Dynamics of massive matter escape into extra space

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Brane solution described in [M. Gogberashvili, Int. J. Mod. Phys. **D11** (2002) 1635, hep-th/9812296; M. Gogberashvili, Int. J. Mod. Phys. **D11** (2002) 1639, hep-ph/9908347; L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 3370, hep-ph/9905221; L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 4690, hep-th/9906064] has the form

$$ds^2 = e^{-2\kappa|z|}\eta_{\mu\nu}dx^\mu dx^\nu - dz^2 \ , \tag{1}$$

where the parameter k is determined by the bulk cosmological and five-dimensional gravitational constants respectively.

Localization of standard model fields on the brane: [B. Bajc and G. Gabadadze, Phys. Lett. **B474** (2000) 282, hep-th/9912232].

Tunneling into an extra dimension: [S. Dubovsky, V. Rubakov and P. Tinyakov, Phys. Rev. **D62** (2000) 105011; hep-th/0006046].

Decaying cold dark matter into an extra space: [K. Ichiki, P. Garnavich, T. Kajino, G. Mathews and M. Yahiro Phys. Rev. **D68** (2003) 083518; astro-ph/0210052].

The comoving density of cold dark matter was taken to decay over time with a rate

$$\rho a^3 \exp(-2\Gamma t)$$

corresponding to the exponential decay of a metastable state.

Quantum Mechanics: exponential decay cannot last forever if the Hamiltonian is bounded below and cannot occur for small times if, besides that, the energy expectation value of the initial state is finite [C. Chiu, E. Sudarshan and B. Misra, Phys. Rev. **D16** (1977) 520; L. Fonda, G. Ghirardi and A. Rimini, Rept. Prog. Phys. **41** (1978) 587].

Due to four-dimensional Poincare invariance of the RS brane every fields in this background can be decomposed into four-dimensional plane waves

$$\phi \propto \exp(ip_\mu x^\mu)\varphi(z) \ , \tag{2}$$

where coordinate four momentum p_μ coincides with the physical momentum on the brane. For the sake of simplicity we put $\vec{p} = \mathbf{0}$ in what follows. Under this assumption the equation

$$\partial_t^2\phi - e^{2k|z|}\partial_z(e^{-4k|z|}\partial_z\phi) + e^{-2k|z|}\mu^2\phi = 0 \ , \tag{3}$$

gives

$$-\partial_z^2\varphi + 4k\,\mathrm{sgn}(z)\partial_z\varphi + \mu^2\varphi = e^{2k|z|}E^2\varphi \ , \tag{4}$$

where $E \equiv p_0$. The continuum spectrum of eq.(4) starts from $E^2 = 0$

$$\varphi = \sqrt{\frac{E}{2k}e^{2k|z|}}\left[a(E)J_\nu\left(\frac{E}{k}e^{k|z|}\right) + b(E)Y_\nu\left(\frac{E}{k}e^{k|z|}\right)\right] \ ,$$

where $\nu = \sqrt{4 + \mu^2/k^2}$. The normalization condition

$$\int\limits_{-\infty}^{\infty} dz e^{-2k|z|}\varphi(E,\ z)\varphi(E',\ z) = \delta(E - E') \ ,$$

as well as the boundary condition $\partial_z\varphi|_{z=0} = 0$ can be satisfied by taking

$$a(E) = -\frac{A(E)}{\sqrt{1 + A^2(E)}} \ , \quad b(E) = \frac{1}{\sqrt{1 + A^2(E)}} \ ,$$

$$A(E) = \frac{Y_{\nu-1}(E/k) - (\nu - 2)(k/E)Y_\nu(E/k)}{J_{\nu-1}(E/k) - (\nu - 2)(k/E)J_\nu(E/k)} \ .$$

The general solution to Eq.(3) can be written as

$$\begin{aligned}\phi(t,\ z) \ &= \ \int\limits_{-\infty}^{\infty} G_1(t,\ z,\ z')\phi(0,\ z')dz' \\ &+ \ \int\limits_{-\infty}^{\infty} G_2(t,\ z,\ z')\dot{\phi}(0,\ z')dz' \ ,\end{aligned} \tag{5}$$

where

$$G_1(t,\ z,\ z') = e^{-2k|z'|} \int\limits_0^\infty dE \cos(Et) \varphi(E,\ z) \varphi(E,\ z') \ ,$$

$$G_2(t,\ z,\ z') = e^{-2k|z'|} \int\limits_0^\infty dE \frac{\sin(Et)}{E} \varphi(E,\ z) \varphi(E,\ z') \ .$$

