcal{J}^{0k}|\vec{p},\,1\rangle = 0\ ,\qquad [\mathcal{J}^{0k},\mathcal{J}^{0l}]|\vec{p},\,1\rangle = \mathrm{i}\,|\vec{p},\,1\rangle$

Solution $\qquad f^{(1)} = 1\ ,\qquad a = 1$

Second level calculations

$$\left(q^k p^0 - \frac{\mathrm{i}\,p^k}{2p^0} + \frac{\mathrm{i}}{p^0}\,L_{-1}a_1^k + \frac{\mathrm{i}\,f^{(2)}}{2\,p^0}\,L_{-2}a_2^k + \frac{\mathrm{i}\,f^{(1,1)}}{2\,p^0}L_{-1}L_{-1}a_2^k\right)|\vec{p},\,2\rangle$$

with the conditions $L_1\mathcal{J}^{0k}|\vec{p},\,2\rangle = 0,\ \ L_2\mathcal{J}^{0k}|\vec{p},\,2\rangle = 0$ yield

$$f^{(1,1)} = -\frac{1}{e+1}\ ,\qquad f^{(2)} = \frac{e}{e+1}\ ,\quad e = 2(p^0)^2\ ,\quad D = 26$$

Connection to the covariant quantization

Physical states in covariant quantization

$$\hat{L}_m ||\psi_{ph}\rangle = 0 \ , \quad \text{for} \quad m > 0, \qquad \text{and} \quad (\hat{L}_0 - 1)||\psi_{ph}\rangle = 0 \ ,$$

$$\hat{L}_m = L_m - L_m^0 \ , \qquad \text{with} \qquad L_m^0 = \frac{1}{2}\, a_{m-n}^0\, a_n^0 \ ,$$

The physical states of static gauge quantization become physical states of the covariant quantization.

The boost operators in the covariant quantization

$$J^{0k} = p^0 q^k - p^k q^0 + \mathrm{i} \sum_{n>0} \left(\frac{a_{-n}^0 a_n^k}{n} - \frac{a_{-n}^k a_n^0}{n} \right)$$

have no ordering ambiguity. From the expansion

$$a_{-n}^0 ||p^0, \vec{p};\, 0, N\rangle = \sum_{(n_1,\ldots,n_j)} \tilde{f}^{(n_1,\ldots,n_j)}(p^0)\ L_{-n_j}^0 \cdots L_{-n_1}^0\, ||p^0, \vec{p};\, 0, N\rangle \ ,$$

follows $\qquad f^{(n_1,\ldots,n_j)}(p^0) = p^0\, \tilde{f}^{(n_1,\ldots,n_j)}(p^0)$

AdS string in static gauge

Parametrization

$$Y^{0'} = \sqrt{R^2 + X^2}\,\sin(\theta/R)\,, \quad Y^0 = \sqrt{R^2 + X^2}\,\cos(\theta/R)\,, \quad Y^k = X^k\,,$$

$$X^2 := X^k X^k.$$

Action

$$S = \int \mathsf{d}\tau \int_0^\pi \frac{\mathsf{d}\sigma}{\pi}\, \left[\mathcal{P}_k\,\dot{X}^k + \frac{1}{2}\left(h^{kl}\mathcal{P}_k\mathcal{P}_l + f_{kl}\,X'^{\,k}X'^{\,l}\right)\right]\,,$$

The matrix

$$h_{kl} = \frac{R^2}{R^2 + X^2}\delta_{kl} - \frac{R^2\,X^k\,X^l}{(R^2 + X^2)^2}$$

corresponds to the induced metric tensor obtained by the projection of a semi-sphere on a plane: the plane is tangent to the sphere at the pole and the projection is made from the center of the sphere.

Summary

We discussed the quantization of a particle in $(N+1)$-dimensional static spacetimes in the coordinate representation. The key point of this analysis is the construction of the squared energy operator which has the form of a non-relativistic particle Hamiltonian in a curved N-dimensional space.

The description of a quantum particle in terms of position dependent wave functions seems to be most natural. But in order to respect the relativistic principles, the generators for transformations involving time (energy, boosts) become non-local.

The ordering ambiguities are fixed by the spacetime isometries.

For the isometry algebra of AdS_{N+1} we found a new representation in terms of operators acting on functions depending on N-dimensional space coordinates.

We have proposed a new treatment of the bosonic strings in static gauge. It has been shown that the string dynamics is D dimensional Minkowski space can be described by $D-1$ component conformal free-field theory, restricted on the constraint surface $L_m = 0$, $m \neq 0$.

The structure of the boost operators has been found on the basis of classical calculations. This structure defines the boosts up to some energy dependent coefficients. These coefficients can be calculated to any desirable level, but their closed form is still missing. Simple low level calculations of the commutation relations of the boost operators define the string mass spectrum and the space-time critical dimension.

We have shown the equivalence between the static gauge and the covariant quantization. This equivalence shows that the coefficients we were looking for in the static gauge quantization are just the expansion coefficients of the oscillator excitations in terms of the Virasoro excitations.

AdS strings description in static gauge exhibits a new coset WZW structure, which differs from the Pohlmeyer reduction.