To compute $\phi(t,\ z)$ one needs to know the initial data $\phi(0,\ z)$, $\dot{\phi}(0,\ z)$. In what follows we take the following condition $\dot{\phi}(0,\ z) = iE_0 \phi(0,\ z)$ corresponding to the free particle on the brane (2) with energy E_0. Correspondingly the Eq.(5) takes the form

$$\phi(t,\ z) = \int\limits_{-\infty}^\infty \left[G_1(t,\ z,\ z') + iE_0 G_2(t,\ z,\ z')\right] \phi(0,\ z') dz' \ . \qquad (6)$$

At $t = 0$ the particle is confined on the brane. The probability of the particle to remain on the brain at instant t is given by

$$|\langle\phi(0)|\phi(t)\rangle|^2 \ ,$$

where $\phi(0)$ denotes the brane localized initial state and the scalar product is understood with the measure $\exp(-2k|z|)$

$$\langle\phi(0)|\phi(t)\rangle = \int\limits_{-\infty}^\infty dz e^{-2k|z|} \phi^*(0,\ z) \phi(t,\ z) \ .$$

From Eq.(6) one finds

$$Re \langle\phi(0,\ z)|\phi(t,\ z)\rangle = Re \left[\int\limits_0^\infty dE e^{-iEt} |C(E)|^2\right] \ , \qquad (7)$$

$$Im \langle\phi(0,\ z)|\phi(t,\ z)\rangle = -E_0 Im \left[\int\limits_0^\infty dE \frac{e^{-iEt}}{E} |C(E)|^2\right] \ , \qquad (8)$$

where

$$C(E) = \langle\phi(0)|\varphi(E)\rangle \ .$$

In the above E is real variable, but in the following it is understood that E can be complex. The function $|C(E)|^2$ has the simple poles determined by the equation

$$i \pm A(E) = 0 \ .$$

The equation

$$A(E) = i \ \Rightarrow \ \frac{E}{k} H^{(1)}_{\nu-1}\left(\frac{E}{k}\right) + (2-\nu) H^{(1)}_{\nu}\left(\frac{E}{k}\right) = 0 \ ,$$

under assumption $\mu/k, E/k \ll 1$ gives the pole in the fourth quadrant of a complex E plane, $E = E_0 - i\Gamma$, where

$$E_0 = \frac{\mu}{\sqrt{2}} \ , \quad \frac{\Gamma}{E_0} = \frac{\pi}{8}\left(\frac{E_0}{k}\right)^2 \ .$$

Since, when $t > 0$ and $|E| \to \infty$, $e^{-iEt} \to 0$ in the fourth quadrant, for evaluating of integrals entering Eqs.(7, 8) one can deform the integration counter to the imaginary axis. In this way one finds

$$\int_0^\infty dE e^{-iEt} \left|C(E)\right|^2 \ = \\ \text{residue term} \ - \ i \int_0^\infty dE e^{-Et} \left|C(-iE)\right|^2 . \qquad (9)$$

From this equation one sees that the second term gives an imaginary contribution and correspondingly $Re \left\langle \phi(0,\ z) | \phi(t,\ z) \right\rangle$ is determined by the residue term only. In the same way one finds that $Im \left\langle \phi(0,\ z) | \phi(t,\ z) \right\rangle$ is also determined only by the residue term. Following to the quantum mechanical formalism, and by taking into account that $\mu/k,\ E_0/k \ll 1$ and the localization width of the scalar is $\sim k^{-1}$, one can simply take

$$\phi(0,\ z) = \begin{cases} \sqrt{k}/\sqrt{1-e^{-1}} \ , & \text{for} \ \ |z| \leq (2k)^{-1} \ , \\ 0 \ , & \text{for} \ \ |z| > (2k)^{-1} \ . \end{cases} \qquad (10)$$

Thus, the probability of finding the particle on the brane equals one when $t = 0$. By evaluating the residue term and using Eq.(10) one finds

$$\left| \left\langle \phi(0) | \phi(t) \right\rangle \right|^2 = 0.04 \times e^{-2\Gamma t} \ .$$

The minimal representation of spinors in five dimensions can be chosen to be four dimensional. The five-dimensional Minkowskian gamma matrices can be chosen as follows

$$\begin{array}{l}\Gamma^\mu = \gamma^\mu \ , \\ \Gamma^z = -i\gamma^5 \ , \end{array} \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \ , \ \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ ,$$

where $\sigma^\mu = (\mathbf{1} \ , \ \vec{\sigma}) \ , \ \bar{\sigma}^\mu = (\mathbf{1} \ , \ - \vec{\sigma})$ with σ_i the three Pauli matrices. The Dirac equation in the background

$$ds^2 = e^{-2\sigma(z)}\eta_{\mu\nu}dx^\mu dx^\nu - dz^2 \ ,$$

for the fermion coupled with the domain wall Φ reads

$$\left[i\Gamma^z(\partial_z - 2\sigma'(z)) + ie^{\sigma(z)}\Gamma^\mu\partial_\mu - g\Phi\right]\psi = 0 \ . \tag{11}$$

Solutions to this equation are linear combination of wave functions of the form

$$\psi = \exp(-ip_\mu x^\mu)\chi \ .$$

We do not expect the three momentum $\vec{p}$ to affect the overall picture of particle tunneling into an extra space. Therefore, as in the case of scalar field, we assume $\vec{p} = \mathbf{0}$. Under this assumption the Eq.(11) takes the form

$$\left(i\partial_0 - H\right)\psi = 0 \ , \tag{12}$$

where

$$H = e^{-\sigma}\left[\gamma^0 g\Phi - \gamma^0\gamma^5(\partial_z - 2\sigma')\right] \ . \tag{13}$$

So that the time evolution of initially brane localized fermion $\psi(0, \ z)$ is given by

$$\psi(t, \ z) = \int_0^\infty dE e^{-iEt}\chi(E, \ z)\left\langle\chi(E)|\psi(0)\right\rangle \ , \tag{14}$$

where the scalar product is understood with the measure $e^{-3k|z|}$ and

$$H\chi = E\chi \ . \tag{15}$$

In terms of the right- and left-handed components

$$\chi_R \equiv \frac{1+\gamma^5}{2}\chi \ , \quad \chi_L \equiv \frac{1-\gamma^5}{2}\chi \ ,$$

the Eq.(15) takes the form

$$(\partial_z - 2\sigma' - g\Phi)\,\chi_R = -Ee^{\sigma}\chi_L\ , \tag{16}$$

$$(\partial_z - 2\sigma' + g\Phi)\,\chi_L = Ee^{\sigma}\chi_R\ , \tag{17}$$

where $E \equiv p_0$. After eliminating χ_R from Eqs.(16, 17) one obtains a second order equation for χ_L

$$\Big[\partial_z^2 - 5\sigma'\partial_z - 2\sigma'' + 6(\sigma')^2 - g\sigma'\Phi \\ + g\Phi' - (g\Phi)^2 + E^2e^{2\sigma}\Big]\,\chi_L = 0\ . \tag{18}$$

For the domain wall profile in the thin wall limit one can take

$$\Phi = v\,\mathrm{sgn}(z)\ .$$

To the right ($z > 0$) and left ($z < 0$) of the brane one gets

$$\Big[\partial_z^2 - 5k\partial_z - gkv + 6k^2 - g^2v^2 + E^2e^{2kz}\Big]\,\chi_L = 0\ , \\ \Big[\partial_z^2 + 5k\partial_z - gkv + 6k^2 - g^2v^2 + E^2e^{-2kz}\Big]\,\chi_L = 0\ . \tag{19}$$

The solution of Eq.(19) reads

$$\chi_L = \sqrt{\frac{Ee^{5k|z|}}{2k}}\left[a(E)J_\nu\left(\frac{E}{k}e^{k|z|}\right) + b(E)Y_\nu\left(\frac{E}{k}e^{k|z|}\right)\right]\ , \tag{20}$$

where $\nu = (2gv+k)/2k$. From Eqs.(16,17) one finds the following boundary condition

$$\chi_L'(0) + (gv - 2k)\chi_L(0) = 0\ ,$$

where the prime stands for the derivative with respect to $|z|$. To satisfy this boundary condition as well as the normalization condition

$$\int\limits_{-\infty}^{\infty} dze^{-3k|z|}\chi_L(E,\ z)\chi_L(E',\ z) = \delta\left(E - E'\right)\ ,$$

the coefficients $a(E),\ b(E)$ should have the form

$$a(E) = \frac{Y_{\nu-1}(E/k)}{\sqrt{Y_{\nu-1}^2(E/k) + J_{\nu-1}^2(E/k)}}\ ,$$

$$b(E) = -\sqrt{1-a^2(E)} \ .$$

The solution to Eq.(19) for $E=0$

$$\psi_L \ \propto \ \ e^{(2k-gv)|z|} \ , \tag{21}$$

is localized on the brane as long as $k < 2gv$ with the localization width $2/(2gv-k)$. Under assumption that the initial state is given by the left-handed particle localized on the brane one finds

$$\langle \psi_L(0)|\psi_L(t)\rangle = \int\limits_0^\infty dE e^{-iEt} \left|\langle \chi_L(E)|\psi_L(0)\rangle\right|^2 \ . \tag{22}$$

The function $\left|\langle \chi_L(E)|\psi_L(0)\rangle\right|^2$ may have the poles E_n in the fourth quadrant of complex E plane determined by the zeros of $H^{(1)}_{\nu-1}(E/k)H^{(2)}_{\nu-1}(E/k)$ indicating the presence of resonances in the spectrum of KK modes. For evaluating the integral (22) one can deform the integration contour to the imaginary axis as it was done in the previous case. In this way one gets

$$\langle \psi_L(0)|\psi_L(t)\rangle = \\ \sum_n c_n e^{-iE_n t} - i \int\limits_0^\infty dE e^{-Et} \left|\langle \chi_L(-iE)|\psi_L(0)\rangle\right|^2 \ , \tag{23}$$

where

$$c_n = -2\pi i \lim_{E\to E_n} (E-E_n) \left|\langle \chi_L(E)|\psi_L(0)\rangle\right|^2 \ .$$

So that the probability to find the particle on the brane at time t is given by

$$\left|\langle \psi_L(0)|\psi_L(t)\rangle\right|^2 = \sum_n |c_n|^2 e^{2Im(E_n)t} + \text{interference terms} \ .$$

One can easily check that

$$\left.\frac{d\,\langle \psi_L(0)|\psi_L(t)\rangle}{dt}\right|_{t=0} = 0 \ .$$

Namely, by using Eqs.(12,13) and taking into account that $\psi_L(0,\ z)$ is an even function of z one sees that in this expression the odd function is integrated over a symmetric region. So that the evolution of the decay process requires some time to reach the regime of exponential decay. For large values of time $t \gg k^{-1}$ the second term in Eq.(23) becomes dominant for which in this limit only modes with $E \ll k$ are relevant. Therefore, the non-escape probability behaves asymptotically as

$$|\langle\psi_L(0)|\psi_L(t)\rangle|^2 \sim (kt)^{-4\nu+4} \ .$$

Thus, after a long time the decay proceeds with respect to the power law. One can choose the index ν in such a way $(2 < \nu < 5)$ the Hankel function $H^{(1)}_{\nu-1}(\omega)$ to have exactly one zero in the fourth quadrant of complex ω plane. This is the only zero of the product $H^{(1)}_{\nu-1}(\omega)H^{(2)}_{\nu-1}(\omega)$ located in the fourth quadrant since for real values of ν, $H^{(1)}_{\nu}(\omega)^* = H^{(2)}_{\nu}(\omega^*)$. From the trajectories of zeros one finds that if ν is close to 2.5 $(\nu > 2.5)$ the function $H^{(1)}_{\nu-1}(E/k)$ has the zero $E = E_0 - i\Gamma$ in the fourth quadrant such that $E_0/k \ll 1$ and $\Gamma \sim k$. Using Eq.(21) for the initial state of the low lying KK resonances one can simply take

$$\psi_L(0) = \begin{cases} \sqrt{\frac{2gv-k}{2(1-e^{-1})}}e^{(2k-gv)|z|} \ , & |z| \leq (2gv-k)^{-1} \ , \\ 0 \ , & |z| > (2gv-k)^{-1} \ . \end{cases} \qquad (24)$$

So that at $t = 0$ the particle is known to be on the brane, $|z| \leq (2gv-k)^{-1}$, with probability one. Using Eq.(24) and the previous consideration one can perform the concrete calculations for different low lying KK resonances.

Conclusion

From the present consideration one sees that if the transverse equation contains the second order time derivative the corresponding decay law has the exponential form as long as $\Gamma \ll E_0$. So that under this assumption the decay of massive initially brane localized metastable modes for scalar, vector and gravitational fields follows to the exponential law. It should be stressed that the choice $\dot{\phi}(0,\ z) = iE_0\phi(0,\ z)$ which is essential to this result is not unique. However we find this condition to be natural as it corresponds to the free particle on the brane with energy E_0 corresponding to the metastable state.

In contrast, the decay of massive quasilocalized fermion modes does not follow completely to the exponential law. As in the standard quantum mechanical case, the decay initially is slower than exponential, then comes the exponential region and after a long time it obeys a power law.